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INVERSE SOURCE PROBLEMS IN ELECTRODYNAMICS

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Abstract. This paper concerns inverse source problems for the time-dependent Maxwell equations. The electric current density is assumed to be the product of a spatial function and a temporal function. We prove uniqueness and stability in determining the spatial or temporal function from the electric field, which is measured on a sphere or at a point over a finite time interval.

1. Introduction.

Consider the time-dependent Maxwell equations in a homogeneous medium for $\boldsymbol{x} \in \mathbb{R}^3,\ t > 0$:

$$\begin{aligned} &\mu\partial_t \boldsymbol{H}(\boldsymbol{x},t) + \nabla \times \boldsymbol{E}(\boldsymbol{x},t) = 0, \\ &\varepsilon\partial_t \boldsymbol{E}(\boldsymbol{x},t) - \nabla \times \boldsymbol{H}(\boldsymbol{x},t) = \boldsymbol{F}(\boldsymbol{x},t), \end{aligned} \tag{1.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields, respectively, $\boldsymbol{F}$ is the electric current density, ε and μ are the dielectric permittivity and the magnetic permeability, respectively. Since the medium is homogeneous, without loss of generality, we assume that $\varepsilon = \mu = 1$. Eliminating the magnetic field $\boldsymbol{H}$ from (1.1), we obtain the Maxwell system for the electric field $\boldsymbol{E}$:

$$\partial_t^2 \boldsymbol{E}(\boldsymbol{x},t) + \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x},t)) = \partial_t \boldsymbol{F}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \tag{1.2}$$

which is supplemented by the homogeneous initial conditions:

$$\boldsymbol{E}(\boldsymbol{x},0) = \partial_t \boldsymbol{E}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{1.3}$$

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This paper concerns the inverse source problem of determining the electric current density. Assuming that the source is a separable function, we consider the following two inverse problems:

(i) Inverse problem one (IP1). The electric current density is assumed to take the form

$$\boldsymbol{F}(\boldsymbol{x},t)=\boldsymbol{J}(\boldsymbol{x})g(t),$$

where g is a known scalar function satisfying $g(t)=0, t\in(-\infty,0]\cup[T_0,\infty]$, where $T_0>0$, and $g\in C([0,T_0]), g'\in L^2([0,T_0])$, and $\boldsymbol{J}$ is an unknown vector function satisfying $\boldsymbol{J}\in H^1(\mathbb{R}^3)^3, \operatorname{supp}(\boldsymbol{J})\subset B_{\hat{R}}$, where $B_{\hat{R}}=\{\boldsymbol{x}\in\mathbb{R}^3:|x|<\hat{R}\}$ is the ball with a radius $\hat{R}>0$. In addition, we assume that $\nabla\cdot\boldsymbol{J}=0$ in $\mathbb{R}^3$. Let $R>\hat{R}$ and $\Gamma_R=\{\boldsymbol{x}\in\mathbb{R}^3:|\boldsymbol{x}|=R\}$. Denote $T=T_0+\hat{R}+R$. The inverse problem is to determine $\boldsymbol{J}$ from the measurement $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}, \boldsymbol{x}\in\Gamma_R, t\in(0,T)$, where $\boldsymbol{\nu}$ is the unit normal vector on Γ_R.

(ii) Inverse problem two (IP2). The electric current density has the form

$$\boldsymbol{F}(\boldsymbol{x},t)=J(\boldsymbol{x})\boldsymbol{g}(t),$$

where J is a given scalar function satisfying $\operatorname{supp}(J)\subset B_{\hat{R}}$ and $\boldsymbol{g}$ is an unknown vector function which is assumed to satisfy $\boldsymbol{g}\in H^1([0,T])^3$. The inverse problem is to determine $\boldsymbol{g}$ from the interior electric field $\boldsymbol{E}(\boldsymbol{x}_0,t), t\in(0,T)$ for some $\boldsymbol{x}_0\in\operatorname{supp}(J)$ or from the tangential trace of the electric field on Γ_R, i.e., from the data $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}, \boldsymbol{x}\in\Gamma_R, t\in(0,T)$.

Inverse source problems have significant applications in many scientific areas such as antenna synthesis and design, biomedical engineering, medical imaging, and optical tomography [7, 3, 14, 15, 20, 27]. The spatial function is usually designed for mathematically approximating a pulsed signal transmitted by antennas, whereas the temporal function is used to model a source profile. Typical physical examples of determining the spatial function (IP1) include passive coherent location (PCL) and detection of launched rockets. The recovery of the temporal function (IP2) can be used for signal pattern recognition.

In general, there is no uniqueness for the inverse source problems with a single frequency data, due to the existence of non-radiating sources, e.g., see [16, 19] for acoustic problems. In [1], the authors considered the inverse source problem for time-harmonic Maxwell's equations and characterized the non-radiating sources with a single frequency. A non-uniqueness example was discussed there for recovering a volume current source in an inhomogeneous medium. It was also proved in [1] that surface currents or dipole sources can be uniquely determined by surface measurements at a fixed frequency. In [4], the authors showed uniqueness and stability, and presented an inversion scheme to reconstruct dipole sources based on the low-frequency asymptotic analysis of the time-harmonic Maxwell equations. A monograph for a formulation with impulsive inputs can be found in [33]. Most of these works mentioned above dealt with the time-harmonic wave equations with a single frequency. In this paper, we make use of dynamical data over a finite time interval to overcome the non-uniqueness issue in inverse source problems. It is known from Huygens' principle that dynamical data implies multi-frequency near-field data in the Fourier domain. This approach was largely motivated by recent studies on uniqueness and stability in recovering sources with multiple frequencies [9, 10, 11, 13] and closely related arguments for the time-dependent Lamé system in linear elasticity [8].

