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Analysis of time-domain elastic scattering by an unbounded structure

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This paper is devoted to the analysis of the time-domain elastic wave scattering problem in an unbounded structure. The transparent boundary condition is developed to reformulate the scattering problem into an initial-boundary value problem in an infinite slab. The well posedness and stability are established for the reduced problem in both the frequency and time domains. Our proofs are based on the energy method, the Lax-Milgram theorem, and the inversion theorem of the Laplace transform. Moreover, a priori estimates with explicit dependence on the time are achieved for the elastic displacement by taking special test functions for the time-domain variational problems of the Navier equation.

KEYWORDS
a priori estimates, Laplace transform, stability, time-domain elastic wave equation, unbounded rough surfaces

1 | INTRODUCTION

This paper is concerned with the mathematical analysis of the time-domain elastic wave scattering by an unbounded structure in two dimensions. An unbounded surface is referred to as a nonlocal perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane. The scattering problems in unbounded structures for acoustic, electromagnetic, and elastic waves have been of great interests to physicists, engineers, and applied mathematicians. These problems have significant applications in various scientific areas such as optics, acoustics, radio wave propagation, seismology, and radar techniques.[1-5] In particular, diffraction phenomena for the propagation of elastic waves through unbounded interfaces have many applications in geophysics and seismology. For instance, the problem of elastic pulse transmission and reflection through the Earth is fundamental to the investigation of earthquakes and the utility of controlled explosions in search for oil and ore bodies.[6-8]

The problem addressed in this work belongs to the class of unbounded rough surface scattering problems, which are quite challenging because of unbounded structures. The usual Sommerfeld (for acoustic waves), Kupradze-Sommerfeld (for elastic waves), or Silver-Müller (for electromagnetic waves) radiation condition is no longer valid.[9,10] The typical Fredholm alternative argument is not applicable either because of the lack of compactness results. The time-harmonic problems have been widely studied for the wave scattering by unbounded structures. We refer to Chandler-Wilde et al,[11] Chandler-Wilde and Monk,[12] Chandler-Wilde and Zhang,[13] Lechleiter and Ritterbusch,[14] and Li and Shen[15] for the two-dimensional Helmholtz equation, Haddar and Lechleiter,[16] Li et al,[17] and Li et al[18] for the three-dimensional Maxwell equations, and Arens[19,20] for the elastic wave equation. Despite so many studies conducted so far, it is still unclear what the least restrictive conditions are for those physical parameters and geometrical shapes to assure the well posedness of the scattering problems in unbounded structures.

The time-domain scattering problems have recently attracted considerable attention because of their capability of capturing wideband signals and modeling more general material and nonlinearity.[21-25] These unique features motivate us to turn our effort from seeking the best possible conditions for those physical parameters to studying directly the time-domain problems. Compared with the time-harmonic problems, the time-domain problems are less studied because of the additional challenge of the temporal dependence. The analysis can be found in Chen and Nédélec[26] and Wang B and Wang LL[27] for the time-domain acoustic and electromagnetic obstacle scattering problems. We refer to Li et al[28] and Gao and Li[29] for the analysis of the time-dependent electromagnetic scattering from an open cavity and a periodic structure, respectively. Compared with Maxwell equation in Gao and Li[30] and the acoustic-elastic interaction scattering problems in Bao et al[31] and Gao et al,[32] the elastic wave problem appears to be more complicated because of the coexistence of compressional and shear waves that propagate at different speeds. What differs dramatically from the acoustic and electromagnetic wave equations is that the transparent boundary operator is derived from the Helmholtz decomposition. The essential difficulty is to show the positive definite of the operator, which is crucial in establishing the well posedness and stability of the problem. We define an admissible set for the Lamé parameters to handle this issue. The set allows some feasible parameters. But it remains an open problem on how to remove this limitation and show the same results for more general media.

The rest of the paper is organized as follows. In Section 2, we present the model problem of the time-domain elastic scattering by an unbounded structure. The transparent boundary condition (TBC) is introduced to reformulate the problem into an initial-boundary value problem in infinite slab. Two auxiliary problems are studied in Section 3. The well posedness and stability of the reduced problem are established in the frequency domain. The well posedness of the time-domain elastic wave equation with the Dirichlet boundary condition is presented. Section 4 is devoted to the well posedness and stability of the reduced time-domain elastic wave equation and a priori estimates of the solution. We conclude the paper with some remarks in Section 5.

2 | PROBLEM FORMULATION

In this section, we introduce a mathematical model and define some notations for the elastic scattering by an unbounded structure.

2.1 | Elastic wave equation

Let us first specify the problem geometry, which is shown in Figure 1. The problem is assumed to be invariant in the z-direction. Let S_1 and S_2 be two Lipschitz continuous surfaces, which are embedded in the slab

$$\Omega = \{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \,:\, h_2 < y < h_1\},$$

where h_1 and h_2 are two constants. Such a geometric assumption is weaker than that used in Arens[19,20] for unbounded rough surfaces. The region between the surfaces S_1 and S_2 may be filled with an isotropic inhomogeneous elastic medium, which is characterized by the variable Lamé parameters $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ and the variable density $\rho(\boldsymbol{x})$. The regions above the surface S_1 and below the surface S_2 are assumed to be filled with isotropic homogeneous elastic media. Let $\Omega_1 = \left\{\boldsymbol{x} \in \mathbb{R}^2 \,:\, y > h_1)\right\}$ and $\Omega_2 = \left\{\boldsymbol{x} \in \mathbb{R}^2 \,:\, y < h_2\right\}$. Define $\Gamma_1 = \{y = h_1\}$ and $\Gamma_2 = \{y = h_2\}$. Hence, the surfaces S_1 and S_2 divide Ω into three connected components.

The displacement of the wave field $\boldsymbol{u} = (u_1, u_2)^\top$ is governed by the time-domain elastic wave equation:

$$\rho(\boldsymbol{x})\partial_t^2 \boldsymbol{u}(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x}, t)) = \boldsymbol{j}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathbb{R}^2, t > 0, \tag{2.1}$$

where $\boldsymbol{j}$ is the external force, which is assumed to have a compact support contained in $\Omega \times (0, T)$ for some $T > 0$, the stress tensor $\boldsymbol{\sigma}(\boldsymbol{u})$ is given by the generalized Hook law:

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\epsilon(\boldsymbol{u}) + \lambda \mathrm{tr}(\epsilon(\boldsymbol{u}))\mathbf{I}, \quad \epsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top). \tag{2.2}$$

Here $\mathbf{I}$ is the 2×2 identity matrix, and $\nabla \boldsymbol{u}$ is the displacement gradient tensor given by

$$\nabla \boldsymbol{u} = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix}.$$

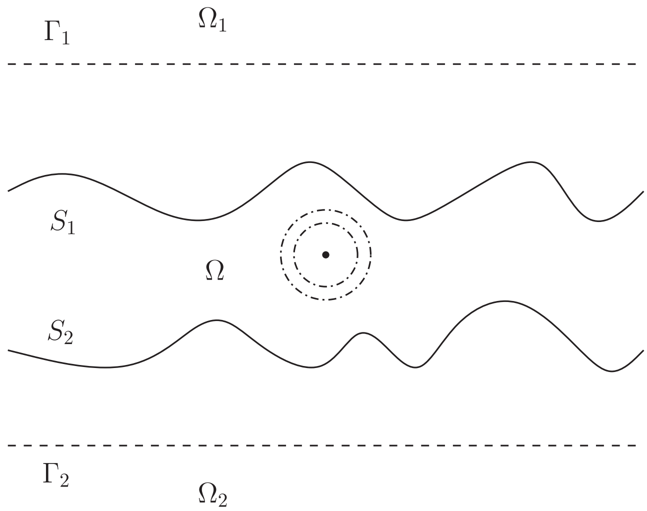

FIGURE 1 Problem geometry of the elastic scattering by an unbounded structure

The density $\rho(\boldsymbol{x}) \in L^\infty(\mathbb{R}^2)$ and the Lamé parameters $\mu(\boldsymbol{x}) \in L^\infty(\mathbb{R}^2)$, $\lambda(\boldsymbol{x}) \in L^\infty(\mathbb{R}^2)$. They satisfy

$$\rho(\boldsymbol{x}) > 0, \quad \mu(\boldsymbol{x}) > 0, \quad \mu(\boldsymbol{x}) + \lambda(\boldsymbol{x}) > 0, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

Since the media are homogeneous in Ω_j, there exist constants μ_j, λ_j, ρ_j such that

$$\mu(\boldsymbol{x}) = \mu_j,\ \lambda(\boldsymbol{x}) = \lambda_j,\ \rho(\boldsymbol{x}) = \rho_j, \quad \boldsymbol{x} \in \Omega_j,\ j = 1, 2.$$

Substituting (2.2) into (2.1) yields

$$\rho\partial_t^2\boldsymbol{u} - \nabla \cdot (\mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)) - \nabla\,(\lambda\nabla \cdot \boldsymbol{u}) = \boldsymbol{j} \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+. \tag{2.3}$$

The system is constrained by the initial conditions:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}|_{t=0} = \boldsymbol{u}_1,$$

where $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are also assumed to be compactly supported in Ω. Because of the unbounded problem geometry, it is no longer valid to impose the classical Kupradze-Sommerfeld radiation condition (see, eg, Kupradze et al[33]). We employ the following radiation condition: The wave field is required to be bounded outgoing in $\Omega_j, j = 1, 2$ as $y \to \pm\infty$. The specific radiation condition is given in Section 2.3.

2.2 | Function spaces and Laplace transform

We introduce some Sobolev spaces. For $u \in L^2(\Gamma_j)$, we denote by $\hat{u}$ the Fourier transform of u, ie,

$$\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} u(x)e^{-\mathrm{i}x\xi}\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

Define the functional space

$$H^\nu(\mathbb{R}) = \left\{ u(x) \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + \xi^2)^\nu |\hat{u}|^2 \mathrm{d}\xi < \infty \right\},$$

whose norm is defined by

$$\|u\|_{H^\nu(\mathbb{R})} = \left[\int_{\mathbb{R}} (1 + \xi^2)^\nu |\hat{u}|^2 \mathrm{d}\xi \right]^{1/2}.$$

It is clear that the dual space of $H^\nu(\mathbb{R})$ is $H^{-\nu}(\mathbb{R})$ with respect to the scalar product in $L^2(\mathbb{R})$ defined by

$$\langle u, v \rangle = \int_{\mathbb{R}} \hat{u}\bar{\hat{v}}\mathrm{d}\xi.$$

Define a trace functional space

$$H^{1/2}(\Gamma_j) = \{u(x) : u(x) \in H^{1/2}(\mathbb{R})\}$$

and a dual paring

$$\langle u, v\rangle_{\Gamma_j} = \int_{\Gamma_j} u\bar{v}\mathrm{d}x = \int_{\mathbb{R}} \hat{u}(\xi)\hat{v}(\xi)\mathrm{d}\xi.$$

Denote by $H^{-1/2}(\Gamma_j)$ the dual space of $H^{1/2}(\Gamma_j)$, ie, $H^{-1/2}(\Gamma_j) = (H^{1/2}(\Gamma_j))'$. The norm on the space $H^{-1/2}(\Gamma_j)$ is defined by

$$\|u\|_{H^{-1/2}(\Gamma_j)} = \sup_{v\in H^{1/2}(\Gamma_j)} \frac{|\langle u, v\rangle_{\Gamma_j}|}{\|v\|_{H^{1/2}(\Gamma_j)}}.$$

We define the Sobolev space $H^{\nu}(\Omega) = \{D^{\alpha}u \in L^2(\Omega)\text{for all}|\alpha| \leq \nu\}$, which is the Banach space for the norm

$$\|u(x,z)\|_{H^{\nu}(\Omega)} = \left[\int_{h_2}^{h_1} \sum_{l+m\leq\nu} \left(\int_{\mathbb{R}} (1+\xi^2)^l |D_y^m \hat{u}(\xi, y)|^2 \mathrm{d}\xi\right) \mathrm{d}y\right]^{1/2}.$$

Here $l, m \in \mathbb{N}$ and D_y^m is the mth derivative with respect to y. These norms given in the spatial-frequency domain are equivalent to the usual Sobolev norms in the entire spatial domain because of the Parseval identity.

Let $H^1(\Omega)^2 = H^1(\Omega) \times H^1(\Omega)$ be a Cartesian produce space, which is equipped with the norm

$$\|\boldsymbol{u}\|_{H^1(\Omega)^2} = \left[\|u_1\|^2_{H^1(\Omega)} + \|u_2\|^2_{H^1(\Omega)}\right]^{1/2}.$$

Denote $H^{\nu}(\mathbb{R})^2 = H^{\nu}(\mathbb{R}) \times H^{\nu}(\mathbb{R})$ with the norm

$$\|\boldsymbol{u}\|_{H^{\nu}(\mathbb{R})^2} = \left[\|u_1\|^2_{H^{\nu}(\mathbb{R})} + \|u_2\|^2_{H^{\nu}(\mathbb{R})}\right]^{1/2}.$$

It is also easy to verify that $H^{-\nu}(\mathbb{R})^2$ is the dual space of $H^{\nu}(\mathbb{R})^2$ for any ν with respect to the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int_{\mathbb{R}} \hat{\boldsymbol{u}} \cdot \bar{\hat{\boldsymbol{v}}}\mathrm{d}\xi.$$

Next, we introduce some properties of the Laplace transform. For any $s = s_1 + is_2$ with $s_1 > \sigma_0 > 0, s_2 \in \mathbb{R}$, and $\mathrm{i} = \sqrt{-1}$, define by $\breve{\boldsymbol{u}}(s)$ the Laplace transform of the vector field $\boldsymbol{u}(t)$, ie,

$$\breve{\boldsymbol{u}}(s) = \mathscr{L}(\boldsymbol{u})(s) = \int_0^{\infty} e^{-st}\boldsymbol{u}(t)\mathrm{d}t.$$

It follows from the integration by parts that

$$\int_0^t \boldsymbol{u}(\tau)\mathrm{d}\tau = \mathscr{L}^{-1}(s^{-1}\breve{\boldsymbol{u}}(s)),$$

where $\mathscr{L}^{-1}$ is the inverse Laplace transform. It can be verified from the inverse Laplace transform that

$$\boldsymbol{u}(t)= \mathscr{F}^{-1}\left(e^{s_1 t}\mathscr{L}(\boldsymbol{u})(s_1+s_2)\right),$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2.

Recall the Plancherel or Parseval identity for the Laplace transform (cf Cohen[34, (2.46)]):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{\boldsymbol{u}}(s)\breve{\boldsymbol{v}}(s)\mathrm{d}s_2 = \int_0^{\infty} e^{-2s_1 t}\boldsymbol{u}(t)\boldsymbol{v}(t)\mathrm{d}t, \quad \forall\, s_1 > \sigma_0 > 0, \tag{2.4}$$

where $\breve{\boldsymbol{u}} = \mathscr{L}(\boldsymbol{u}), \breve{\boldsymbol{v}} = \mathscr{L}(\boldsymbol{v})$, and σ_0 is abscissa of convergence for the Laplace transform of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Hereafter, the expression $a \lesssim b$ stands for $a \leq Cb$, where C is a positive constant and its specific value is not required but should be clear from the context.

The following lemma (cf Trèves[35, Theorem 43.1]) is an analogue of the Paley-Wiener-Schwarz theorem for the Fourier transform of distributions with compact supports in the case of the Laplace transform.

Lemma 2.1. *Let $\breve{\boldsymbol{h}}(s)$ be a holomorphic function in the half plane $s_1 > \sigma_0$ and be valued in the Banach space $\mathbb{E}$. The following two conditions are equivalent:*

(1) there is a distribution $\breve{\boldsymbol{h}} \in \mathcal{D}'_+(\mathbb{E})$ whose Laplace transform is equal to $\breve{\boldsymbol{h}}(s)$;
(2) there is a real σ_1 with $\sigma_0 \leq \sigma_1 < \infty$ and an integer $m \geq 0$ such that for all complex numbers s with $\mathrm{Re} s = s_1 > \sigma_1$, the estimate $\|\breve{\boldsymbol{h}}(s)\|_{\mathbb{E}} \lesssim (1+|s|)^m$ holds,

where $\mathcal{D}'_+(\mathbb{E})$ is the space of distributions on the real line that vanish identically in the open negative half line.

2.3 | Transparent boundary conditions

We introduce exact time-domain TBCs to formulate the scattering problem into the following initial-boundary value problem:

$$\begin{cases} \rho\partial_t^2\boldsymbol{u} - \nabla\cdot(\mu(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^\top)) - \nabla(\lambda\nabla\cdot\boldsymbol{u}) = \boldsymbol{j} & \text{in } \Omega,\ t>0, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{u}|_{t=0} = \boldsymbol{u}_1 & \text{in } \Omega, \\ \mu_j\partial_y\boldsymbol{u} + (\lambda_j+\mu_j)(0,\ 1)^\top\nabla\cdot\boldsymbol{u} = (-1)^{j-1}\mathscr{T}_j[\boldsymbol{u}] & \text{on } \Gamma_j,\ t>0, \end{cases} \tag{2.5}$$

where $\mathscr{T}_j, j=1,2$, are the time-domain transparent boundary operators.

In what follows, we derive the formulation of the operator $\mathscr{T}_j$ and show some of its properties. Since the external force $\boldsymbol{j}$ and the initial conditions $\boldsymbol{u}_0, \boldsymbol{u}_1$ are supported in Ω, the medium is homogeneous in Ω_j, the Navier equation 2.3 reduces to

$$\begin{cases} \rho_j\partial_t^2\boldsymbol{u} - \left(\mu_j\Delta\boldsymbol{u} + (\lambda_j+\mu_j)\nabla\nabla\cdot\boldsymbol{u}\right) = 0 & \text{in } \Omega_j,\ t>0, \\ \boldsymbol{u}|_{t=0} = \partial_t\boldsymbol{u}|_{t=0} = 0 & \text{in } \Omega_j. \end{cases} \tag{2.6}$$

The idea is to solve (2.6) analytically and then find the relation between the Dirichlet data and the Neumann data on Γ_j.

We introduce some notations. Let $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))^\top$ and $u(\boldsymbol{x})$ be a vector and scalar function, respectively. Introduce a scalar curl operator and a vector operator :

$$\mathrm{curl}\boldsymbol{u}(\boldsymbol{x}) = \partial_x u_2(\boldsymbol{x}) - \partial_y u_1(\boldsymbol{x}), \quad \boldsymbol{curl}u(\boldsymbol{x}) = (\partial_y u(\boldsymbol{x}), -\partial_x u(\boldsymbol{x}))^\top.$$

It is clear to note that the two components of the wave field are coupled in the Navier equation, which is the essential difficulty to derive an analytic solution for (2.6) in Ω_j. To decouple them, it is crucial to introduce the Helmholtz decomposition to split the wave field into its compressional part and shear part.

For any solution $\boldsymbol{u}(\boldsymbol{x},t)$ of the Navier equation (2.6) in Ω_j, the Helmholtz decomposition reads

$$\boldsymbol{u}(\boldsymbol{x},t) = \nabla\varphi_j(\boldsymbol{x},t) + \boldsymbol{curl}\psi_j(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Omega_j,\ t>0, \tag{2.7}$$

where φ_j and ψ_j are called the compression and shear scalar potential functions in Ω_j, respectively. Substituting (2.7) into (2.6) yields

$$\begin{cases} \rho_j\partial_t^2\varphi_j - (\lambda_j+2\mu_j)\Delta\varphi_j = 0 & \text{in } \Omega_j,\ t>0, \\ \varphi_j\big|_{t=0} = \partial_t\varphi_j\big|_{t=0} = 0 & \text{in} \Omega_j \end{cases} \tag{2.8}$$

and

$$\begin{cases} \rho_j\partial_t^2\psi_j - \mu_j\Delta\psi_j = 0 & \text{in } \Omega_j,\ t>0, \\ \psi_j\big|_{t=0} = \partial_t\psi_j\big|_{t=0} = 0 & \text{in } \Omega_j, \end{cases} \tag{2.9}$$

where the initial conditions of φ_j and ψ_j follow from the fact that $\boldsymbol{u}_0, \boldsymbol{u}_1$ are compactly supported in Ω.

Taking the Laplace transform of (2.8) and (2.9) and using the initial conditions yield

$$\Delta\breve{\varphi}_j - \left(\frac{\rho_j}{\lambda_j+2\mu_j}\right)s^2\breve{\varphi}_j = 0, \quad \Delta\breve{\psi}_j - \left(\frac{\rho_j}{\mu_j}\right)s^2\breve{\psi}_j = 0 \quad \text{in} \quad \Omega_j, \tag{2.10}$$

where $\breve{\varphi}_j = \mathscr{L}(\varphi_j), \breve{\psi}_j = \mathscr{L}(\psi_j)$ are the Laplace transform of φ_j and ψ_j with respect to t, respectively.

Taking the Fourier transform of (2.10) with respect to x yields the second ordinary differential equations

$$\begin{cases} \frac{d^2\hat{\check{\varphi}}_1}{dy^2} - \left(\left(\frac{\rho_1}{\lambda_1+2\mu_1}\right)s^2+\xi^2\right)\hat{\check{\varphi}}_1 = 0, \quad \frac{d^2\hat{\check{\psi}}_1}{dy^2} - \left(\left(\frac{\rho_1}{\mu_1}\right)s^2+\xi^2\right)\hat{\check{\psi}}_1 = 0, & y > h_1, \\ \hat{\check{\varphi}}_1(\xi,y) = \hat{\check{\varphi}}_1(\xi,h_1), \quad \hat{\check{\psi}}_1(\xi,y) = \hat{\check{\psi}}_1(\xi,h_1), & y = h_1, \end{cases}$$

and

$$\begin{cases} \frac{d^2\hat{\check{\varphi}}_2}{dy^2} - \left(\left(\frac{\rho_2}{\lambda_2+2\mu_2}\right)s^2+\xi^2\right)\hat{\check{\varphi}}_2 = 0, \quad \frac{d^2\hat{\check{\psi}}_2}{dy^2} - \left(\left(\frac{\rho_2}{\mu_2}\right)s^2+\xi^2\right)\hat{\check{\psi}}_2 = 0, & y < h_2, \\ \hat{\check{\varphi}}_2(\xi,y) = \hat{\check{\varphi}}_2(\xi,h_2), \quad \hat{\check{\psi}}_2(\xi,y) = \hat{\check{\psi}}_2(\xi,h_2), & y = h_2. \end{cases}$$

Solving the above equations and using the bounded outgoing conditions in Ω_j, we obtain

$$\hat{\check{\varphi}}_1(\xi,y) = \hat{\check{\varphi}}_1(\xi,h_1)e^{-\beta_1(\xi)(y-h_1)}, \quad \hat{\check{\psi}}_1(\xi,y) = \hat{\check{\psi}}_1(\xi,h_1)e^{-\gamma_1(\xi)(y-h_1)}, \quad y > h_1,$$

and

$$\hat{\check{\varphi}}_2(\xi,y) = \hat{\check{\varphi}}_2(\xi,h_2)e^{\beta_2(\xi)(y-h_2)}, \quad \hat{\check{\psi}}_2(\xi,y) = \hat{\check{\psi}}_2(\xi,h_2)e^{\gamma_2(\xi)(y-h_2)}, \quad y < h_2,$$

where

$$\beta_j^2(\xi) = \left(\frac{\rho_j}{\lambda_j+2\mu_j}\right)s^2+\xi^2, \quad \mathrm{Re}(\beta_j(\xi)) > 0 \tag{2.11}$$

and

$$\gamma_j^2(\xi) = \left(\frac{\rho_j}{\mu_j}\right)s^2+\xi^2, \quad \mathrm{Re}(\gamma_j(\xi)) > 0. \tag{2.12}$$

Hence, we have the solutions of (2.10):

$$\check{\varphi}_j(\boldsymbol{x},s) = \int_{\mathbb{R}} \hat{\check{\varphi}}_j(\xi,h_j)e^{(-1)^j\beta_j(\xi)(y-h_j)}e^{\mathrm{i}x\xi}d\xi, \quad \boldsymbol{x}\in\Omega_j, \tag{2.13}$$

and

$$\check{\psi}_j(\boldsymbol{x},s) = \int_{\mathbb{R}} \hat{\check{\psi}}_j(\xi,h_j)e^{(-)^j\gamma_j(\xi)(y-h_j)}e^{\mathrm{i}x\xi}d\xi, \quad \boldsymbol{x}\in\Omega_j. \tag{2.14}$$

Taking the Laplace transform of the Helmholtz decomposition (2.7) yields

$$\check{\boldsymbol{u}}(\boldsymbol{x},s) = \nabla\check{\varphi}_j(\boldsymbol{x},s) + \boldsymbol{curl}\check{\psi}_j(\boldsymbol{x},s), \quad \boldsymbol{x}\in\Omega_j. \tag{2.15}$$

Combining (2.13) to (2.15) gives

$$\begin{aligned} \check{\boldsymbol{u}}(\boldsymbol{x},s) = &\int_{\mathbb{R}} (\mathrm{i}\xi, -\beta_1(\xi))^\top \hat{\check{\varphi}}_1(\xi,h_1)e^{-\beta_1(\xi)(y-h_1)}e^{\mathrm{i}x\xi}d\xi \\ &+ \int_{\mathbb{R}} (-\gamma_1(\xi), -\mathrm{i}\xi)^\top \hat{\check{\psi}}_1(\xi,h_1)e^{-\gamma_1(\xi)(y-h_1)}e^{\mathrm{i}x\xi}d\xi, \quad \boldsymbol{x}\in\Omega_1, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \check{\boldsymbol{u}}(\boldsymbol{x},s) = &\int_{\mathbb{R}} (\mathrm{i}\xi,\ \beta_2(\xi))^\top \hat{\check{\varphi}}_2(\xi,h_2)e^{\beta_2(\xi)(y-h_2)}e^{\mathrm{i}x\xi}d\xi \\ &+ \int_{\mathbb{R}} (\gamma_2(\xi), -\mathrm{i}\xi)^\top \hat{\check{\psi}}_2(\xi,h_2)e^{\gamma_2(\xi)(y-h_2)}e^{\mathrm{i}x\xi}d\xi, \quad \boldsymbol{x}\in\Omega_2. \end{aligned} \tag{2.17}$$

On the other hand, taking the Fourier transform of $\check{\boldsymbol{u}}(\boldsymbol{x},s)$ with respect to x and evaluating equalities (2.16) and (2.17) at $y = h_j$, respectively, we obtain a linear system of algebraic equations for $\hat{\check{\varphi}}_j(\xi,h_j)$ and $\hat{\check{\psi}}_j(\xi,h_j)$:

$$\begin{bmatrix} \mathrm{i}\xi & -\gamma_1(\xi) \\ -\beta_1(\xi) & -\mathrm{i}\xi \end{bmatrix} \begin{bmatrix} \hat{\check{\varphi}}_1(\xi,h_1) \\ \hat{\check{\psi}}_1(\xi,h_1) \end{bmatrix} = \begin{bmatrix} \hat{\check{u}}_1(\xi,h_1) \\ \hat{\check{u}}_2(\xi,h_1) \end{bmatrix}$$

and

$$\begin{bmatrix} \mathrm{i}\xi & \gamma_2(\xi) \\ \beta_2(\xi) & -\mathrm{i}\xi \end{bmatrix} \begin{bmatrix} \hat{\check{\varphi}}_2(\xi,h_2) \\ \hat{\check{\psi}}_2(\xi,h_2) \end{bmatrix} = \begin{bmatrix} \hat{\check{u}}_1(\xi,h_2) \\ \hat{\check{u}}_2(\xi,h_2) \end{bmatrix}.$$

It follows from Cramer's rule that

$$\begin{cases} \hat{\varphi}_1(\xi, h_1) = \frac{1}{\chi_1(\xi)} \left(-\mathrm{i}\xi \hat{u}_1(\xi, h_1) + \gamma_1(\xi)\hat{u}_2(\xi, h_1) \right), \\ \hat{\psi}_1(\xi, h_1) = \frac{1}{\chi_1(\xi)} \left(\beta_1(\xi)\hat{u}_1(\xi, h_1) + \mathrm{i}\xi \hat{u}_2(\xi, h_1) \right), \end{cases} \tag{2.18}$$

and

$$\begin{cases} \hat{\varphi}_2(\xi, h_2) = \frac{1}{\chi_2(\xi)} \left(-\mathrm{i}\xi \hat{u}_1(\xi, h_2) - \gamma_2(\xi)\hat{u}_2(\xi, h_2) \right), \\ \hat{\psi}_2(\xi, h_2) = \frac{1}{\chi_2(\xi)} \left(-\beta_2(\xi)\hat{u}_1(\xi, h_2) + \mathrm{i}\xi \hat{u}_2(\xi, h_2) \right), \end{cases} \tag{2.19}$$

where $\chi_j(\xi) = \xi^2 - \beta_j(\xi)\gamma_j(\xi), j = 1, 2$.