Recently, many efforts have been made on inverse source scattering problems by using multi-frequency data to obtain uniqueness and to achieve increasing stability for the Helmholtz, Navier, and Maxwell equations in the frequency domain [2, 9, 10, 11, 17, 24, 35, 23]. In [13], an attempt was made to bridge the connection of stability estimate between the Helmholtz equation and the wave equation. The spatial source function was transformed to be the inhomogeneous initial conditions for the wave equation. In this work, we consider directly how to determine the temporal and spatial source functions for the time-dependent Maxwell equations. We refer to [6, 25, 34, 26, 22] for the method of applying Carleman estimates to inverse source problems for hyperbolic systems, and to [32, 31] for other formulation of inverse problems which are related to Maxwell's equations. Our approach of converting the time-dependent problem into a multi-frequency problem in the frequency domain seems natural. It differs greatly from the method of using Carleman estimates. Compared to existing references, our work has several new contributions. Firstly, we present an interesting result in Theorem 2.6, which shows that sparse boundary observations of the electric field can also be used for extracting information of the support of a source term. The proof is motivated by recent studies in the acoustic case [2]. The result gives new ideas for reconstructing the geometrical shape of an obstacle from dynamical data. Secondly, we provide new and simple techniques in Theorems 2.8 and 3.4 for estimating source terms from dynamical boundary observations.

There are still several questions which remain open. Our arguments rely heavily on the divergence free condition of the source term (see Remark 2.2) and the boundedness of the support of the source. The divergence-free condition indeed simplifies our arguments by helping to eliminate a class of non-radiating sources (for which the uniqueness does not hold). It is not trivial to remove this assumption and obtain the same results within the framework of this paper. When the source is not compactly supported, we think that it might be possible to apply the Laplace transform instead of the Fourier transform. In that case one should investigate the asymptotic behavior of the radiated electric wave field at large time instead of using Huygen's principle. On the other hand, it would also be interesting to investigate numerical methods to reconstruct sources and other kind of source terms such as moving point sources. We shall report the progress on these directions in separate works.

The remaining part of this paper is organized as follows. In section 2, we study IP1 and present the uniqueness and stability results for recovering the spatial function. In particular, we show unique determination of the maximum and minimum distance between one observation point and the support of the spatial function, and provide novel mathematical techniques for deriving the stability estimate with boundary observations. In section 3, we discuss IP2 and show the unique determination of the temporal function by using boundary measurement and the stability estimate with an interior measurement at a single point only.

2. IP1: Determination of the spatial function.

2.1. Preliminaries. We first introduce the electrodynamic Green tensor $\mathbb{G}$ to the system (1.1) and then present an estimate for the electric field in terms of the regularity of $\boldsymbol{J}$.

The derivation of the electrodynamic Green tensor in a spatially homogeneous media has a long history. We refer to [28, 18] for detailed discussions and the representation in different forms. Here we provide a simplified derivation of $\mathbb{G}$ for the reader's convenience by reducing the Maxwell equations into a single vector wave equation. Consider the Maxwell system

$$\begin{cases}\partial_t\mathbb{H}(\boldsymbol{x},t)+\nabla\times\mathbb{G}(\boldsymbol{x},t)=0,\\ -\partial_t\mathbb{G}(\boldsymbol{x},t)+\nabla\times\mathbb{H}(\boldsymbol{x},t)=\delta(t)\delta(\boldsymbol{x})\mathbb{I},\end{cases}$$

where $\mathbb{G}$ and $\mathbb{H}$ are the electric Green tensor and the magnetic Green tensor, respectively, $\mathbb{I}$ is the 3×3 identity matrix, and δ is the Dirac distribution. It is easy to verify that $\mathbb{G}$ satisfies

$$\partial_t^2\mathbb{G}(\boldsymbol{x},t)+\nabla\times(\nabla\times\mathbb{G}(\boldsymbol{x},t))=-\delta'(t)\delta(\boldsymbol{x})\,\mathbb{I}. \tag{2.1}$$

The homogeneous initial conditions are imposed:

$$\mathbb{G}(\boldsymbol{x},0)=\partial_t\mathbb{G}(\boldsymbol{x},0)=0,\quad |\boldsymbol{x}|\neq 0.$$

To find an analytical expression of $\mathbb{G}$, we introduce a vector potential $\boldsymbol{\Phi}$ and a matrix potential $\mathbb{A}$ such that

$$\begin{cases}\mathbb{H}=\nabla\times\mathbb{A},\\ \mathbb{G}=-\partial_t\mathbb{A}-\nabla\boldsymbol{\Phi},\\ \nabla\cdot\mathbb{A}+\partial_t\boldsymbol{\Phi}=0.\end{cases}$$

The last equation is known as the Lorentz gauge condition [21]. Substituting the above equations and gauge condition into (2.1), we may verify that $\mathbb{A}$ satisfies the following wave equation:

$$\partial_t^2\mathbb{A}-\Delta\mathbb{A}=\delta(t)\delta(\boldsymbol{x})\mathbb{I}.$$

It is well-known that

$$G(\boldsymbol{x},t)=\frac{1}{4\pi|\boldsymbol{x}|}\delta(|\boldsymbol{x}|-t)$$

is the fundamental solution of the wave equation in $\mathbb{R}^3\times[0,\infty)$, i.e., it satisfies

$$\partial_t^2G(\boldsymbol{x},t)-\Delta G(\boldsymbol{x},t)=\delta(t)\delta(\boldsymbol{x})$$

and the homogeneous initial conditions. Therefore, we have

$$\mathbb{A}=G(\boldsymbol{x},t)\mathbb{I}.$$

On the other hand, it follows from the Lorentz gauge condition that we get

$$\partial_t\boldsymbol{\Phi}(\boldsymbol{x},t)=-\nabla\cdot\mathbb{A}(\boldsymbol{x},t)=-\nabla G(\boldsymbol{x},t).$$