Lemma 2.2. *For any $\xi \in \mathbb{R}$, we have $\chi_j(\xi) \neq 0$. Moreover, the following estimate holds*

$$|\chi_j(\xi)| \sim \frac{\rho_j}{2} \left(\frac{|s|^2}{\lambda_j + 2\mu_j} + \frac{|s|^2}{\mu_j} \right) \quad \textit{as } \xi \to \infty.$$

Proof. Let $\beta_j(\xi) = a_j + ib_j, \gamma_j(\xi) = c_j + id_j$ with $a_j > 0, c_j > 0, j = 1, 2$. Recalling the definitions of $\beta_j(\xi)$ and $\gamma_j(\xi)$ in (2.11) and (2.12), we have

$$a_j^2 - b_j^2 = \rho_j \frac{s_1^2 - s_2^2}{\lambda_j + 2\mu_j} + \xi^2, \tag{2.20}$$

$$a_j b_j = \rho_j \frac{s_1 s_2}{\lambda_j + 2\mu_j}, \tag{2.21}$$

$$c_j^2 - d_j^2 = \rho_j \frac{s_1^2 - s_2^2}{\mu_j} + \xi^2, \tag{2.22}$$

$$c_j d_j = \rho_j \frac{s_1 s_2}{\mu_j}. \tag{2.23}$$

By the definition of $\chi_j(\xi)$, we obtain

$$\chi_j(\xi) = \xi^2 - (a_j c_j - b_j d_j) - \mathrm{i}(a_j d_j + b_j c_j),$$

which gives

$$\begin{aligned} |\chi(\xi)|^2 &= \left(\xi^2 - (a_j c_j - b_j d_j)\right)^2 + (a_j d_j + b_j c_j)^2 \\ &= \xi^4 - 2a_j c_j \xi^2 + a_j^2 c_j^2 + 2b_j d_j \xi^2 + (a_j^2 + b_j^2) d_j^2 + b_j^2 c_j^2. \end{aligned}$$

Plugging (2.21) and (2.23) into the above equality gives

$$|\chi_j(\xi)|^2 = \left(\xi^2 - a_j c_j\right)^2 + 2\rho_j^2 \frac{s_1^2 s_2^2 \xi^2}{a_j c_j (\lambda_j + 2\mu_j)\mu_j} + (a_j^2 + b_j^2) d_j^2 + b_j^2 c_j^2 > 0,$$

where we have used the fact that $s = s_1 + is_2$ with $s_1 > 0$. Furthermore,

$$\begin{aligned} |\chi_j(\xi)| &= |\xi^2 - \beta_j(\xi)\gamma_j(\xi)| = \left| \xi^2 \left(1 - \left(1 + \frac{\rho_j s^2}{(\lambda_j + 2\mu_j)\xi^2} \right)^{1/2} \right) \left(1 + \frac{\rho_j s^2}{\mu_j \xi^2} \right)^{1/2} \right| \\ &\sim \frac{\rho_j}{2} \left(\frac{|s^2|}{\lambda_j + 2\mu_j} + \frac{|s|^2}{\mu_j} \right) \quad \text{as} \quad \xi \to \infty, \end{aligned}$$

which completes the proof. □

Substituting (2.18) into (2.16) and (2.19) into (2.17), we obtain

$$\begin{aligned}\check{\boldsymbol{u}}(\boldsymbol{x},s) &= \int_{\mathbb{R}} \frac{1}{\chi_1(\xi)} \begin{bmatrix} \xi^2 & \mathrm{i}\xi\gamma_1(\xi) \\ \mathrm{i}\xi\beta_1(\xi) & -\beta_1(\xi)\gamma_1(\xi) \end{bmatrix} \hat{\boldsymbol{u}}(\xi,h_1) e^{-\beta_1(\xi)(y-h_1)} e^{\mathrm{i}x\xi} \mathrm{d}\xi \\ &+ \int_{\mathbb{R}} \frac{1}{\chi_1(\xi)} \begin{bmatrix} -\beta_1(\xi)\gamma_1(\xi) & -\mathrm{i}\xi\gamma_1(\xi) \\ -\mathrm{i}\xi\beta_1(\xi) & \xi^2 \end{bmatrix} \hat{\boldsymbol{u}}(\xi,h_1) e^{-\gamma_1(\xi)(y-h_1)} e^{\mathrm{i}x\xi} \mathrm{d}\xi, \quad \boldsymbol{x} \in \Omega_1, \end{aligned} \tag{2.24}$$

$$\begin{aligned}\check{\boldsymbol{u}}(\boldsymbol{x},s) &= \int_{\mathbb{R}} \frac{1}{\chi_2(\xi)} \begin{bmatrix} \xi^2 & -\mathrm{i}\xi\gamma_2(\xi) \\ -\mathrm{i}\xi\beta_2(\xi) & -\beta_2(\xi)\gamma_2(\xi) \end{bmatrix} \hat{\boldsymbol{u}}(\xi,h_2) e^{\beta_2(\xi)(y-h_2)} e^{\mathrm{i}x\xi} \mathrm{d}\xi \\ &+ \int_{\mathbb{R}} \frac{1}{\chi_2(\xi)} \begin{bmatrix} -\beta_2(\xi)\gamma_2(\xi) & \mathrm{i}\xi\gamma_2(\xi) \\ \mathrm{i}\xi\beta_2(\xi) & \xi^2 \end{bmatrix} \hat{\boldsymbol{u}}(\xi,h_2) e^{\gamma_2(\xi)(y-h_2)} e^{\mathrm{i}x\xi} \mathrm{d}\xi, \quad \boldsymbol{x} \in \Omega_2. \end{aligned} \tag{2.25}$$

Given a vector field $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))^\top$, we define two boundary operators

$$\begin{aligned}\mathscr{B}_1[\boldsymbol{u}] &= \mu_1\partial_y\boldsymbol{u} + (\lambda_1+\mu_1)(0,\ 1)^\top\nabla\cdot\boldsymbol{u} \\ &= (\mu_1\partial_y u_1,\ (\lambda_1+\mu_1)\partial_x u_1 + (\lambda_1+2\mu_1)\partial_y u_2)^\top \quad \text{on } \Gamma_1 \end{aligned} \tag{2.26}$$

and

$$\begin{aligned}\mathscr{B}_2[\boldsymbol{u}] &= -\mu_2\partial_y\boldsymbol{u} + (\lambda_2+\mu_2)(0,-1)^\top\nabla\cdot\boldsymbol{u} \\ &= (-\mu_2\partial_y u_1, -(\lambda_2+\mu_2)\partial_x u_1 - (\lambda_2+2\mu_2)\partial_y u_2)^\top \quad \text{on } \Gamma_2. \end{aligned} \tag{2.27}$$

Combining (2.24) to (2.27), we deduce the explicit expression for the boundary operator $\mathscr{B}_j$:

$$\mathscr{B}_j[\boldsymbol{u}] = \int_{\mathbb{R}} \boldsymbol{M}_j(\xi)\hat{\boldsymbol{u}}(\xi,h_j)e^{\mathrm{i}x\xi}\mathrm{d}\xi \quad \text{on } \Gamma_j, \tag{2.28}$$

where the 2×2 matrix

$$\boldsymbol{M}_j(\xi) = \frac{1}{\chi_j}\begin{bmatrix} \mu_j\beta_j(\gamma_j^2-\xi^2) & (-1)^{j-1}\mathrm{i}\mu_j\xi\left((\gamma_j^2-\xi^2)+\chi_j\right) \\ (-1)^{j-1}\mathrm{i}\xi\left((\lambda_j+2\mu_j)(\xi^2-\beta_j^2)-\mu_j\chi_j\right) & (\lambda_j+2\mu_j)\gamma_j\left(\beta_j^2-\xi^2\right) \end{bmatrix}.$$

Recalling the definitions of β_j and γ_j in (2.11) and (2.12), we get

$$\boldsymbol{M}_j(\xi) = \frac{1}{\chi_j}\begin{bmatrix} \beta_j\rho_j s^2 & \mathrm{i}(-1)^{j-1}\xi\left(\rho_j s^2+\mu_j\chi_j\right) \\ -\mathrm{i}(-1)^{j-1}\xi\left(\rho_j s^2+\mu_j\chi_j\right) & \gamma_j\rho_j s^2 \end{bmatrix}. \tag{2.29}$$

The following trace result in $H^{1/2}(\Gamma_j)^2$ is useful in subsequent analysis.

Lemma 2.3. *There exists a positive constants* $C_1 = \max\{\sqrt{2(h_1-h_2)^{-1}+1}, \sqrt{2}\}$ *such that*

$$\|\boldsymbol{u}\|_{H^{1/2}(\Gamma_j)^2} \le C_1\|\boldsymbol{u}\|_{H^1(\Omega)^2}, \quad \forall \boldsymbol{u} \in H^1(\Omega)^2.$$

Proof. First we have

$$\begin{aligned}(h_1-h_2)|\hat{\boldsymbol{u}}(\xi,h_j)|^2 &= \int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y + \int_{h_2}^{h_1}\int_y^{h_j}\tfrac{\mathrm{d}}{\mathrm{d}\tau}|\hat{\boldsymbol{u}}(\xi,\tau)|^2\mathrm{d}\tau\mathrm{d}y \\ &\le \int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y + (h_1-h_2)\int_{h_2}^{h_1}2|\hat{\boldsymbol{u}}(\xi,y)||\hat{\boldsymbol{u}}'(\xi,y)|\mathrm{d}y, \end{aligned}$$

which implies

$$\begin{aligned}(1+\xi^2)^{1/2}|\hat{\boldsymbol{u}}(\xi,h_j)|^2 \le (h_1-h_2)^{-1}(1+\xi^2)^{1/2} &\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y \\ &+ \int_{h_2}^{h_1}2(1+\xi^2)^{1/2}|\hat{\boldsymbol{u}}(\xi,y)||\hat{\boldsymbol{u}}'(\xi,y)|\mathrm{d}y. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$
\begin{aligned}
(1+\xi^2)^{1/2}|\hat{\boldsymbol{u}}(\xi,h_j)|^2 &\le (h_1-h_2)^{-1}(1+|\xi|)\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y \\
&\quad +(1+\xi^2)\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y+\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}'(\xi,y)|^2\mathrm{d}y \\
&\le C_1^2\left((1+\xi^2)\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}(\xi,y)|^2\mathrm{d}y+\int_{h_2}^{h_1}|\hat{\boldsymbol{u}}'(\xi,y)|^2\mathrm{d}y\right).
\end{aligned}
$$

Combining the above estimate and the definitions of the norm, we get

$$
\begin{aligned}
\|\boldsymbol{u}(x,y)\|^2_{H^{1/2}(\Gamma_j)^2} &= \int_{\mathbb{R}}(1+\xi^2)^{1/2}|\hat{\boldsymbol{u}}(\xi,h_j)|^2\mathrm{d}\xi \\
&\le C_1^2\int_{h_2}^{h_1}\left(\int_{\mathbb{R}}(1+\xi^2)|\hat{\boldsymbol{u}}(\xi,y)|^2+|\hat{\boldsymbol{u}}'(\xi,y)|^2\mathrm{d}\xi\right)\mathrm{d}y = C_1^2\|\boldsymbol{u}(x,y)\|^2_{H^1(\Omega)^2},
\end{aligned}
$$

which completes the proof. □

Lemma 2.4. *The boundary operator $\mathscr{B}_j : H^{1/2}(\Gamma_j)^2 \to H^{-1/2}(\Gamma_j)^2$ is continuous, ie,*

$$
\|\mathscr{B}_j\boldsymbol{u}\|_{H^{-1/2}(\Gamma_j)^2} \lesssim \|\boldsymbol{u}\|_{H^{1/2}(\Gamma_j)^2}, \quad \forall\, \boldsymbol{u}\in H^{1/2}(\Gamma_j)^2.
$$

Proof. It follows from the definition of β_j, γ_j in (2.11) and (2.12) and Lemma 2.2 that we get

$$
\begin{aligned}
&|\beta_j(\xi)\rho_j s^2| \sim |s|^2\rho_j|\xi|, \quad |\gamma_j(\xi)\rho_j s^2| \sim |s|^2\rho_j|\xi|, \\
&|\chi_j(\xi)| \sim |s|^2\frac{\rho_j}{2}\left(\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu_j}\right), \quad |\xi(\rho_j s^2+\mu\chi_j(\xi))| \sim |s|^2\rho_j|\xi| \quad \text{as } |\xi|\to\infty.
\end{aligned}
$$

Denote by $\|\boldsymbol{M}_j\|_2$ the Euclidean norm of matrix $\boldsymbol{M}_j$. It follows from (2.29) that

$$
\|\boldsymbol{M}_j(\xi)\|_2 \sim |\xi| \quad \text{as } |\xi|\to\infty.
$$

Hence, we have

$$
\begin{aligned}
\|\mathscr{B}_j\boldsymbol{u}\|^2_{H^{-1/2}(\Gamma_j)^2} &= \int_{\mathbb{R}}(1+\xi^2)^{-1/2}|\boldsymbol{M}_j(\xi)\hat{\boldsymbol{u}}(\xi,h_j)|^2\mathrm{d}\xi \\
&\lesssim \int_{\mathbb{R}}(1+\xi^2)^{1/2}|\hat{\boldsymbol{u}}(\xi,h_j)|^2\mathrm{d}\xi \\
&= \|\boldsymbol{u}\|^2_{H^{1/2}(\Gamma_j)^2},
\end{aligned}
$$

which completes the proof. □

Lemma 2.5. *For $(\lambda_j,\mu_j)\in S_{\lambda_j,\mu_j}$, which is given in (2.34), the following estimate holds*

$$
-Re\langle s^{-1}\mathscr{B}_j\boldsymbol{u},\ \boldsymbol{u}\rangle_{\Gamma_j} \ge 0, \quad j=1,2, \quad \forall\,\boldsymbol{u}\in H^{1/2}(\Gamma_j)^2,
$$

where $s=s_1+\mathrm{i}s_2, s_1\ge\sigma_0>0, s_2\in\mathbb{R}$.

Proof. Let

$$
\boldsymbol{m}_j(\xi)=s^{-1}\boldsymbol{M}_j(\xi)=\frac{1}{\chi_j(\xi)}\begin{bmatrix}\beta_j(\xi)\rho_j s & \mathrm{i}(-1)^{j-1}\xi\left(\rho_j s+\mu s^{-1}\chi_j(\xi)\right)\\ -\mathrm{i}(-1)^{j-1}\xi\left(\rho_j s+\mu_j s^{-1}\chi_j(\xi)\right) & \gamma_j(\xi)\rho_j s\end{bmatrix}.
$$

Define

$$
\mathscr{M}_j(\xi)=-\frac{1}{2}(\boldsymbol{m}_j(\xi)+\boldsymbol{m}_j^*(\xi)), \tag{2.30}
$$

where $\boldsymbol{m}_j^*(\xi)$ is the adjoint of the matrix $\boldsymbol{m}_j(\xi)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ in $\mathbb{C}^2$. A simple calculation yields

$$
\mathscr{M}_j(\xi)=\begin{bmatrix}\mathscr{M}_{11}^{(j)}(\xi) & \mathscr{M}_{12}^{(j)}(\xi)\\ \mathscr{M}_{21}^{(j)}(\xi) & \mathscr{M}_{22}^{(j)}(\xi)\end{bmatrix}=\begin{bmatrix}-\rho_j\mathrm{Re}\left(\frac{s\beta_j(\xi)}{\chi_j(\xi)}\right) & -\mathrm{i}(-1)^{j-1}\mathrm{Re}\left(\frac{\rho_j s\xi}{\chi_j(\xi)}+\frac{\mu_j\xi}{s}\right)\\ \mathrm{i}(-1)^{j-1}\mathrm{Re}\left(\frac{s\rho_j\xi}{\chi_j(\xi)}+\frac{\mu_j\xi}{s}\right) & -\rho_j\mathrm{Re}\left(\frac{s\gamma_j(\xi)}{\chi_j(\xi)}\right)\end{bmatrix}. \tag{2.31}
$$

We show that the first leading principle element of $\mathscr{M}_j(\xi)$ is positive. By (2.20) to (2.23), we obtain

$$\begin{aligned}\mathscr{M}_{11}^{(j)}(\xi) &= -\rho_j \mathrm{Re}\left(\frac{s\beta_j(\xi)}{\chi_j(\xi)}\right) = -\frac{\rho_j}{|\chi_j(\xi)|^2}\mathrm{Re}\left(s\beta_j(\xi)\bar{\chi}_j(\xi)\right) \\ &= -\frac{\rho_j}{|\chi_j(\xi)|^2}\mathrm{Re}\left[(a_j+\mathrm{i}b_j)(s_1+\mathrm{i}s_2)(\xi^2-(a_jc_j-b_jd_j)+\mathrm{i}(a_jd_j+c_jb_j))\right] \\ &= \frac{\rho_j}{|\chi_j(\xi)|^2}\left[(c_js_1+d_js_2)\left(a_j^2+b_j^2\right)+(b_js_2-a_js_1)\xi^2\right] \\ &= \frac{\rho_j}{|\chi_j(\xi)|^2}\left[\frac{s_1}{c_j}\left(c_j^2+\frac{\rho_js_2^2}{\mu_j}\right)\left(a_j^2+b_j^2\right)+\frac{s_1}{a_j}\left(\frac{\rho_js_2^2}{\lambda_j+2\mu_j}-a_j^2\right)\xi^2\right].\end{aligned} \tag{2.32}$$

We discuss the sign of the above equality in different cases.