Recall that the derivative of the Heaviside step function coincides with the Dirac distribution, i.e.,$H'(t)=\delta(t)$. Then we obtain

$$\boldsymbol{\Phi}(\boldsymbol{x},t)=\nabla\Big(\frac{1}{4\pi|\boldsymbol{x}|}H(|\boldsymbol{x}|-t)\Big).$$

Consequently, it follows from the relation $\mathbb{G}=-\partial_t\mathbb{A}-\nabla\boldsymbol{\Phi}$ that the electrodynamic Green tensor can be expressed as

$$\mathbb{G}(\boldsymbol{x},t)=\frac{1}{4\pi|\boldsymbol{x}|}\delta'(|\boldsymbol{x}|-t)\mathbb{I}-\nabla\nabla^\top\Big(\frac{1}{4\pi|\boldsymbol{x}|}H(|\boldsymbol{x}|-t)\Big). \tag{2.2}$$

Denote by $\hat{\mathbb{G}}(\boldsymbol{x},\kappa)$ the Fourier transform of $\mathbb{G}(\boldsymbol{x},t)$ with respect to the time variable t, i.e.,

$$\hat{\mathbb{G}}(\boldsymbol{x},\kappa)=\int_{\mathbb{R}}\mathbb{G}(\boldsymbol{x},t)e^{-\mathrm{i}\kappa t}\mathrm{d}t.$$

Then it follows from (2.1) and (2.2) that

$$\nabla \times (\nabla \times \hat{\mathbb{G}}) - \kappa^2 \hat{\mathbb{G}} = -\mathrm{i}\kappa\delta(\boldsymbol{x})\mathbb{I}, \quad \boldsymbol{x} \in \mathbb{R}^3.$$

The expression of $\hat{\mathbb{G}}$ takes the form (see e.g., [29])

$$\hat{\mathbb{G}}(\boldsymbol{x}, \kappa) = -\mathrm{i}\kappa \left(g(\boldsymbol{x}, \kappa)\mathbb{I} + \frac{1}{\kappa^2}\nabla\nabla^\top g(\boldsymbol{x}, \kappa) \right), \tag{2.3}$$

where g is the fundamental solution of the three-dimensional Helmholtz equation and is given by

$$g(\boldsymbol{x}, \kappa) = \frac{1}{4\pi}\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}|}}{|\boldsymbol{x}|}.$$

It is clear to note that $\hat{\mathbb{G}}(\boldsymbol{x}, \kappa)$ satisfies the Silver–Müller radiation condition in the frequency domain.

The following lemma states that the electric field $\boldsymbol{E}$ over B_R vanishes after a finite time. Physically, this phenomenon can be interpreted by Huygens' principle.

Lemma 2.1. *Let $T = T_0 + \hat{R} + R$ be given in IP1. Then $\boldsymbol{E}(\boldsymbol{x}, t) = 0, \forall \boldsymbol{x} \in B_R, t > T$.*

Proof. Using the Green tensor (2.2), we have

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x}, t) &= \int_0^\infty \int_{\mathbb{R}^3} \mathbb{G}(\boldsymbol{x} - \boldsymbol{y}, t - s)\boldsymbol{J}(\boldsymbol{y})g(s)\mathrm{d}\boldsymbol{y}\mathrm{d}s \\
&= \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|}\delta'(|\boldsymbol{x} - \boldsymbol{y}| - (t - s))\boldsymbol{J}(\boldsymbol{y})g(s)\mathrm{d}\boldsymbol{y}\mathrm{d}s \\
&\quad - \int_0^\infty \int_{\mathbb{R}^3} \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}^\top \Big(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} H(|\boldsymbol{x} - \boldsymbol{y}| + s - t) \Big) \boldsymbol{J}(\boldsymbol{y})g(s)\mathrm{d}\boldsymbol{y}\mathrm{d}s \\
&= -\int_{B_{\hat{R}}} \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} g'(t - |\boldsymbol{x} - \boldsymbol{y}|)\boldsymbol{J}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad - \int_0^{T_0} \int_{B_{\hat{R}}} \Big(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} H(|\boldsymbol{x} - \boldsymbol{y}| + s - t) \Big) \nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}^\top \cdot \boldsymbol{J}(\boldsymbol{y})g(s)\mathrm{d}\boldsymbol{y}\mathrm{d}s.
\end{aligned}$$

For $t > T = T_0 + R + \hat{R}$, one can easily observe that

$$g'(t - |\boldsymbol{x} - \boldsymbol{y}|) = 0, \quad H(|\boldsymbol{x} - \boldsymbol{y}| + s - t) = 0$$

hold uniformly for all $\boldsymbol{x} \in B_R,\ \boldsymbol{y} \in B_{\hat{R}}, s \in (0, T_0)$, which imply the result. ☐

Remark 2.2. We present some remarks on the assumptions for the source term in the proof of Lemma 2.1.

(i) The assumption that the source term $\boldsymbol{J}$ has compact support is necessary to show Huygens' principle in Lemma 2.1.

(ii) It follows from [1] that the source term can be decomposed into a sum of radiation source and non-radiating source. Hence, by assuming that $\nabla \cdot \boldsymbol{J} = 0$, we may eliminate a class of non-radiating sources in order to ensure the uniqueness of IP1. Moreover, even though (1.2) can be reduced into a simple vector wave equation under the divergence free condition, we may not use the results in [6] and [23] for the hyperbolic wave equation and Helmholtz equation directly. The reason is that for the Maxwell equation, it is more practical to use the tangential trace of the electric field $\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{\nu}$ on the boundary Γ_R as a data rather than the total field $\boldsymbol{E}(\boldsymbol{x}, t)$ on the boundary Γ_R.