(I) If $\frac{\rho_j s_2^2}{\lambda_j+2\mu_j} \geq a_j^2$, obviously we have $\mathscr{M}_{11}^{(j)}(\xi) > 0$.

(II) If $\frac{\rho_j s_2^2}{\lambda_j+2\mu_j} < a_j^2$, we have two possibilities:

(II.a) If $|s_2| < s_1$, by 2.20, we get $a_j^2+b_j^2 = \rho_j\frac{s_1^2-s_2^2}{\lambda_j+2\mu_j}+2b_j^2+\xi^2 > \xi^2$. Subtracting (2.20) from (2.22) yields

$$c_j^2-a_j^2 = \left(d_j^2-b_j^2\right)+\rho_j\left(s_1^2-s_2^2\right)\left(\frac{1}{\mu_j}-\frac{1}{\lambda_j+2\mu_j}\right).$$

Obviously, if $|b_j| \leq d_j$, we obtain $c_j > a_j$. If $|b_j| > |d_j|$, it follows from (2.21) and (2.23) that

$$\frac{a_j|b_j|}{c_j|d_j|} = \frac{\mu_j}{\lambda_j+2\mu_j} < 1, \tag{2.33}$$

which gives $a_j < c_j$. Combining above estimate with (2.32), we get

$$\mathscr{M}_{11}^{(j)}(\xi) \geq \frac{\rho_j}{|\chi_j(\xi)|^2}\left[s_1(c_j-a_j)\xi^2+\frac{\rho_js_1}{c_j}\frac{s_2^2}{\mu_j}\left(a_j^2+b_j^2\right)+\frac{\rho_js_1}{a_j}\frac{s_2^2}{\lambda_j+2\mu_j}\xi^2\right] > 0.$$

(II.b) If $s_1 \leq |s_2| \leq \sqrt{(\lambda_j+2\mu_j)/\rho_j}\,a_j$, we also have two cases:

(II.b.i) If $|b_j| \geq |d_j|$, by (2.33), we obtain $c_j > a_j$. Subtracting (2.22) from (2.20) yields

$$c_j^2-a_j^2-\left(d_j^2-b_j^2\right) = \frac{\lambda_j+\mu_j}{\mu_j(\lambda_j+2\mu_j)}\left(s_1^2-s_2^2\right) < 0,$$

which implies $c_j^2-a_j^2 < d_j^2-b_j^2 \leq 0$. This is in contradiction with $c_j > a_j$.

(II.b.ii) We only have to consider $|b_j| < |d_j|$. We discuss it further in two cases:

(II.b.ii.1) If $a_j \geq c_j$, it follows from (2.20) to (2.23) that

$$\begin{aligned}\mathscr{M}_{11}^{(j)}(\xi) &= \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[\left(c_j+\frac{\rho_js_2^2}{\mu_jc_j}\right)\left(a_j^2+b_j^2\right)+\frac{\rho_js_2^2}{(\lambda_j+2\mu_j)a_j}\xi^2-a_j\xi^2\right] \\ &> \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[c_j\left(a_j^2+b_j^2\right)+\frac{\rho_js_2^2}{\mu_jc_j}a_j^2-a_j\xi^2\right] \\ &= \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[c_ja_j^2+\frac{\rho_js_2^2}{\mu_jc_j}\left(a_j^2+b_j^2\right)-a_j\left(c_j^2-d_j^2-\frac{\rho_j(s_1^2-s_2^2)}{\mu_j}\right)\right] \\ &> \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[c_ja_j(a_j-c_j)+\frac{\rho_js_2^2a_j}{\mu_jc_j}(a_j-c_j)+\frac{a_j\rho_js_1^2}{\mu_j}\right] > 0.\end{aligned}$$

(II.b.ii.2) If $a_j < c_j$, it also follows from (2.20) to (2.23) that

$$\begin{aligned}\mathscr{M}_{11}^{(j)}(\xi) &= \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[\left(c_j+\frac{\rho_js_2^2}{\mu_jc_j}\right)(a_j^2+b_j^2)+\frac{\rho_js_2^2}{(\lambda_j+2\mu_j)a_j}\xi^2-a_j\xi^2\right] \\ &> \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[c_j(a_j^2+b_j^2)+\frac{\rho_js_2^2}{\mu_jc_j}a_j^2-a_j\xi^2\right] \\ &= \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[c_j(a_j^2+b_j^2)+\frac{\rho_js_2^2}{\mu_jc_j}a_j^2-a_j\left(a_j^2-b_j^2-\frac{\rho_j(s_1^2-s_2^2)}{\lambda_j+2\mu_j}\right)\right] \\ &> \frac{\rho_js_1}{|\chi_j(\xi)|^2}\left[(c_j-a_j)a_j^2+\frac{a_j\rho_js_2^2}{c_j}\left(\frac{a_j}{\mu_j}-\frac{c_j}{\lambda_j+2\mu_j}\right)\right] > 0,\end{aligned}$$

where we have used the fact that

$$\frac{|b_j|}{|d_j|}=\frac{c_j/(\lambda_j+2\mu_j)}{a_j/\mu_j}<1.$$

Thus, the first leading principle element of $\mathscr{M}_j(\xi)$ satisfies

$$\mathscr{M}_{11}^{(j)}(\xi)>0.$$

Next, we prove the determinant of $\mathscr{M}_j(\xi)$ is positive. It is easy to compute the determinant of matrix $\mathscr{M}_j(\xi)$ (cf (2.31)):

$$\mathscr{D}_j(\xi):=\rho_j^2\mathrm{Re}\left(s\frac{\beta_j(\xi)}{\chi_j(\xi)}\right)\mathrm{Re}\left(\frac{s\gamma_j(\xi)}{\chi_j(\xi)}\right)-\left(\mathrm{Re}\left(\frac{s\rho_j\xi}{\chi_j(\xi)}+\frac{\mu_j\xi}{s}\right)\right)^2.$$

We define the admissible set:

$$S_{\lambda_j,\mu_j}=\left\{(\lambda_j,\mu_j)\in\mathbb{R}\times\mathbb{R}^+:\ \mu_j>0,\lambda_j+\mu_j>0,\quad \mathscr{D}_j(\xi)>0\text{ for }\epsilon_0<|\xi|<M_0\right\},\tag{2.34}$$

where the constants ϵ_0 and M_0 will be given in the following proof.

Next, we consider $\mathscr{D}_j(\xi)$ in three different cases.

(i) When $\xi=0$. It follows from the definitions of $\beta_j,\gamma_j,\chi_j(\xi)$ that

$$\beta_j(\xi)=s\sqrt{\frac{\rho_j}{\lambda_j+2\mu_j}},\quad \gamma_j(\xi)=s\sqrt{\frac{\rho_j}{\mu_j}},\quad \chi_j(\xi)=-\rho_js^2\sqrt{\frac{1}{\mu_j(\lambda_j+2\mu_j)}}.$$

Then

$$\mathscr{D}_j(\xi)=\rho_j\sqrt{\mu_j(\lambda_j+2\mu_j)}>0.$$

Since $\mathscr{D}_j(\xi)$ is continuous with respect to ξ. Thus, there exists a constant ϵ_0, such that

$$\mathscr{D}_j(\xi)>0,\quad |\xi|\le\epsilon_0.$$

(ii) When $|\xi|$ is large enough. It follows the definitions of β_j,γ_j that

$$\begin{aligned}\beta_j(\xi)&=\sqrt{\tfrac{\rho_js^2}{\lambda_j+2\mu_j}+\xi^2}=|\xi|\sqrt{1+\tfrac{\rho_js^2}{(\lambda_j+2\mu_j)\xi^2}}=|\xi|\left(1+\tfrac{\rho_js^2}{2(\lambda_j+2\mu_j)\xi^2}+O\left(\tfrac{1}{\xi^2}\right)\right),\\ \gamma_j(\xi)&=\sqrt{\rho_j\tfrac{s^2}{\mu_j}+\xi^2}=|\xi|\sqrt{1+\tfrac{\rho_js^2}{\mu_j\xi^2}}=|\xi|\left(1+\tfrac{\rho_js^2}{2\mu_j\xi^2}+O\left(\tfrac{1}{\xi^2}\right)\right).\end{aligned}$$

Since

$$\begin{aligned}\chi_j(\xi)&=\xi^2-\beta_j(\xi)\gamma_j(\xi)=\xi^2-\sqrt{\tfrac{\rho_js^2}{\lambda_j+2\mu_j}+\xi^2}\sqrt{\tfrac{\rho_js^2}{\mu_j}+\xi^2}\\ &=\xi^2\left(1-\sqrt{1+\tfrac{\rho_js^2}{(\lambda_j+2\mu_j)\xi^2}}\sqrt{1+\tfrac{\rho_js^2}{\mu_j\xi^2}}\right)\\ &=-\tfrac{\rho_js^2}{2}\left(\tfrac{1}{\lambda_j+2\mu_j}+\tfrac{1}{\mu_j}\right)+O\left(\tfrac{1}{\xi^2}\right),\end{aligned}$$

the first leading principle element of $\mathscr{M}_j(\xi)$ is

$$\begin{aligned}\mathscr{M}_{11}^{(j)}(\xi)&=\rho_j\mathrm{Re}\left(-\tfrac{s\beta_j(\xi)}{\chi_j(\xi)}\right)=\mathrm{Re}\left(\frac{2|\xi|\left(1+O\left(\frac{1}{|\xi|}\right)\right)}{s\left(\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu_j}\right)}\right)\\ &=\frac{2|\xi|s_1}{|s|^2\left(\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu_j}\right)}(1+O\left(\frac{1}{|\xi|}\right)>0.\end{aligned}$$

It remains to verify that

$$\det\mathscr{M}_j(\xi)=\mathscr{M}_{11}^{(j)}\mathscr{M}_{22}^{(j)}-\mathscr{M}_{12}^{(j)}\mathscr{M}_{21}^{(j)}>0.$$

Note $\mu_j > 0, \lambda_j + \mu_j > 0$. A simple calculation yields

$$\begin{aligned}\det \mathscr{M}_j(\xi) &= \left(\frac{4\xi^2 s_1^2}{|s|^4\left(\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu}\right)^2} - \frac{\xi^2 s_1^2}{|s|^4}\left(\frac{-2}{\left(\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu_j}\right)}+\frac{1}{\frac{1}{\mu_j}}\right)^2\right)\left(1+O\left(\frac{1}{|\xi|}\right)\right)\\ &= \frac{\xi^2 s_1^2}{|s|^4}\mu_j\left(\frac{4}{\frac{1}{\lambda_j+2\mu_j}+\frac{1}{\mu_j}} - \frac{1}{\frac{1}{\mu_j}}\right)\left(1+O\left(\frac{1}{|\xi|}\right)\right) > 0 \quad \text{as } |\xi| \to \infty.\end{aligned}$$

Thus, there exists a constant M_0 large enough such that $\det \mathscr{M}_j(\xi) > 0$ for $|\xi| \geq M_0$.

(iii) When $\epsilon_0 < |\xi| < M_0$, since $(\lambda_j, \mu_j) \in S_{\lambda_j,\mu_j}$, we can get

$$\det \mathscr{M}_j(\xi) > 0 \quad \text{for} \quad \epsilon_0 < |\xi| < M_0.$$

It follows from Sylvester's rule that matrix $\mathscr{M}_j(\xi)$ is positive. Following the definition of the matrix in (2.30), we obtain

$$-\mathrm{Re}\langle s^{-1}\mathscr{B}_j \boldsymbol{u}, \boldsymbol{u}\rangle_{\Gamma_j} = \int_{\mathbb{R}} \mathscr{M}_j(\xi)\hat{\boldsymbol{u}}(\xi, h_j)\bar{\hat{\boldsymbol{u}}}(\xi, h_j)\mathrm{d}\xi \geq 0,$$

which completes the proof. □

Remark 2.6. It can be verified that the admissible S_{λ_j,μ_j} is nonempty. We give two examples:

(i) When $\lambda_j + \mu_j = o(1)$ for any fixed μ_j, it follows from the definitions of $\beta_j, \gamma_j, \chi_j$ that

$$\begin{aligned}\gamma_j^2(\xi) &= \beta_j^2(\xi) + \frac{\rho_j s^2}{\mu_j(\lambda_j+2\mu_j)}(\lambda_j+\mu_j) = \beta_j^2(\xi) + o(1),\\ \chi_j(\xi) &= \xi^2 - \beta_j(\xi)\gamma_j(\xi) = -\frac{\rho_j s^2}{\mu_j} + o(1),\end{aligned}$$

which gives

$$\begin{aligned}\mathscr{D}_j(\xi) &= \left(\rho_j \mathrm{Re}\left(\frac{s\gamma_j(\xi)}{\chi_j(\xi)}\right) + o(1)\right)^2 - (o(1))^2\\ &= \left(-\frac{\mu_j}{|s^2|}\left(c_j s_1 + \frac{s_1}{\mu_j c_j}\rho_j s_2^2\right) + o(1)\right)^2 > 0, \quad \xi \in \mathbb{R}.\end{aligned}$$

Thus, we obtain $(\lambda_j, \mu_j) \in S_{\lambda_j,\mu_j}$ for sufficiently small $\lambda_j + \mu_j$.

(ii) When μ_j is large enough, we have

$$\gamma_j^2(\xi) = \beta_j^2(\xi) + \frac{\rho_j s^2}{\mu_j(1+\frac{\mu_j}{\lambda - j + \mu_j})} = \beta_j^2(\xi) + o(1), \quad \chi_j(\xi) = -\frac{\rho_j s^2}{\mu_j} + o(1).$$

Similarly, we get $\mathscr{D}_j(\xi) > 0$, which implies $(\lambda_j, \mu_j) \in S_{\lambda_j,\mu_j}$ for sufficiently large μ_j.

Following from Lemmas 2.1 and 2.4, we obtain the existence of the inverse Laplace transform of operator $\mathscr{B}_j$. Taking the inverse Laplace transform of (2.26) and (2.27) yields the transparent boundary operators in the time domain:

$$(-1)^{j-1}\mathscr{T}_j[\boldsymbol{u}] = \mu_j \partial_y \boldsymbol{u} + (\lambda_j + \mu_j)(0,1)^\top(\nabla \cdot \boldsymbol{u}) \quad \text{on } \Gamma_j,\ t > 0,$$

where $\mathscr{T}_j = \mathscr{L}^{-1} \circ \mathscr{B}_j \circ \mathscr{L}$. The TBCs help to reduce the scattering problem from $\mathbb{R}^2$ into the slab Ω.

3 | ANALYSIS OF TWO AUXILIARY PROBLEMS

In this section, we make necessary preparations for the proof of the main results by considering two auxiliary problems related to the scattering problem (2.5).

3.1 Time-harmonic elastic wave equation with a complex wavenumber

This section is devoted to the mathematical study of a time-harmonic elastic scattering problem with a complex wavenumber, which may be viewed as a frequency version of the initial-boundary problem of the Naiver equation under the Laplace transform.

Consider the auxiliary boundary value problem:

$$\begin{cases} \nabla\cdot\left(\mu s^{-1}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\top})\right)+\nabla\left(\lambda s^{-1}\nabla\cdot\boldsymbol{u}\right)-s\rho\boldsymbol{u}=-s^{-1}\boldsymbol{k} & \text{in } \Omega,\\ \mu_j\partial_y\boldsymbol{u}+(\lambda_j+\mu_j)(0,\quad 1)^{\top}(\nabla\cdot\boldsymbol{u})=(-1)^{j-1}\mathscr{B}_j[\boldsymbol{u}] & \text{on } \Gamma_j,\end{cases} \tag{3.1}$$

where $s=s_1+is_2$ with $s_1,s_2\in\mathbb{R}, s_1>0$, and $\boldsymbol{k}$ is assumed to be supported in Ω.

Multiplying a test function $\boldsymbol{v}\in H^1(\Omega)^2$ and integrating by parts, we arrive at the variational formulation: to find $\boldsymbol{u}\in H^1(\Omega)^2$ such that

$$a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{v})=\int_{\Omega}s^{-1}\boldsymbol{k}\cdot\bar{\boldsymbol{v}}\mathrm{d}\boldsymbol{x},\quad \forall\boldsymbol{v}\in H^1(\Omega)^2, \tag{3.2}$$

where the sesquilinear form

$$\begin{aligned} a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{v})=\int_{\Omega}&\left(\mu s^{-1}(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{v}})+(\lambda+\mu)s^{-1}(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})+s\rho\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\right)\mathrm{d}\boldsymbol{x}\\ &-\sum_{j=1}^{2}\langle s^{-1}\mathscr{B}_j[\boldsymbol{u}],\boldsymbol{u}\rangle_{\Gamma_j}.\end{aligned}$$

Here $A:B=\operatorname{tr}\left(AB^{\top}\right)$ is the Frobenius inner product of square matrices A and B. For any $\boldsymbol{u}\in H^1(\Omega)^2$, define the norm

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^{2\times2}}:=\left(\sum_{j=1}^{2}\int_{\Omega}|\nabla u_j|^2\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

It is easy to verify that

$$\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)^{2\times2}}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)}\lesssim\|\boldsymbol{u}\|^2_{H^1(\Omega)^2}.$$

Theorem 3.1. *For $(\lambda_j,\mu_j)\in S_{\lambda_j,\mu_j}$, the variational problem 3.2 has a unique solution $\boldsymbol{u}\in H^1(\Omega)^2$, which satisfies*

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^{2\times2}}+\|\nabla\cdot\boldsymbol{u}\|_{L^2(\Omega)}+\|s\boldsymbol{u}\|_{L^2(\Omega)^2}\lesssim s_1^{-1}\|\boldsymbol{k}\|_{L^2(\Omega)^2}.$$

Proof. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}|a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{v})|\le&\frac{\mu_{\max}}{|s|}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^{2\times2}}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)^{2\times2}}+\frac{(\lambda+\mu)_{\max}}{|s|}\|\nabla\cdot\boldsymbol{u}\|_{L^2(\Omega)}\|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)}\\ &+|s|\rho_{\max}\|\boldsymbol{u}\|_{L^2(\Omega)^2}\|\boldsymbol{v}\|_{L^2(\Omega)^2}+\frac{1}{|s|}\sum_{j=1}^{2}\|\mathscr{B}_j\boldsymbol{u}\|_{H^{-1/2}(\Gamma_j)^2}\|\boldsymbol{v}\|_{H^{1/2}(\Gamma_j)^2}\\ \lesssim&\|\boldsymbol{u}\|_{H^1(\Omega)^2}\|\boldsymbol{v}\|_{H^1(\Omega)^2}+\sum_{j=1}^{2}\|\mathscr{B}_j\boldsymbol{u}\|_{H^{-1/2}(\Gamma_j)^2}\|\boldsymbol{v}\|_{H^{1/2}(\Gamma_j)^2}.\end{aligned}$$

Applying Lemmas 2.3 and 2.4 yields

$$|a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{v})|\lesssim\|\boldsymbol{u}\|_{H^1(\Omega)^2}\|\boldsymbol{v}\|_{H^1(\Omega)^2},$$

which shows that the sesquilinear form is bounded.

A simple calculation yields

$$a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{u})=\int_{\Omega}\left(\mu s^{-1}(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{u}})+(\lambda+\mu)s^{-1}|\nabla\cdot\boldsymbol{u}|^2+s\rho|\boldsymbol{u}|^2\right)\mathrm{d}\boldsymbol{x}-\sum_{j=1}^{2}\langle s^{-1}\mathscr{B}[\boldsymbol{u}],\boldsymbol{u}\rangle_{\Gamma_j}. \tag{3.3}$$

Taking the real part of (3.3) and using Lemma 2.5, we obtain

$$\operatorname{Re}\left(a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{u})\right) \gtrsim \frac{s_1}{|s|^2}\left(\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)} + \|s\boldsymbol{u}\|^2_{L^2(\Omega)^2}\right). \tag{3.4}$$

It follows from the Lax-Milgram lemma that the variational problem (3.2) has a unique solution $\boldsymbol{u} \in H^1(\Omega)^2$. Moreover, we have from (3.2) that

$$|a_{\mathrm{TH}}(\boldsymbol{u},\boldsymbol{u})| \le \frac{1}{|s|^2}\|\boldsymbol{k}\|_{L^2(\Omega)^2}\|s\boldsymbol{u}\|_{L^2(\Omega)^2}. \tag{3.5}$$

Combining (3.4) and (3.5) leads to

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)} + \|s\boldsymbol{u}\|^2_{L^2(\Omega)^2} \lesssim s_1^{-1}\|\boldsymbol{k}\|_{L^2(\Omega)^2}\|s\boldsymbol{u}\|_{L^2(\Omega)^2},$$

which completes the proof after applying the Cauchy-Schwarz inequality. □

3.2 | Time-domain Navier equation with the Dirichlet boundary condition

Consider the initial-boundary value problem for the time-domain Naiver equation with the Dirichlet boundary condition on Γ_j:

$$\begin{cases} \rho\partial_t^2\boldsymbol{U} - \nabla\cdot\left(\mu(\nabla\boldsymbol{U}+\nabla\boldsymbol{U}^{\top})\right) - \nabla(\lambda\nabla\cdot\boldsymbol{U}) = 0 & \text{in } \Omega,\ t>0,\\ \boldsymbol{U}|_{t=0} = \boldsymbol{u}_0, \quad \partial_t\boldsymbol{U}|_{t=0} = \boldsymbol{u}_1 & \text{in } \Omega,\\ \boldsymbol{U} = 0 & \text{on } \Gamma_j, \quad t>0, \end{cases} \tag{3.6}$$

where $\boldsymbol{u}_0, \boldsymbol{u}_1$ are assumed to be compactly supported in Ω.

Let $\check{\boldsymbol{U}} = \mathscr{L}(\boldsymbol{U})$. Taking the Laplace transform of 3.6, we obtain the boundary value problem:

$$\begin{cases} \nabla\cdot\left(\mu s^{-1}(\nabla\check{\boldsymbol{U}}+\nabla\check{\boldsymbol{U}}^{\top})\right) + \nabla\left(\lambda s^{-1}\nabla\cdot\check{\boldsymbol{U}}\right) - s\rho\check{\boldsymbol{U}} = -\check{\boldsymbol{q}} & \text{in } \Omega,\\ \check{\boldsymbol{U}} = 0 & \text{on } \Gamma_j, \end{cases} \tag{3.7}$$

where $\check{\boldsymbol{q}} = \rho(\boldsymbol{u}_0 + s^{-1}\boldsymbol{u}_1)$. The variational formulation of 3.7 is to find $\check{\boldsymbol{U}} \in H^1(\Omega)^2$ such that

$$a_{\mathrm{TD}}(\check{\boldsymbol{U}},\boldsymbol{v}) = \int_\Omega \check{\boldsymbol{q}}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{v}\in H^1(\Omega)^2, \tag{3.8}$$

where the sesquilinear form

$$a_{\mathrm{TD}}(\check{\boldsymbol{U}},\boldsymbol{v}) = \int_\Omega \left(\mu s^{-1}(\nabla\check{\boldsymbol{U}}) : (\nabla\bar{\boldsymbol{v}}) + (\lambda+\mu)s^{-1}(\nabla\cdot\check{\boldsymbol{U}})(\nabla\cdot\bar{\boldsymbol{v}}) + s\rho\check{\boldsymbol{U}}\bar{\boldsymbol{v}}\right)\mathrm{d}\boldsymbol{x}.$$

Following the same proof as that for Theorem 3.1, we can show the well posedness of the variational problem 3.8 and its stability, which are stated below. The proof is omitted for brevity.

Lemma 3.2. *The variational problem 3.8 has a unique solution $\check{\boldsymbol{U}} \in H^1(\Omega)^2$, which satisfies*

$$\|\nabla\check{\boldsymbol{U}}\|_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot\check{\boldsymbol{U}}\|_{L^2(\Omega)} + \|s\check{\boldsymbol{U}}\|_{L^2(\Omega)^2} \lesssim s_1^{-1}|s|\|\boldsymbol{u}_0\|_{L^2(\Omega)^2} + s_1^{-1}\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}.$$

Theorem 3.3. *The initial-boundary value problem 3.6 has a unique solution $\boldsymbol{U}$, which satisfies the estimates*

$$\|\partial_t\boldsymbol{U}\|_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{U}\|_{L^2(\Omega)} + \|\nabla\boldsymbol{U}\|_{L^2(\Omega)^{2\times 2}} \lesssim \|\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_0\|_{L^2(\Omega)} + \|\nabla\boldsymbol{u}_0\|_{L^2(\Omega)^{2\times 2}},$$

$$\begin{aligned}\|\partial_t^2\boldsymbol{U}\|_{L^2(\Omega)^2} &+ \|\nabla\cdot(\partial_t\boldsymbol{U})\|_{L^2(\Omega)} + \|\nabla(\partial_t\boldsymbol{U})\|_{L^2(\Omega)^{2\times 2}}\\ &\lesssim \|\Delta\boldsymbol{u}_0\|_{L^2(\Omega)^2} + \|\nabla\nabla\cdot\boldsymbol{u}_0\|_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_1\|_{L^2(\Omega)} + \|\nabla\boldsymbol{u}_1\|_{L^2(\Omega)^{2\times 2}},\end{aligned}$$

$$\begin{aligned}\|\partial_t^3\boldsymbol{U}\|_{L^2(\Omega)^2} &+ \|\nabla\cdot(\partial_t^2\boldsymbol{U})\|_{L^2(\Omega)} + \|\nabla(\partial_t^2\boldsymbol{U})\|_{L^2(\Omega)^{2\times 2}}\\ &\lesssim \|\Delta\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\nabla\nabla\cdot\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\nabla\cdot(\Delta\boldsymbol{u}_0 + \nabla\nabla\cdot\boldsymbol{u}_0)\|_{L^2(\Omega)}\\ &\quad + \|\nabla(\Delta\boldsymbol{u}_0 + \nabla\nabla\cdot\boldsymbol{u}_0)\|_{L^2(\Omega)^{2\times 2}}.\end{aligned}$$

Proof. Let $\breve{\boldsymbol{U}} = \mathscr{L}(\boldsymbol{U})$. By Lemma 3.2, we have

$$\|\nabla\breve{\boldsymbol{U}}\|_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot\breve{\boldsymbol{U}}\|_{L^2(\Omega)} + \|s\breve{\boldsymbol{U}}\|_{L^2(\Omega)^2} \lesssim s_1^{-1}|s|\|\boldsymbol{u}_0\|_{L^2(\Omega)^2} + s_1^{-1}\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}.$$

It follows from Trèves[35], Lemma 44.1 that $\breve{\boldsymbol{U}}$ is a holomorphic function of s on the half plane $s_1 > \sigma_0 > 0$, where σ_0 is any positive constant. Hence, we have from Lemma 2.1 that the inverse Laplace transform of $\breve{\boldsymbol{U}}$ exists and is supported in $[0,\infty]$.