Recalling $\nabla \cdot \boldsymbol{J} = 0$, taking the divergence on both sides of (1.2) and using the initial conditions (1.3), we have

$$\partial_t^2 \Big(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) \Big) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0$$

and

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t \Big(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, 0) \Big) = 0.$$

Therefore, $\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. In view of the identify $\nabla \times \nabla \times = -\Delta + \nabla\nabla\cdot$, we obtain from (1.2) that

$$\begin{cases} \partial_t^2 \boldsymbol{E}(\boldsymbol{x}, t) - \Delta \boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{J}(\boldsymbol{x}) g'(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{2.4}$$

To state the regularity of the solution for the initial value problem (2.4), we recall the definition of spaces involving time variables. Given the Banach space X with norm $||\cdot||_X$, the space $C([0,T];X)$ consists of all continuous functions $f : [0,T] \to X$ with the norm

$$||f||_{C([0,T];X)} := \max_{t \in [0,T]} ||f(t, \cdot)||_X.$$

The Sobolev space $W^{m,p}(0,T;X)$, where both m and p are positive integers such that $1 \le m < \infty, 1 \le p < \infty$, comprises all functions $f : L^p(0,T;X)$ such that $\partial_t^k f$ $(k = 1, 2, \cdots, m)$ exists in the weak sense and belongs to $L^p(0,T;X)$. Further, the norm of $W^{m,p}(0,T;X)$ can be characterized by

$$||f||_{W^{m,p}(0,T;X)} := \left(\int_0^T \sum_{k=0}^{m} ||\partial_t^k f(t, \cdot)||_X^p \mathrm{d}t \right)^{1/2}.$$

We denote $H^m = W^{m,2}$.

Lemma 2.3. *Let $\boldsymbol{J} \in H^p(\mathbb{R}^3)^3$ with $p > 0$ be supported on B_R. The initial value problem (2.4) admits a unique solution*

$$\boldsymbol{E} \in C(0,T; H^{p+1}(\mathbb{R}^3))^3 \cap H^\tau(0,T; H^{p-\tau+1}(\mathbb{R}^3))^3$$

for $\tau = 1, 2$, which satisfies

$$\|\boldsymbol{E}\|_{C(0,T;H^{p+1}(\mathbb{R}^3))^3} + \|\boldsymbol{E}\|_{H^\tau(0,T;H^{p-\tau+1}(\mathbb{R}^3))^3} \le C \|g'\|_{L^2(0,T)} \|\boldsymbol{J}\|_{H^p(\mathbb{R}^3)^3}, \tag{2.5}$$

where C is a positive constant depending on R.

Proof. Taking the Fourier transform of (2.4) with respect to the spatial variable $\boldsymbol{x}$, we obtain

$$\begin{cases} \partial_t^2 \hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) + \mathbf{A}(\boldsymbol{\xi}) \hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = g'(t) \hat{\boldsymbol{J}}(\boldsymbol{\xi}), \\ \hat{\boldsymbol{E}}(\boldsymbol{\xi}, 0) = \partial_t \hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = 0, \end{cases} \tag{2.6}$$

where $\boldsymbol{\xi} \in \mathbb{R}^3$, $\mathbf{A}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 \mathbb{I}$. By Duhamel's principle, it is clear to note that the unique solution of (2.6) is

$$\hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = \int_0^t g'(s) |\boldsymbol{\xi}|^{-1} \sin((t-s)|\boldsymbol{\xi}|) \hat{\boldsymbol{J}}(\boldsymbol{\xi}) \mathrm{d}s. \tag{2.7}$$

For all $t \in [0,T]$ and $s \in [0,t]$, define

$$\boldsymbol{K}(t-s, \cdot) := \boldsymbol{\xi} \mapsto |\boldsymbol{\xi}|^{-1} \sin((t-s)|\boldsymbol{\xi}|) \hat{\boldsymbol{J}}(\boldsymbol{\xi}).$$

Then we can rewrite (2.7) as

$$\hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = \int_0^t g'(s)\boldsymbol{K}(t-s, \boldsymbol{\xi})\,\mathrm{d}s.$$

Moreover, since $\boldsymbol{J}$ has compact support, its Fourier transform $\hat{\boldsymbol{J}}$ is bounded. Hence, by using the Plancherel theorem we can obtain the following estimate

$$\begin{aligned}
\|\boldsymbol{K}(t-s,\cdot)\|^2_{L^2(\mathbb{R}^3)^3} &= \int_{\mathbb{R}^3} |\boldsymbol{\xi}|^{-2}|\sin((t-s)|\boldsymbol{\xi}|)|^2|\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2\mathrm{d}\boldsymbol{x} \\
&\leq \int_{B_R} |\boldsymbol{\xi}|^{-2}|\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3\setminus B_R} |\boldsymbol{\xi}|^{-2}|\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2\mathrm{d}\boldsymbol{x} \\
&\leq \|\hat{\boldsymbol{J}}\|^2_{L^\infty(\mathbb{R}^3)^3}\int_{B_R} |\boldsymbol{\xi}|^{-2}\mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{R}^3\setminus B_R} |R|^{-2}|\hat{\boldsymbol{J}}|^2\mathrm{d}\boldsymbol{\xi} \\
&\leq C_1\|\boldsymbol{J}\|^2_{L^2(\mathbb{R}^3)^3} + C_2\|\boldsymbol{J}\|^2_{L^2(\mathbb{R}^3)^3} \leq C\|\boldsymbol{J}\|^2_{L^2(\mathbb{R}^3)^3},
\end{aligned} \tag{2.8}$$

where C_1, C_2, C are positive constants depending on R. Following similar arguments for $p > 0$ we have

$$\|(1+|\boldsymbol{\xi}|^2)^{\frac{p+1}{2}}\boldsymbol{K}(t-s,\cdot)\|^2_{L^2(\mathbb{R}^3)^3} \leq C\int_{\mathbb{R}^3}(1+|\boldsymbol{\xi}|^2)^p|\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2\mathrm{d}\boldsymbol{\xi} \leq C\,\|\boldsymbol{J}\|^2_{H^p(\mathbb{R}^3)^3}. \tag{2.9}$$

Combing estimates (2.8)–(2.9) yields that $\boldsymbol{E} \in C([0,T]; H^{p+1}(\mathbb{R}^3))^3$.