Next, we prove the stability. Define the energy function

$$e_1(t) = \|\sqrt{\rho}\partial_t\boldsymbol{U}\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{U}\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{U}\|^2_{L^2(\Omega)^{2\times 2}}.$$

It follows from 3.6 and the integration by parts that

$$\begin{aligned}
e_1(t) - e_1(0) &= \int_0^t e'(\tau)\mathrm{d}\tau \\
&= 2\mathrm{Re}\int_0^t\int_\Omega\left(\rho\partial_t^2\boldsymbol{U}\cdot\partial_t\bar{\boldsymbol{U}} + (\lambda+\mu)(\partial_t(\nabla\cdot\boldsymbol{U}))(\nabla\cdot\bar{\boldsymbol{U}}) + \mu(\partial_t\nabla\boldsymbol{U}) : \nabla\bar{\boldsymbol{U}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= 2\mathrm{Re}\int_0^t\int_\Omega\Big(\left(\nabla\cdot\left(\mu(\boldsymbol{x})(\nabla\boldsymbol{U}(\boldsymbol{x}) + \nabla\boldsymbol{U}^\top(\boldsymbol{x}))\right) + \nabla\left(\lambda(\boldsymbol{x})\nabla\cdot\boldsymbol{U}(\boldsymbol{x})\right)\right)\cdot\partial_t\bar{\boldsymbol{U}} \\
&\quad + (\lambda+\mu)(\partial_t(\nabla\cdot\boldsymbol{U}))(\nabla\cdot\bar{\boldsymbol{U}}) + \mu(\partial_t\nabla\boldsymbol{U}) : \nabla\bar{\boldsymbol{U}}\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= 2\mathrm{Re}\int_0^t\int_\Omega(-\mu\nabla\boldsymbol{U} : (\partial_t\nabla\bar{\boldsymbol{U}}) - (\lambda+\mu)(\nabla\cdot\boldsymbol{U})(\partial_t(\nabla\cdot\bar{\boldsymbol{U}})) \\
&\quad + (\lambda+\mu)(\partial_t(\nabla\cdot\boldsymbol{U}))(\nabla\cdot\bar{\boldsymbol{U}}) + \mu(\partial_t\nabla\boldsymbol{U}) : \nabla\bar{\boldsymbol{U}})\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\sqrt{\rho}\partial_t\boldsymbol{U}(\cdot,t)\|^2_{L^2(\Omega)^2} &+ \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{U}(\cdot,t)\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{U}(\cdot,t)\|^2_{L^2(\Omega)^{2\times 2}} \\
&= \|\sqrt{\rho}\boldsymbol{u}_1\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{u}_0\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{u}_0\|^2_{L^2(\Omega)^{2\times 2}},
\end{aligned}$$

which implies

$$\|\partial_t\boldsymbol{U}\|_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{U}\|_{L^2(\Omega)} + \|\nabla\boldsymbol{U}\|_{F(\Omega)} \lesssim \|\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_0\|_{L^2(\Omega)} + \|\nabla\boldsymbol{u}_0\|_{L^2(\Omega)^{2\times 2}}.$$

Taking the first and second partial derivatives of (3.6) with respect to t yields

$$\begin{cases}
\rho\partial_t^2(\partial_t\boldsymbol{U}) - \nabla\cdot\left(\mu(\nabla(\partial_t\boldsymbol{U}) + \nabla(\partial_t\boldsymbol{U})^\top)\right) - \nabla\left(\lambda\nabla\cdot(\partial_t\boldsymbol{U})\right) = 0 & \text{in } \Omega, \quad t>0, \\
(\partial_t\boldsymbol{U})|_{t=0} = \boldsymbol{u}_1 & \text{in } \Omega, \\
\partial_t(\partial_t\boldsymbol{U})\big|_{t=0} = \rho^{-1}\left(\nabla\cdot(\mu\nabla\boldsymbol{u}_0 + \nabla\boldsymbol{u}_0^\top) + \nabla(\lambda\nabla\cdot\boldsymbol{u}_0)\right) & \text{in } \Omega, \\
\partial_t\boldsymbol{U} = 0 & \text{on } \Gamma_j,\ t>0,
\end{cases}$$

and

$$\begin{cases}
\rho\partial_t^2(\partial_t^2\boldsymbol{U}) - \nabla\cdot\left(\mu(\nabla(\partial_t^2\boldsymbol{U}) + \nabla(\partial_t^2\boldsymbol{U}^\top))\right) - \nabla\left(\lambda\nabla\cdot(\partial_t^2\boldsymbol{U})\right) = 0 & \text{in } \Omega,\ t>0, \\
(\partial_t^2\boldsymbol{U})\big|_{t=0} = \rho^{-1}\left(\nabla\cdot(\mu\nabla\boldsymbol{u}_0 + \nabla\boldsymbol{u}_0^\top) + \nabla(\lambda\nabla\cdot\boldsymbol{u}_0)\right) & \text{in } \Omega, \\
\partial_t(\partial_t^2\boldsymbol{U})\big|_{t=0} = \rho^{-1}\left(\nabla\cdot(\mu\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_1^\top) + \nabla(\lambda\nabla\cdot\boldsymbol{u}_1)\right) & \text{in } \Omega, \\
\partial_t^2\boldsymbol{U} = 0 & \text{on } \Gamma_j,\ t>0.
\end{cases}$$

Considering the energy functions

$$e_2(t) = \|\sqrt{\rho}\partial_t^2\boldsymbol{U}\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot(\partial_t\boldsymbol{U})\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla(\partial_t\boldsymbol{U})\|^2_{L^2(\Omega)^{2\times 2}}$$

and

$$e_3(t) = \|\sqrt{\rho}\partial_t^3\boldsymbol{U}\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot(\partial_t^2\boldsymbol{U})\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla(\partial_t^2\boldsymbol{U})\|^2_{L^2(\Omega)^{2\times 2}}.$$

We may follow the same steps as those proving the first inequality to derive the other two inequalities. □

4 | THE REDUCED PROBLEM

In this section, we present the main results, which include the well posedness, stability, and a priori estimates for the scattering problem (2.5).

4.1 | Well posedness

Let $\boldsymbol{e}(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{U}(\boldsymbol{x},t)$, where $\boldsymbol{u}$ satisfies 2.5 and $\boldsymbol{U}$ satisfies (3.6). It follows from (2.5) and (3.6) that $\boldsymbol{e}$ satisfies the following system:

$$\begin{cases} \rho\partial_t^2\boldsymbol{e}-\nabla\cdot\left(\mu(\nabla\boldsymbol{e}+\nabla\boldsymbol{e}^\top)\right)-\nabla(\lambda\nabla\cdot\boldsymbol{e})=\boldsymbol{j} & \text{in }\Omega,\ t>0,\\ \boldsymbol{e}|_{t=0}=\partial_t\boldsymbol{e}|_{t=0}=0 & \text{in }\Omega,\\ \mu_j\partial_y\boldsymbol{e}+(\lambda_j+\mu_j)(0,1)^\top\nabla\cdot\boldsymbol{e}=(-1)^{j-1}\mathscr{T}_j[\boldsymbol{e}]+\boldsymbol{\eta}_j & \text{on }\Gamma_j,\ t>0, \end{cases} \tag{4.1}$$

where $\boldsymbol{\eta}_j=(-1)^j\left(\mu_j\partial_y\boldsymbol{U}+(\lambda_j+\mu_j)(0,1)^\top\nabla\cdot\boldsymbol{U}\right)$.

Let $\breve{\boldsymbol{e}}=\mathscr{L}(\boldsymbol{e})$. Taking the Laplace transform of (4.1), we obtain

$$\begin{cases} \nabla\cdot\left(\mu s^{-1}(\nabla\breve{\boldsymbol{e}}+\nabla\breve{\boldsymbol{e}}^\top)\right)+\nabla(\lambda\nabla\breve{\boldsymbol{e}})-s\rho\breve{\boldsymbol{e}}=-s^{-1}\breve{\boldsymbol{j}} & \text{in }\Omega,\\ \mu_j s^{-1}\partial_y\breve{\boldsymbol{e}}+(\lambda_j+\mu_j)s^{-1}(0,1)^\top\nabla\cdot\breve{\boldsymbol{e}}+(-1)^j\mathscr{B}_j[\breve{\boldsymbol{e}}]=\breve{\boldsymbol{\eta}}_j & \text{on }\Gamma_j. \end{cases} \tag{4.2}$$

Lemma 4.1. *For $(\lambda_j,\mu_j)\in S_{\lambda_j,\mu_j}$, the problem (4.2) has a unique weak solution $\breve{\boldsymbol{e}}(\boldsymbol{x})\in H^1(\Omega)^2$, which satisfies*

$$\|\nabla\breve{\boldsymbol{e}}\|_{L^2(\Omega)^{2\times2}}+\|\nabla\cdot\breve{\boldsymbol{e}}\|_{L^2(\Omega)}+\|s\boldsymbol{e}\|_{L^2(\Omega)^2}\lesssim s_1^{-1}\left(\|\breve{\boldsymbol{j}}\|_{L^2(\Omega)^2}+\sum_{j=1}^2\|\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}+\sum_{j=1}^2\|s\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}\right). \tag{4.3}$$

Proof. The well posedness of the solution $\breve{\boldsymbol{e}}(\boldsymbol{x})\in H^1(\Omega)^2$ follows directly from Theorem 3.1. Moreover, we have

$$a_{\mathrm{TH}}(\breve{\boldsymbol{e}},\breve{\boldsymbol{e}})=\sum_{j=1}^2\langle s^{-1}\breve{\boldsymbol{\eta}}_j,\breve{\boldsymbol{e}}\rangle_{\Gamma_j}+\int_\Omega s^{-1}\breve{\boldsymbol{j}}\cdot\breve{\boldsymbol{e}}\mathrm{d}\boldsymbol{x}.$$

It follows from the coercivity of a_{TH} in 3.4 and the trace theorem in Lemma 2.3 that

$$\begin{aligned} &\frac{s_1}{|s|^2}\left(\|\nabla\breve{\boldsymbol{e}}\|^2_{L^2(\Omega)^{2\times2}}+\|\nabla\cdot\breve{\boldsymbol{e}}\|^2_{L^2(\Omega)}+\|s\boldsymbol{e}\|^2_{L^2(\Omega)^2}\right)\\ &\quad\lesssim|s|^{-1}\|s^{-1}\breve{\boldsymbol{j}}\|_{L^2(\Omega)^2}\|s\breve{\boldsymbol{e}}\|_{L^2(\Omega)^2}+\sum_{j=1}^2\|s^{-1}\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}\|\breve{\boldsymbol{e}}\|_{H^{1/2}(\Gamma_j)^2}\\ &\quad\lesssim|s|^{-1}\|s^{-1}\breve{\boldsymbol{j}}\|_{L^2(\Omega)^2}\|s\breve{\boldsymbol{e}}\|_{L^2(\Omega)^2}+\sum_{j=1}^2\|s^{-1}\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}\|\breve{\boldsymbol{e}}\|_{H^1(\Omega)^2}\\ &\quad\lesssim|s|^{-1}\|s^{-1}\breve{\boldsymbol{j}}\|_{L^2(\Omega)^2}\|s\breve{\boldsymbol{e}}\|_{L^2(\Omega)^2}+|s|^{-1}\sum_{j=1}^2\|s^{-1}\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}\|s\breve{\boldsymbol{e}}\|_{L^2(\Omega)^2}\\ &\qquad+\sum_{j=1}^2\|s^{-1}\breve{\boldsymbol{\eta}}_j\|_{H^{-1/2}(\Gamma_j)^2}\|\nabla\breve{\boldsymbol{e}}\|_{L^2(\Omega)^{2\times2}}, \end{aligned}$$

which gives the estimate (4.3) after applying the Cauchy-Schwarz inequality. □

To show the well posedness of the reduced problem (2.5), we assume that

$$\boldsymbol{u}_0(\boldsymbol{x}),\boldsymbol{u}_1(\boldsymbol{x})\in H^2(\Omega)^2,\quad \boldsymbol{j}(\boldsymbol{x},t)\in H^1\left(0,T;L^2(\Omega)\right)^2. \tag{4.4}$$

Theorem 4.2. *The problem (2.5) has a unique solution $\boldsymbol{u}(\boldsymbol{x},t)$ for $(\lambda_j,\mu_j)\in S_{\lambda_j,\mu_j}$. Moreover, it satisfies*

$$\boldsymbol{u}(\boldsymbol{x},t)\in L^2\left(0,T;H^1(\Omega)\right)^2\cap H^1(0,T;L^2(\Omega))^2$$

and the stability estimate

$$\begin{aligned} &\max_{t\in[0,T]}\left(\|\partial_t\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}+\|\nabla(\partial_t\boldsymbol{u}(\cdot,t))\|_{L^2(\Omega)^{2\times2}}\right)\\ &\quad\lesssim\|\nabla\cdot\boldsymbol{u}_0\|_{L^2(\Omega)^2}+\|\nabla\boldsymbol{u}_0\|_{L^2(\Omega)^{2\times2}}+\|\Delta\boldsymbol{u}_0\|_{L^2(\Omega)}+\|\nabla\nabla\cdot\boldsymbol{u}_0\|_{L^2(\Omega)^2}\\ &\qquad+\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}+\|\nabla\cdot\boldsymbol{u}_1\|_{L^2(\Omega)}+\|\nabla\boldsymbol{u}_1\|_{L^2(\Omega)^{2\times2}}+\|\boldsymbol{j}\|_{H^1(0,T;L^2(\Omega))^2}. \end{aligned} \tag{4.5}$$

Proof. Recall the decomposition $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{e}$, where $\boldsymbol{U}$ satisfies (3.6) and $\boldsymbol{e}$ satisfies (4.1). Since

$$\begin{aligned}
&\int_0^T \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&\le \int_0^T e^{-2s_1(t-T)} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&= e^{2s_1 T}\int_0^T e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&\lesssim \int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t,
\end{aligned}$$

it suffices to estimate the integral

$$\int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t.$$

Taking the Laplace transform of (4.1) and applying Lemma 4.1, it leads to

$$\|\nabla \breve{\boldsymbol{e}}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \breve{\boldsymbol{e}}\|^2_{L^2(\Omega)} + \|s\boldsymbol{e}\|^2_{L^2(\Omega)^2} \lesssim s_1^{-2}\left(\|\breve{\boldsymbol{j}}\|^2_{L^2(\Omega)^2} + \sum_{j=1}^{2}\|\breve{\boldsymbol{\eta}}_j\|^2_{H^{-1/2}(\Gamma_j)^2} + \sum_{j=1}^{2}\|s\breve{\boldsymbol{\eta}}_j\|^2_{H^{-1/2}(\Gamma_j)^2}\right). \tag{4.6}$$

It follows from Trèves[35, Lemma 44.1] that $\breve{\boldsymbol{e}}$ is a holomorphic function of s on the half plane $s_1 > \sigma_0 > 0$, where σ_0 is any positive constant. Hence, we have from Lemma 2.1 that the inverse Laplace transform of $\breve{\boldsymbol{e}}$ exists and is supported in $[0, \infty]$.