On the other hand, for almost every $\boldsymbol{\xi} \in \mathbb{R}^3$, we have

$$\partial_t\hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = \int_0^t g'(s)\partial_t\boldsymbol{K}(t-s,\boldsymbol{\xi})\mathrm{d}s = \int_0^t g'(s)\cos((t-s)|\boldsymbol{\xi}|)\hat{\boldsymbol{J}}(\boldsymbol{\xi})\mathrm{d}s.$$

Hence we can obtain by following similar arguments for (2.8) that

$$\|(1+|\boldsymbol{\xi}|^2)^{\frac{p}{2}}\partial_t\boldsymbol{K}(t-s,\cdot)\|_{L^2(\mathbb{R}^3)^3} \leq C\|\boldsymbol{J}\|_{H^p(\mathbb{R}^3)^3}, \tag{2.10}$$

which means that $\boldsymbol{E} \in H^1(0,T; H^p(\mathbb{R}^3))^3$. Similarly, we also have

$$\partial_t^2\hat{\boldsymbol{E}}(\boldsymbol{\xi}, t) = g'(t)\hat{\boldsymbol{J}}(\boldsymbol{\xi}) + \int_0^t g'(s)\partial_t^2\boldsymbol{K}(t-s,\boldsymbol{\xi})\mathrm{d}s.$$

It follows from similar arguments that we get

$$\|(1+|\boldsymbol{\xi}|^2)^{\frac{p-1}{2}}\partial_t^2\boldsymbol{K}(t-s,\cdot)\|_{L^2(\mathbb{R}^3)^3} \leq C\|\boldsymbol{J}\|_{H^p(\mathbb{R}^3)^3}, \tag{2.11}$$

which means that $\boldsymbol{E} \in H^2(0,T; H^{p-1}(\mathbb{R}^3))^3$. Therefore, we obtain

$$\boldsymbol{E} \in H^\tau(0,T; H^{p-\tau+1}(\mathbb{R}^3))^3, \quad \tau = 1, 2,$$

which completes the proof. □

Assuming that the temporal function g is given, we present a Fourier approach to determine the unknown spatial function $\boldsymbol{J}$ in the subsequent two subsections. Our arguments rely on the Fourier transform and are motivated by recent studies on inverse source problems for the time-harmonic elastic and electromagnetic wave equations [10].

2.2. **Uniqueness.** First we consider the uniqueness for IP1.

Theorem 2.4. *The spatial source function* $\boldsymbol{J}$ *can be uniquely determined by the data set* $\{\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu} : \boldsymbol{x}\in\Gamma_R, t\in(0,T)\}$.

Proof. It suffices to show $\boldsymbol{J}=0$ in B_R if $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}(\boldsymbol{x})=0$ for all $x\in\Gamma_R, t\in(0,T)$. Recalling Lemma 2.1, we have $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}(\boldsymbol{x})=0$ for all $\boldsymbol{x}\in\Gamma_R, t\in\mathbb{R}^+$. Combing this with the fact that $\boldsymbol{E}(\boldsymbol{x},t)=0$ for $t\le 0$, we deduce that $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}(\boldsymbol{x})=0$ for all $\boldsymbol{x}\in\Gamma_R, t\in\mathbb{R}$. Defining by $\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)$ the Fourier transform of $\boldsymbol{E}(\boldsymbol{x},t)$ with respect to the time t, i.e.,

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)=\int_{\mathbb{R}}\boldsymbol{E}(\boldsymbol{x},t)e^{-\mathrm{i}\kappa t}\mathrm{d}t,\quad \forall\boldsymbol{x}\in\Gamma_R,\kappa\in\mathbb{R}^+,$$

we have

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}(\boldsymbol{x})=0,\quad \forall\boldsymbol{x}\in\Gamma_R,\kappa\in\mathbb{R}^+.$$

Then the equation (1.2) becomes

$$\nabla\times(\nabla\times\hat{\boldsymbol{E}})-\kappa^2\hat{\boldsymbol{E}}=\mathrm{i}\kappa\hat{g}\boldsymbol{J}\quad\text{in }\mathbb{R}^3. \tag{2.12}$$

From (2.3) we obtain

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)=-\hat{g}(\kappa)\int_{\mathbb{R}^3}\hat{\mathbb{G}}(\boldsymbol{x}-\boldsymbol{y},\kappa)\boldsymbol{J}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{2.13}$$

Since $\mathrm{supp}(\boldsymbol{J})\subset B_{\hat{R}}$, it is clear to note that $\hat{\boldsymbol{E}}$ satisfies the Silver–Müller radiation condition:

$$\lim_{r\to\infty}((\nabla\times\hat{\boldsymbol{E}})\times\boldsymbol{x}-\mathrm{i}\kappa r\hat{\boldsymbol{E}})=0,\quad r=|\boldsymbol{x}|,$$

for any fixed frequency $\kappa>0$. Let $\hat{\boldsymbol{E}}\times\boldsymbol{\nu}$ and $\hat{\boldsymbol{H}}\times\boldsymbol{\nu}$ be the tangential trace of the electric and the magnetic fields in the frequency domain, respectively. In the Fourier domain, there exists a capacity operator $T: H^{-1/2}(\mathrm{div},\Gamma_R)\to H^{-1/2}(\mathrm{div},\Gamma_R)$ such that the following transparent boundary condition can be imposed on Γ_R (see the appendix in section 3.2 for details):

$$\hat{\boldsymbol{H}}\times\boldsymbol{\nu}=T(\hat{\boldsymbol{E}}\times\boldsymbol{\nu})\quad\text{on }\Gamma_R, \tag{2.14}$$

which implies that $\hat{\boldsymbol{H}}\times\boldsymbol{\nu}$ can be computed once $\hat{\boldsymbol{E}}\times\boldsymbol{\nu}$ is available on Γ_R. The transparent boundary condition (2.14) can be equivalently written as

$$(\nabla\times\hat{\boldsymbol{E}})\times\boldsymbol{\nu}=\mathrm{i}\kappa T(\hat{\boldsymbol{E}}\times\boldsymbol{\nu})\quad\text{on }\Gamma_R. \tag{2.15}$$

Next we introduce the functions $\hat{\boldsymbol{E}}^{\mathrm{inc}}$ and $\hat{\boldsymbol{H}}^{\mathrm{inc}}$ by

$$\hat{\boldsymbol{E}}^{\mathrm{inc}}(x)=\boldsymbol{p}e^{\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\quad\text{and}\quad\hat{\boldsymbol{H}}^{\mathrm{inc}}(x)=\boldsymbol{q}e^{\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}, \tag{2.16}$$

where $\boldsymbol{d}:=\boldsymbol{d}(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)^\top$ is the unit propagation vector, and $\boldsymbol{p},\boldsymbol{q}$ are two unit polarization vectors satisfying $\boldsymbol{p}(\theta,\varphi)\cdot\boldsymbol{d}(\theta,\varphi)=0, \boldsymbol{q}(\theta,\varphi)=\boldsymbol{p}(\theta,\varphi)\times\boldsymbol{d}(\theta,\varphi)$ for any fixed $\theta\in[0,\pi],\varphi\in[0,2\pi]$. It is easy to verify that $\hat{\boldsymbol{E}}^{\mathrm{inc}}$ and $\hat{\boldsymbol{H}}^{\mathrm{inc}}$ satisfy the homogeneous time-harmonic Maxwell equations in $\mathbb{R}^3$:

$$\nabla\times(\nabla\times\hat{\boldsymbol{E}}^{\mathrm{inc}})-\kappa^2\hat{\boldsymbol{E}}^{\mathrm{inc}}=0 \tag{2.17}$$

and

$$\nabla\times(\nabla\times\hat{\boldsymbol{H}}^{\mathrm{inc}})-\kappa^2\hat{\boldsymbol{H}}^{\mathrm{inc}}=0. \tag{2.18}$$

Let $\boldsymbol{\xi}=-\kappa\boldsymbol{d}$ with $|\boldsymbol{\xi}|=\kappa\in(0,\infty)$. We have from (2.16) that $\hat{\boldsymbol{E}}^{\mathrm{inc}}=\boldsymbol{p}e^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}$ and $\hat{\boldsymbol{H}}^{\mathrm{inc}}=\boldsymbol{q}e^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}$. Multiplying both sides of (2.12) by $\hat{\boldsymbol{E}}^{\mathrm{inc}}$ and using the integration by parts over B_R and (2.17), we have from $\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}=0$ and the transparent boundary condition (2.15) that

$$
\begin{aligned}
&\mathrm{i}\kappa\hat{g}(\kappa)\int_{B_R}\boldsymbol{p}e^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\cdot\boldsymbol{J}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
&=\int_{\Gamma_R}\nu\times(\nabla\times\hat{\boldsymbol{E}}(x,\kappa))\cdot\hat{\boldsymbol{E}}^{\mathrm{inc}}-\nu\times(\nabla\times\hat{\boldsymbol{E}}^{\mathrm{inc}})\cdot\hat{\boldsymbol{E}}(x,\kappa)\mathrm{d}S\\
&=-\int_{\Gamma_R}\Big(\mathrm{i}\kappa T(\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu})\cdot\hat{\boldsymbol{E}}^{\mathrm{inc}}+(\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu})\cdot(\nabla\times\hat{\boldsymbol{E}}^{\mathrm{inc}})\Big)\,\mathrm{d}S\\
&=0. \qquad (2.19)
\end{aligned}
$$

Hence we obtain

$$\hat{g}(\kappa)\boldsymbol{p}\cdot\hat{\boldsymbol{J}}(\xi)=0$$

for all $\kappa\in\mathbb{R}^+$.

Similarly, we may deduce from (2.18) and the integration by parts that

$$\hat{g}(\kappa)\boldsymbol{q}\cdot\hat{\boldsymbol{J}}(\xi)=0$$

for all $\kappa\in\mathbb{R}^+$. On the other hand, since $\boldsymbol{J}$ is compactly supported in B_R and $\nabla\cdot\boldsymbol{J}=0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}\boldsymbol{d}e^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\cdot\boldsymbol{J}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}&=-\frac{1}{\mathrm{i}\kappa}\int_{\mathbb{R}^3}\nabla e^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\cdot\boldsymbol{J}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
&=\frac{1}{\mathrm{i}\kappa}\int_{\mathbb{R}^3}e^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\nabla\cdot\boldsymbol{J}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=0,
\end{aligned}
$$

which implies that $\boldsymbol{d}\cdot\hat{\boldsymbol{J}}(\boldsymbol{\xi})=0$. Since $\boldsymbol{p},\boldsymbol{q},\boldsymbol{d}$ are orthonormal vectors, they form an orthonormal basis in $\mathbb{R}^3$. It follows from the previous identities that

$$\hat{g}(\kappa)\hat{\boldsymbol{J}}(\boldsymbol{\xi})=\hat{g}(\kappa)\Big(\boldsymbol{p}\cdot\hat{\boldsymbol{J}}(\boldsymbol{\xi})\boldsymbol{p}+\boldsymbol{q}\cdot\hat{\boldsymbol{J}}(\boldsymbol{\xi})\boldsymbol{q}+\boldsymbol{d}\cdot\hat{\boldsymbol{J}}(\boldsymbol{\xi})\boldsymbol{d}\Big)=0.$$