Denote by $\boldsymbol{e} = \mathscr{L}^{-1}(\breve{\boldsymbol{e}})$. Since

$$\breve{\boldsymbol{e}} = \mathscr{L}(\boldsymbol{e}) = \mathscr{F}(e^{-s_1 t}\boldsymbol{e}),$$

where $\mathscr{F}$ is the Fourier transform in s_2, we have from the Parseval identity (2.4) and (4.6) that

$$\begin{aligned}
&\int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\|s\breve{\boldsymbol{e}}\|^2_{L^2(\Omega)^2} + \|\nabla \breve{\boldsymbol{e}}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \breve{\boldsymbol{e}}\|^2_{L^2(\Omega)}\right) \mathrm{d}s_2 \\
&\lesssim s_1^{-2}\int_{-\infty}^{\infty} \|\breve{\boldsymbol{j}}\|^2_{L^2(\Omega)^2}\mathrm{d}s_2 + s_1^{-2}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\|\breve{\boldsymbol{\eta}}_j\|^2_{H^{-1/2}(\Gamma_j)^2}\mathrm{d}s_2 + s_1^{-2}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\|s\breve{\boldsymbol{\eta}}_j\|^2_{H^{-1/2}(\Gamma_j)^2}\mathrm{d}s_2.
\end{aligned}$$

Denote by $\boldsymbol{\eta}_{j,0} := \boldsymbol{\eta}_j|_{t=0} = (-1)^j\left(\mu_j\partial_y \boldsymbol{u}_0 + (\lambda_j + \mu_j)(0,1)^\top \nabla\cdot \boldsymbol{u}_0\right)$. Since $\boldsymbol{u}_0$ is supported in Ω, it leads to $\boldsymbol{\eta}_{j,0} = 0$ on Γ_j. We obtain $\mathscr{L}(\partial_t \boldsymbol{\eta}_j) = s\breve{\boldsymbol{\eta}}_j, j = 1, 2$. Hence,

$$\begin{aligned}
&\int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&\lesssim s_1^{-2}\int_{-\infty}^{\infty}\left(\|\mathscr{L}(\boldsymbol{j})\|^2_{L^2(\Omega)^2} + \sum_{j=1}^{2}\|\mathscr{L}(\boldsymbol{\eta}_j)\|^2_{H^{-1/2}(\Gamma_j)^2} + \sum_{j=1}^{2}\|\mathscr{L}(\partial_t\boldsymbol{\eta}_j)\|^2_{H^{-1/2}(\Gamma_j)^2}\right)\mathrm{d}s_2.
\end{aligned}$$

Using the Parseval identity again gives

$$\begin{aligned}
&\int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{e}\|^2_{L^2(\Omega)^2} + \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot \boldsymbol{e}\|^2_{L^2(\Omega)}\right) \mathrm{d}t \\
&\lesssim s_1^{-2}\int_0^\infty e^{-2s_1 t}\left(\|\boldsymbol{j}\|^2_{L^2(\Omega)^2} + \sum_{j=1}^{2}\|\boldsymbol{\eta}_j\|^2_{H^{-1/2}(\Gamma_j)^2} + \sum_{j=1}^{2}\|\partial_t\boldsymbol{\eta}_j\|^2_{H^{-1/2}(\Gamma_j)^2}\right)\mathrm{d}t,
\end{aligned}$$

which shows that

$$\boldsymbol{e} \in L^2\left(0, T; H^1(\Omega)^2\right) \cap H^1\left(0, T; L^2(\Omega)^2\right).$$

Next, we prove the stability. Let $\tilde{\boldsymbol{u}}$ be the extension of $\boldsymbol{u}$ with respect to t in $\mathbb{R}$ such that $\tilde{\boldsymbol{u}} = 0$ outside the interval $[0, t]$. By the Parseval identity (2.4) and Lemma 2.5, we get

$$
\begin{aligned}
\mathrm{Re}\int_0^t e^{-2s_1t}\int_{\Gamma_j}\mathscr{T}_j[\boldsymbol{u}]\cdot\partial_t\bar{\boldsymbol{u}}\mathrm{d}x\mathrm{d}t &= \mathrm{Re}\int_{\Gamma_h}\int_0^\infty e^{-2s_1t}\mathscr{T}_j[\tilde{\boldsymbol{u}}]\cdot\partial_t\bar{\tilde{\boldsymbol{u}}}\mathrm{d}t\mathrm{d}x\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\langle\mathscr{B}_j[\breve{\tilde{\boldsymbol{u}}}], s\breve{\tilde{\boldsymbol{u}}}\rangle_{\Gamma_j}\mathrm{d}s_2\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|s|^2\mathrm{Re}\langle s^{-1}\mathscr{B}_j[\breve{\tilde{\boldsymbol{u}}}](s), \breve{\tilde{\boldsymbol{u}}}\rangle_{\Gamma_j}\mathrm{d}s_2 \le 0,
\end{aligned}
$$

which yields after taking $s_1 \to 0$ that

$$
\mathrm{Re}\int_0^t\int_{\Gamma_j}\mathscr{T}_j[\boldsymbol{u}]\cdot\partial_t\bar{\boldsymbol{u}}\mathrm{d}x\mathrm{d}t \le 0. \tag{4.7}
$$

For any $0 < t < T$, consider an energy function

$$
E_1(t) = \|\sqrt{\rho}\partial_t\boldsymbol{u}\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{u}\|^2_{L^2(\Omega)^{2\times2}}.
$$

It is easy to note that

$$
\begin{aligned}
\int_0^t E_1'(\tau)\mathrm{d}\tau = &\left(\|\sqrt{\rho}\partial_t\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^{2\times2}}\right)\\
&-\left(\sqrt{\rho}\|\boldsymbol{u}_1\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot\boldsymbol{u}_0\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla\boldsymbol{u}_0\|^2_{L^2(\Omega)^{2\times2}}\right).
\end{aligned} \tag{4.8}
$$

On the other hand, it follows from (2.5) and (4.7) that

$$
\begin{aligned}
\int_0^t E_1'(\tau)\mathrm{d}\tau &= 2\mathrm{Re}\int_0^t\int_\Omega\left(\rho\partial_t^2\boldsymbol{u}\cdot\partial_t\bar{\boldsymbol{u}} + (\lambda+\mu)(\partial_t(\nabla\cdot\boldsymbol{u}))(\nabla\bar{\boldsymbol{u}}) + \mu(\partial_t(\nabla\boldsymbol{u})) : \nabla\bar{\boldsymbol{u}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&= 2\mathrm{Re}\int_0^t\int_\Omega\left(\left(\nabla\cdot(\mu(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^\top)) + \nabla(\lambda\nabla\cdot\boldsymbol{u}) + \boldsymbol{j}\right)\cdot\partial_t\bar{\boldsymbol{u}}\right.\\
&\quad\left. + (\lambda+\mu)(\partial_t(\nabla\cdot\boldsymbol{u}))(\nabla\bar{\boldsymbol{u}}) + \mu(\partial_t(\nabla\boldsymbol{u})) : \nabla\bar{\boldsymbol{u}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&= 2\mathrm{Re}\int_0^t\sum_{j=1}^2\int_{\Gamma_j}\mathscr{T}_j[\boldsymbol{u}]\cdot\partial_t\bar{\boldsymbol{u}}\mathrm{d}x\mathrm{d}t + 2\mathrm{Re}\int_0^t\int_\Omega\boldsymbol{j}\cdot\partial_t\bar{\boldsymbol{u}}\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\le 2\mathrm{Re}\int_0^t\int_\Omega\boldsymbol{j}\cdot\partial_t\bar{\boldsymbol{u}}\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\le 2\max_{t\in[0,T]}\|\partial_t\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\|\boldsymbol{j}\|_{L^1(0,T;L^2(\Omega))^2}.
\end{aligned}
$$

By Young inequality and (4.8), we get

$$
\begin{aligned}
&\max_{t\in[0,T]}\left(\|\partial_t\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^{2\times2}}\right)\\
&\quad\lesssim \|\boldsymbol{u}_1\|^2_{L^2(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_0\|^2_{L^2(\Omega)} + \|\nabla\boldsymbol{u}_0\|^2_{L^2(\Omega)^{2\times2}} + |\boldsymbol{j}|^2_{L^1(0,T;L^2(\Omega)^2)}.
\end{aligned} \tag{4.9}
$$

Taking the derivative of (2.5) with respect to t and using the assumptions (4.4), we know that $\partial_t\boldsymbol{u}(\boldsymbol{x},t)$ satisfies the same equations with the source and the initial conditions replaced by $\partial_t\boldsymbol{j}$, $\partial_t\boldsymbol{u}|_{t=0} = \boldsymbol{u}_1, \partial_t(\partial_t\boldsymbol{u})|_{t=0} = \rho^{-1}\left(\nabla\cdot(\mu\nabla\boldsymbol{u}_0 + \nabla\boldsymbol{u}_0^\top) + \nabla(\lambda\nabla\cdot\boldsymbol{u}_0)\right)$. Hence, we may consider a similar energy function

$$
E_2(t) = \|\sqrt{\rho\partial_t^2}\boldsymbol{u}\|^2_{L^2(\Omega)^2} + \|\sqrt{\lambda+\mu}\nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(\Omega)} + \|\sqrt{\mu}\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(\Omega)^{2\times2}}.
$$

We may repeat similar steps to obtain

$$
\begin{aligned}
&\max_{t\in[0,T]}\left(\|\partial_t^2\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^2} + \|\nabla\cdot(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(\Omega)} + \|\nabla(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(\Omega)^{2\times2}}\right)\\
&\quad\lesssim \|\Delta\boldsymbol{u}_0\|^2_{L^2(\Omega)^2} + \|\nabla\nabla\cdot\boldsymbol{u}_0\|^2_{L^2(\Omega)^2}\\
&\qquad + \|\nabla\cdot\boldsymbol{u}_1\|^2_{L^2(\Omega)} + \|\nabla\boldsymbol{u}_1\|^2_{L^2(\Omega)^{2\times2}} + \|\partial_t\boldsymbol{j}\|^2_{L^1(0,T;L^2(\Omega))^2}.
\end{aligned} \tag{4.10}
$$

Combining the above estimate with 4.9, we obtain

$$\max_{t\in[0,T]}\left(\|\partial_t\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^2}+\|\nabla(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(\Omega)^{2\times2}}\right)$$
$$\le\max_{t\in[0,T]}\left(\|\partial_t^2\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega)^2}+\|\nabla\cdot(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(\Omega)}+\|\nabla(\partial_t\boldsymbol{u}(\cdot,t))\|^2_{L^2(\Omega)^{2\times2}}\right), \tag{4.11}$$

which completes the proof of 4.5 after combining the above estimates (4.9) to (4.11). □

4.2 | A priori estimate

In what follows, we derive a priori stability estimate for the wave field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variational problem of 2.5 is to find $\boldsymbol{u}\in H^1(\Omega)^2$ for all $t>0$ such that

$$\int_\Omega \rho\partial_t^2\boldsymbol{u}\cdot\bar{\boldsymbol{w}}\mathrm{d}\boldsymbol{x}=-\int_\Omega(\mu(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{w}})+(\lambda+\mu)(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{w}}))\,\mathrm{d}\boldsymbol{x}$$
$$+\sum_{j=1}^2\langle\mathscr{T}_j[\boldsymbol{u}],\boldsymbol{w}\rangle_{\Gamma_j}+\int_\Omega\boldsymbol{j}\cdot\bar{\boldsymbol{w}}\mathrm{d}\boldsymbol{x},\quad\forall\boldsymbol{w}\in H^1(\Omega)^2. \tag{4.12}$$

Theorem 4.3. *Let $\boldsymbol{u}\in H^1(\Omega)^2$ be the solution of 4.12. Given $\boldsymbol{u}_0,\boldsymbol{u}_1\in L^2(\Omega)^2$, and $\boldsymbol{j}\in L^1\left(0,T;L^2(\Omega)^2\right)$, we have for any $T>0$ that*

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^2)}\lesssim\|\boldsymbol{u}_0\|_{L^2(\Omega)^2}+T\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}+T\|\boldsymbol{j}\|_{L^1(0,T;L^2(\Omega))^2} \tag{4.13}$$

and

$$\|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega)^2)}\lesssim T^{1/2}\|\boldsymbol{u}_0\|_{L^2(\Omega)^2}+T^{3/2}\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}+T^{3/2}\|\boldsymbol{j}\|_{L^1(0,T;L^2(\Omega))^2}. \tag{4.14}$$

Proof. Let $0<\theta<T$ and define an auxiliary function

$$\boldsymbol{\psi}(\boldsymbol{x},t)=\int_t^\theta\boldsymbol{u}(\boldsymbol{x},\tau)\mathrm{d}\tau,\quad\boldsymbol{x}\in\Omega,\quad 0\le t\le\theta.$$

It is clear that

$$\boldsymbol{\psi}(\boldsymbol{x},\theta)=0,\quad\partial_t\boldsymbol{\psi}(\boldsymbol{x},t)=-\boldsymbol{u}(\boldsymbol{x},t). \tag{4.15}$$

For any $\boldsymbol{\phi}(\boldsymbol{x},t)\in L^2(0,\theta;L^2(\Omega)^2)$, we have

$$\int_0^\theta\boldsymbol{\phi}(\boldsymbol{x},t)\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},t)\mathrm{d}t=\int_0^\theta\left(\int_0^t\boldsymbol{\phi}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t. \tag{4.16}$$

Indeed, using the integration by parts and 4.15, we have

$$\begin{aligned}\textstyle\int_0^\theta\boldsymbol{\phi}(\boldsymbol{x},t)\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},t)\mathrm{d}t&=\textstyle\int_0^\theta\left(\boldsymbol{\phi}(\boldsymbol{x},t)\cdot\int_t^\theta\bar{\boldsymbol{u}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\mathrm{d}t\\&=\textstyle\int_0^\theta\int_t^\theta\bar{\boldsymbol{u}}(\boldsymbol{x},\tau)\mathrm{d}\tau\cdot\mathrm{d}\left(\int_0^t\boldsymbol{\phi}(\boldsymbol{x},\varsigma)\mathrm{d}\varsigma\right)\\&=\textstyle\int_t^\theta\bar{\boldsymbol{u}}(\boldsymbol{x},\tau)\mathrm{d}\tau\cdot\int_0^t\boldsymbol{\phi}(\boldsymbol{x},\varsigma)\mathrm{d}\varsigma\Big|_0^\theta+\int_0^\theta\left(\int_0^t\boldsymbol{\phi}(\boldsymbol{x},\varsigma)\mathrm{d}\varsigma\right)\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t\\&=\textstyle\int_0^\theta\left(\int_0^t\boldsymbol{\phi}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t.\end{aligned}$$

Next, we take the test function $\boldsymbol{w}=\boldsymbol{\psi}$ in (4.12) and get

$$\int_\Omega\rho\partial_t^2\boldsymbol{u}\cdot\bar{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x}=-\int_\Omega(\mu(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{\psi}})+(\lambda+\mu)(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{\psi}}))\,\mathrm{d}\boldsymbol{x}$$
$$+\sum_{j=1}^2\langle\mathscr{T}_j[\boldsymbol{u}],\boldsymbol{\psi}\rangle_{\Gamma_j}+\int_\Omega\boldsymbol{j}\cdot\bar{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x}. \tag{4.17}$$

It follows from (4.15) that

$$\begin{aligned}\mathrm{Re}\int_0^\theta\int_\Omega \rho\partial_t^2\boldsymbol{u}\cdot\bar{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x}\mathrm{d}t &= \mathrm{Re}\int_\Omega\rho\int_0^\theta \partial_t(\partial_t\boldsymbol{u}\cdot\bar{\boldsymbol{\psi}}))+\partial_t\boldsymbol{u}\cdot\bar{\boldsymbol{u}}\mathrm{d}t\mathrm{d}\boldsymbol{x}\\ &= \mathrm{Re}\int_\Omega\rho\left((\partial_t\boldsymbol{u}\cdot\bar{\boldsymbol{\psi}})\Big|_0^\theta+\tfrac{1}{2}|\boldsymbol{u}|^2\Big|_0^\theta\right)\mathrm{d}\boldsymbol{x}\\ &= \tfrac{1}{2}\|\sqrt{\rho}\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(\Omega)^2}-\tfrac{1}{2}\|\sqrt{\rho}\boldsymbol{u}_0\|^2_{L^2(\Omega)^2}-\mathrm{Re}\int_\Omega\rho\boldsymbol{u}_1(\boldsymbol{x})\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Integrating (4.17) from $t = 0$ to $t = \theta$ and taking the real parts yield