Since $g\neq 0$, we can always find an interval $(a,b)\in\mathbb{R}^+$ such that $\hat{g}(\kappa)\neq 0$ for $\kappa\in(a,b)$. Hence, we have

$$\hat{\boldsymbol{J}}(\boldsymbol{\xi})=0,\quad \boldsymbol{\xi}=-\kappa\boldsymbol{d}\quad\text{for all}\quad \boldsymbol{d}\in\mathbb{S}^2,\kappa\in\mathbb{R}^+.$$

Knowing that $\hat{\boldsymbol{J}}$ is an analytical function and $\boldsymbol{d}$ is an arbitrary unit vector, we obtain $\hat{\boldsymbol{J}}=0$, which completes the proof by taking the inverse Fourier transform. □

Remark 2.5. The proof of Theorem 2.4 relies essentially on the Fourier transform and Hugen's principle. Since the electric field is analytic in a neighborhood of Γ_R, Theorem 2.4 remains valid if partial tangential data are available on an open subset of Γ_R. It is also possible to prove Theorem 2.4 by first transforming the inverse source problem to an inverse initial value problem and then applying the Fritz–Johns global Holmgren theorem [6].

If the electric field is measured at one point $\boldsymbol{z}\in\Gamma_R$, it is impossible to determine the source function $\boldsymbol{J}$ in general. However, it is interesting to know what kind of information can be extracted from a single receiver. To this end, following [2], we prove that the maximum and minimum distance between $\boldsymbol{z}$ and the support of $\boldsymbol{J}$ can be uniquely determined by using the electric data at a single point $\boldsymbol{z}$.

Theorem 2.6. *Let $D = \operatorname{supp}(\boldsymbol{J})$ be connected and $\nabla \cdot \boldsymbol{J} = 0$. Let $\boldsymbol{z} \in \Gamma_R$ be fixed and assume that $g(t)$ is a given function which does not vanish identically. Define*

$$h_{\boldsymbol{z}} := \inf_{\boldsymbol{y} \in D} |\boldsymbol{z} - \boldsymbol{y}|, \quad H_{\boldsymbol{z}} := \sup_{\boldsymbol{y} \in D} |\boldsymbol{z} - \boldsymbol{y}|.$$

Then the interval $(h_{\boldsymbol{z}}, H_{\boldsymbol{z}})$ is uniquely determined by $\{\boldsymbol{E}(\boldsymbol{z}, t) : t \in (0, T)\}$ at one receiver $\boldsymbol{z} \in \mathbb{R}^3$, $|\boldsymbol{z}| = R$, provided that the zeros of the function

$$F_{\boldsymbol{z}} : \ r \mapsto \int_{|\boldsymbol{z}-\boldsymbol{y}|=r} \boldsymbol{J}(\boldsymbol{y}) \, \mathrm{d}s(\boldsymbol{y}), \quad r > 0$$

form a discrete set in the interval (h_z, H_z).

Proof. Let $\boldsymbol{E}(\boldsymbol{z}, t)$ and $\tilde{\boldsymbol{E}}(\boldsymbol{z}, t)$ be the two wave fields corresponding to two source terms $\boldsymbol{J}$ and $\tilde{\boldsymbol{J}}$. We also assume that $\tilde{\boldsymbol{J}}$ is divergence free and its support $\tilde{D}$ is connected. Denote by $\tilde{h}_{\boldsymbol{z}}$ and $\tilde{H}_{\boldsymbol{z}}$ the minimum and maximum distance between $\boldsymbol{z}$ and $\tilde{D}$, respectively. Assuming that $\boldsymbol{E}(\boldsymbol{z}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{z}, t)$ for $t \in (0, T)$, we need to prove that $\tilde{h}_{\boldsymbol{z}} = h_{\boldsymbol{z}}$ and $\tilde{H}_{\boldsymbol{z}} = H_{\boldsymbol{z}}$. Setting $\boldsymbol{W} = \boldsymbol{E} - \tilde{\boldsymbol{E}}$ and taking the Fourier transform of $\boldsymbol{W}$, we get

$$\begin{aligned} \nabla \times (\nabla \times \hat{\boldsymbol{W}}) - \kappa^2 \hat{\boldsymbol{W}} &= \mathrm{i}\kappa \hat{g} (\boldsymbol{J} - \tilde{\boldsymbol{J}}) \quad \text{in } \mathbb{R}^3, \\ \hat{\boldsymbol{W}}(z, \kappa) &= 0 \qquad \text{for all } \kappa \in \mathbb{R}^+. \end{aligned}$$