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(\Omega)^2}-\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}_0\|^2_{L^2(\Omega)^2}+\mathrm{Re}\int_0^\theta\int_\Omega\mu\left(\nabla\boldsymbol{u}(\boldsymbol{x},t):\nabla\bar{\boldsymbol{\psi}}(\boldsymbol{x},t)\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ &\quad+\mathrm{Re}\int_0^\theta\int_\Omega(\lambda+\mu)(\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t))(\nabla\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},t))\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ &=\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(\Omega)^2}-\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}_0\|^2_{L^2(\Omega)^2}\\ &\quad+\frac{1}{2}\int_\Omega\left(\mu\left|\int_0^\theta\nabla\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t\right|^2_{L^2(\Omega)^{2\times2}}+(\lambda+\mu)\left|\int_0^\theta\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t\right|^2\right)\mathrm{d}\boldsymbol{x}\\ &=\mathrm{Re}\int_\Omega\rho\boldsymbol{u}_1(\boldsymbol{x})\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}+\mathrm{Re}\int_0^\theta\int_\Omega\boldsymbol{j}(\boldsymbol{x},t)\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t+\mathrm{Re}\int_0^\theta\sum_{j=1}^{2}\langle\mathscr{T}_j[\boldsymbol{u}],\boldsymbol{\psi}\rangle_{\Gamma_j}\mathrm{d}t,\end{aligned}\tag{4.18}$$

where we have used the fact that

$$\left|\int_0^\theta\nabla\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t\right|^2_{L^2(\Omega)^{2\times2}}=\int_0^\theta\nabla\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t:\int_0^\theta\nabla\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t.$$

In what follows, we estimate the three terms on the right-hand side of 4.18 separately. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\mathrm{Re}\int_\Omega\rho\boldsymbol{u}_1(\boldsymbol{x})\cdot\bar{\psi}(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}&=\mathrm{Re}\int_\Omega\rho\boldsymbol{u}_1(\boldsymbol{x})\cdot\left(\int_0^\theta\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\ &=\mathrm{Re}\int_0^\theta\int_\Omega\rho\boldsymbol{u}_1(\boldsymbol{x})\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}x\mathrm{d}t\\ &\le\rho_{\max}\|\boldsymbol{u}_1\|_{L^2(\Omega)^2}\int_0^\theta\|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t.\end{aligned}\tag{4.19}$$

For $0 \le t \le \theta \le T$, we have from (4.16) that

$$\begin{aligned}\mathrm{Re}\int_0^\theta\int_\Omega\boldsymbol{j}(\boldsymbol{x},t)\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t&=\mathrm{Re}\int_\Omega\int_0^\theta\left(\int_0^t\boldsymbol{j}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t\mathrm{d}\boldsymbol{x}\\ &=\mathrm{Re}\int_0^\theta\int_0^t\int_\Omega\boldsymbol{j}(\boldsymbol{x},\tau)\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}\tau\mathrm{d}t\\ &\le\int_0^\theta\left(\int_0^t\|\boldsymbol{j}(\cdot,\tau)\|_{L^2(\Omega)^2}\mathrm{d}\tau\right)\|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\\ &\le\int_0^\theta\left(\int_0^\theta\|\boldsymbol{j}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\right)\|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\\ &\le\left(\int_0^\theta\|\boldsymbol{j}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\right)\left(\int_0^\theta\|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\right).\end{aligned}\tag{4.20}$$

It follows from (4.16) that

$$\begin{aligned}\mathrm{Re}\int_0^\theta\langle\mathscr{T}_j[\boldsymbol{u}],\boldsymbol{\psi}\rangle_{\Gamma_j}\mathrm{d}t&=\mathrm{Re}\int_0^\theta\int_{\Gamma_j}\mathscr{T}_j[\boldsymbol{u}]\cdot\bar{\boldsymbol{\psi}}\mathrm{d}x\mathrm{d}t\\ &=\mathrm{Re}\int_{\Gamma_j}\int_0^\theta\left(\int_0^t\mathscr{T}_j[\boldsymbol{u}](\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t\mathrm{d}x.\end{aligned}$$

Let $\tilde{\boldsymbol{u}}$ be the extension of $\boldsymbol{u}$ with respect to t in $\mathbb{R}$ such that $\tilde{\boldsymbol{u}} = 0$ outside the interval $[0, \theta]$. We obtain from the Parseval identity (2.4) and Lemma 2.5 that

$$\begin{aligned}
&\mathrm{Re}\int_{\Gamma_j}\int_0^\theta e^{-2s_1 t}\left(\int_0^t \mathscr{T}_j[\boldsymbol{u}](\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{u}}(t)\mathrm{d}t\mathrm{d}x \\
&\quad= \mathrm{Re}\int_{\Gamma_j}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{T}_j[\tilde{\boldsymbol{u}}](\tau)\mathrm{d}\tau\right)\cdot\bar{\tilde{\boldsymbol{u}}}(t)\mathrm{d}t\mathrm{d}x \\
&\quad= \mathrm{Re}\int_{\Gamma_j}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{L}^{-1}\circ\mathscr{B}_j\circ\mathscr{L}\tilde{\boldsymbol{u}}(\tau)\mathrm{d}\tau\right)\cdot\bar{\tilde{\boldsymbol{u}}}(t)\mathrm{d}t\mathrm{d}x \\
&\quad= \mathrm{Re}\int_{\Gamma_j}\int_0^\infty e^{-2s_1 t}\left(\mathscr{L}^{-1}\circ(s^{-1}\mathscr{B}_j)\circ\mathscr{L}\tilde{\boldsymbol{u}}(t)\cdot\bar{\tilde{\boldsymbol{u}}}(t)\right)\mathrm{d}t\mathrm{d}x \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\int_{\Gamma_h} s^{-1}\mathscr{B}_j\breve{\tilde{\boldsymbol{u}}}(s)\cdot\bar{\breve{\tilde{\boldsymbol{u}}}}(s)\mathrm{d}x\mathrm{d}s_2 \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\langle s^{-1}\mathscr{B}_j\breve{\tilde{\boldsymbol{u}}}, \breve{\tilde{\boldsymbol{u}}}\rangle_{\Gamma_j}\mathrm{d}s_2 \le 0,
\end{aligned}$$

where we have used

$$\int_0^t \boldsymbol{u}(\tau)\mathrm{d}\tau = \mathscr{L}^{-1}(s^{-1}\breve{\boldsymbol{u}}(s)).$$

After taking $s_1 \to 0$, it leads to

$$\mathrm{Re}\int_0^\theta \langle \mathscr{T}_j[\boldsymbol{u}], \boldsymbol{\psi}\rangle_{\Gamma_j}\mathrm{d}t = \mathrm{Re}\int_{\Gamma_j}\int_0^\theta\left(\int_0^t \mathscr{T}_j[\boldsymbol{u}](\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\bar{\boldsymbol{u}}(\boldsymbol{x},t)\mathrm{d}t\mathrm{d}x \le 0. \tag{4.21}$$

Substituting (4.19) to (4.21) into (4.18), we have for any $\theta \in [0, T]$ that

$$\begin{aligned}
\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(\Omega)^2} \le \frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}_0\|^2_{L^2(\Omega)^2} + \left(\rho_{\max}\|\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \int_0^\theta \|\boldsymbol{j}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\right)\cdot \\
\left(\int_0^\theta \|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega)^2}\mathrm{d}t\right).
\end{aligned} \tag{4.22}$$

Taking the L^∞-norm with respect to θ on both sides of (4.22) yields

$$\|\boldsymbol{u}\|^2_{L^\infty(0,T;L^2(\Omega))^2} \lesssim \|\boldsymbol{u}_0\|^2_{L^2(\Omega)^2} + T\left(\|\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\boldsymbol{j}\|_{L^1(0,T;L^2(\Omega))^2}\right)\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))^2},$$

which gives the estimate (4.13) after applying the Young inequality.

Integrating (4.22) with respect to θ from 0 to T and using the Cauchy-Schwarz inequality, we obtain

$$\|\boldsymbol{u}\|^2_{L^2(0,T;L^2(\Omega))^2} \lesssim T\|\boldsymbol{u}_0\|^2_{L^2(\Omega)^2} + T^{3/2}\left(\|\boldsymbol{u}_1\|_{L^2(\Omega)^2} + \|\boldsymbol{j}\|_{L^1(0,T;L^2(\Omega))^2}\right)\|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))^2},$$

which implies the estimate (4.14) by using the Young inequality again. □

5 | CONCLUSION

In this paper, we have considered the time-domain elastic scattering problem in an unbounded structure. Using the Helmholtz decomposition, we present the exact time-domain TBC. Then the scattering problem can be reduced into an initial-boundary value problem. We study two auxiliary problems: One is to establish the well posedness and stability of the reduced problem in s-domain, and the other is to establish the well posedness of the reduced problem in time domain with the Dirichlet boundary condition. In the time domain, we show that the reduced problem has a unique weak solution by using the energy method. The main ingredients of the proofs are the Laplace transform, the Lax-Milgram theorem, and the Parseval identity. Moreover, we obtain a priori estimates with explicit time dependence for the quantities of elastic wave displacement by taking special test functions of the time-domain variational problem for the Navier equation.

The admissible set S_{λ_j,μ_j} plays an important role in the proof of well posedness and stability for the problem. We can only show that the set is nonempty and includes some possible Lamé parameters. We hope to remove the restriction and show the validity of the results for more general cases in the future. Another possible direction is to study the time-domain elastic scattering by an unbounded structure in three dimensions. The major difficulty is on the analysis for the TBC where a 3×3 matrix needs to be considered. The progress will be reported somewhere else.

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REFERENCES

1. Abubakar I. Scattering of plane elastic waves at rough surfaces I. *Proc Cambridge Philos Soc*. 1962;58:136-157.
2. Elfouhaily TM, Guérin CA. A critical survey of approximate scattering wave theories from random rough surfaces. *Waves Random Media*. 2004;14:R1-R40.
3. Ogilvy JA. Theory of wave scattering from random rough surfaces. Adam Hilger, Ltd., Bristol; 1991.
4. Voronovich AG. Non-classical approaches in the theory of wave scattering from rough surfaces. Mathematical and Numerical Aspects of Wave Propagation, (Golden, CO, 1998),84-88, SIAM, Philadelphia; 1998.
5. Warnick KF, Chew WC. Numerical simulation methods for rough surface scattering. *Waves Random Media*. 2001;11:R1-R30.
6. Fokkema J. Reflection and transmission of elastic waves by the spatially periodic interface between two solids (theory of the integral-equation method). *Wave Motion*. 1980;2:375-393.
7. Fokkema J, Van den Berg P. Elastodynamic diffraction by a periodic rough surface (stress-free boundary). *J Acoust Soc Am*. 1977;62:1095-1101.
8. Sherwood J. Elastic wave propagation in a semi-infinite solid medium. *Proc Phys Soc*. 1958;71:207.
9. Arens T, Hohage T. On radiation conditions for rough surface scattering problems. *IMA J Appl Math*. 2005;70:839-847.
10. Zhang B, Chandler-Wilde SN. Acoustic scattering by an inhomogeneous layer on a rigid plate. *SIAM J Appl Math*. 1998;58:1931-1950.
11. Chandler-Wilde SN, Heinemeyer E, Potthast R. Acoustic scattering by mildly rough unbounded surfaces in three dimensions. *SIAM J Appl Dyn Syst*. 2006;66:1002-1026.
12. Chandler-Wilde SN, Monk P. Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces. *SIAM J Math Anal*. 2005;37:598-618.
13. Chandler-Wilde SN, Zhang B. Scattering of electromagnetic waves by rough interfaces and inhomogeneous layers. *SIAM J Math Anal*. 1999;30:559-583.
14. Lechleiter A, Ritterbusch S. A variational method for wave scattering from penetrable rough layers. *IMA J Appl Math*. 2010;75:366-391.
15. Li P, Shen J. Analysis of the scattering by an unbounded rough surface. *Math Methods Appl Sci*. 2012;35:2166-2184.
16. Haddar H, Lechleiter A. Electromagnetic wave scattering from rough penetrable layers. *SIAM J Math Anal*. 2011;43:2418-2443.
17. Li P, Wu H, Zheng W. Electromagnetic scattering by unbounded rough surfaces. *SIAM J Math Anal*. 2011;43:1205-1231.
18. Li P, Zheng G, Zheng W. Maxwell's equations in an unbounded structure. *Math Methods Appl Sci*. 2017;40:573-588.
19. Arens T. Uniqueness for elastic wave scattering by rough surfaces. *SIAM J Math Anal*; 33(2001):461-476.
20. Arens T. Existence of solution in elastic wave scattering by unbounded rough surfaces. *Math. Methods Appl. Sci.* 2002;25:507-528.
21. Chen Q, Monk P. Discretization of the time domain CFIE for acoustic scattering problems using convolution quadrature. *SIAM J Math Anal*. 2014;46:3107-3130.
22. Jin JM, Riley DJ. *Finite Element Analysis of Antennas and Arrays*: John Wiley & Sons; 2009.
23. Li J, Huang Y. Time-domain finite element methods for Maxwell's equations in metamaterials. *Computational Mathematics*, Springer Series. Heidelberg: Springer; 2013.
24. Riley DJ, Jin JM. Finite-element time-domain analysis of electrically and magnetically dispersive periodic structures. *IEEE Trans Antennas Propag*. 2008;56:3501-3509.
25. Wang LL, Wang B, Zhao X. Fast and accurate computation of time-domain acoustic scattering problems with exact nonreflecting boundary conditions. *SIAM J Appl Math*. 2012;72:1869-1898.
26. Chen Z, Nédélec JC. On Maxwell equations with the transparent boundary condition. *J Comput Math*. 2008;26:284-296.
27. Wang B, Wang LL. On l^2-stability analysis of time-domain acoustic scattering problems with exact nonreflecting boundary conditions. *J Math Study*. 2014;47:65-84.
28. Li P, Wang LL, Wood A. Analysis of transient electromagentic scattering from a three-dimensional open cavity. *SIAM J Appl Math*. 2015;75:1675-1699.
29. Gao Y, Li P. Analysis of time-domain scattering by periodic structures. *J Differ Equ*. 2016;261:5094-5118.
30. Gao Y, Li P. Electromagnetic scattering for time-domain Maxwell's equations in an unbounded structure. *Math Models Methods Appl Sci*. 2017;27:1843-1870.
31. Bao G, Gao Y, Li P. Time-domain analysis of an acoustic-elastic interaction problem. *Arch Ration Mech Anal*. 2018;229:835-884.

32. Gao Y, Li P, Zhang B. Analysis of transient acoustic-elastic interaction in an unbounded structure. *SIAM J Math Anal*. 2017;49:3951-3972.
33. Kupradze VD, Gegelia TG, Basheleĭshvili MO, Burchuladze TV. Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. In: Kupradze VD, ed. *Applied Mathematics and Mechanics*, North-Holland Series, vol. 25. Amsterdam-New York: North-Holland Publishing Co.; 1979.
34. Cohen AM. Numerical methods for Laplace transform inversion. *Numerical Methods and Algorithms*. New York: Springer; 2007.
35. Trèves F. Basic linear partial differential equations. *Pure and Applied Mathematics*, Vol. 62. New York-London: Academic Press; 1975.

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