Note that $\hat{\boldsymbol{W}}(\boldsymbol{z}, \kappa), \kappa \in \mathbb{R}^+$ is uniquely determined by the data $\{\boldsymbol{W}(\boldsymbol{z}, t) : t \in (0, T)\}$ and $\hat{\boldsymbol{W}}(\boldsymbol{z}, \kappa) = \overline{\hat{\boldsymbol{W}}}(\boldsymbol{z}, -\kappa)$ for $\kappa < 0$. In view of Green's tensor for the time-harmonic Maxwell equation, it follows that

$$\begin{aligned} \hat{\boldsymbol{W}}(\boldsymbol{z}, \kappa) =& - \hat{g}(\kappa) \int_{\mathbb{R}^3} \hat{\mathbb{G}}(\boldsymbol{z} - \boldsymbol{y}, \kappa) (\boldsymbol{J}(\boldsymbol{y}) - \tilde{\boldsymbol{J}}(\boldsymbol{y})) \, \mathrm{d}\boldsymbol{y} \\ =& - i\kappa \, \hat{g}(\kappa) \int_{\mathbb{R}^3} \left(g(\boldsymbol{z} - \boldsymbol{y}, \kappa) \mathbb{I} + \frac{1}{\kappa^2} \nabla \nabla^\top g(\boldsymbol{z} - \boldsymbol{y}, \kappa) \right) (\boldsymbol{J}(\boldsymbol{y}) - \tilde{\boldsymbol{J}}(\boldsymbol{y})) \, \mathrm{d}\boldsymbol{y} \\ =& 0. \end{aligned}$$

Since $\nabla \cdot \boldsymbol{J} = 0$, we have from the integration by parts that

$$\begin{aligned} 0 &= -i\kappa \, \hat{g}(\kappa) \int_{\mathbb{R}^3} g(\boldsymbol{z} - \boldsymbol{y}, \kappa) \, (\boldsymbol{J}(\boldsymbol{y}) - \tilde{\boldsymbol{J}}(\boldsymbol{y})) \, \mathrm{d}\boldsymbol{y} \\ &= - \frac{i\kappa \, \hat{g}(\kappa)}{4\pi} \int_0^\infty \int_{|\boldsymbol{z}-\boldsymbol{y}|=r} \left(\frac{e^{i\kappa r}}{r} \right) (\boldsymbol{J}(\boldsymbol{y}) - \tilde{\boldsymbol{J}}(\boldsymbol{y})) \mathrm{d}s(\boldsymbol{y}) \mathrm{d}r \\ &= - \frac{i\kappa \, \hat{g}(\kappa)}{4\pi} \int_0^\infty e^{i\kappa r} \left(\frac{F_{\boldsymbol{z}}(r) - \tilde{F}_{\boldsymbol{z}}(r)}{r} \right) \mathrm{d}r. \end{aligned}$$

Here the function $\tilde{F}_{\boldsymbol{z}}$ is defined as the same as $F_{\boldsymbol{z}}$ with $\tilde{\boldsymbol{J}}$ in placed by $\boldsymbol{J}$. We may extend $F_{\boldsymbol{z}}$ and $\tilde{F}_{\boldsymbol{z}}$ from $\mathbb{R}^+$ to $\mathbb{R}$ by zero, since by definition both of them are compactly supported. On the other hand, since g does vanish identically, we can always find an interval $I \subset \mathbb{R}$ such that $|\hat{g}| > 0$ on I. This together with the previous identity implies that the Fourier transform of the one-dimensional function $(F_z - \tilde{F}_z)/r$ vanishes on I. By analyticity we obtain $F_z = \tilde{F}_z$. Recalling the assumption that the zeros of $F_{\boldsymbol{z}}$ in $(h_{\boldsymbol{z}}, H_{\boldsymbol{z}})$ are discrete, we get $\operatorname{supp}(F_{\boldsymbol{z}}) = [h_{\boldsymbol{z}}, H_{\boldsymbol{z}}]$. Analogously, $\operatorname{supp}(\tilde{F}_{\boldsymbol{z}}) = [\tilde{h}_{\boldsymbol{z}}, \tilde{H}_{\boldsymbol{z}}]$. Hence, we deduce from $F_z = \tilde{F}_z$ that their supports coincide, i.e., $H_{\boldsymbol{z}} = \tilde{H}_{\boldsymbol{z}}$ and $h_{\boldsymbol{z}} = \tilde{h}_{\boldsymbol{z}}$, which particularly gives the coincidence of the maximum and minimum distance. $\square$

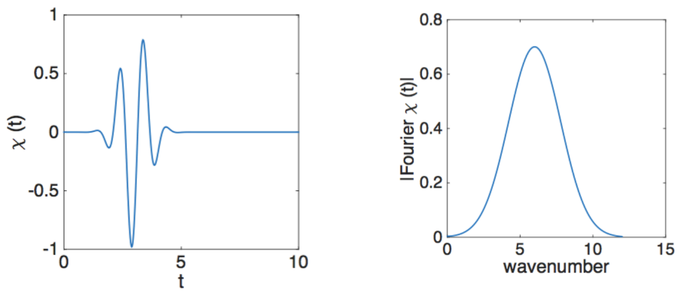

Figure 1. (left): A Gaussian-modulated sinusoidal pulse function χ with $\omega = 6$, $\sigma = 1.6$, $\tau = 3$; (right): Fourier spectrum of χ.

We end up the uniqueness results with the following remarks:

(i) It follows from the proof of Theorem 2.6 that the function $F_{\boldsymbol{z}}$ could be essentially identified. If the Lebesgue measure of the zeros of $F_{\boldsymbol{z}}$ in $(h_{\boldsymbol{z}}, H_{\boldsymbol{z}})$ is not zero, we may construct examples to show that the distance $h_{\boldsymbol{z}}$ and $H_{\boldsymbol{z}}$ can not be uniquely determined. We refer to [2] for discussions in the acoustic case. When $\mathrm{supp}(\boldsymbol{J})$ consists of several disconnected components, one can prove the unique determination of the union of the subintervals formed by the maximum and minimum distance to each connected component of $\boldsymbol{J}$.

(ii) If one component of the electric data $\boldsymbol{E} = (E_1, E_2, E_2)^\top$ is measured at $\boldsymbol{z}$, say e.g., E_j, then the maximum and minimum distance between $\boldsymbol{z}$ and the support of the j-th component of $\boldsymbol{J}$ can be recovered. This follows directly from the proof of Theorem 2.6.

2.3. **Stability estimate.** In this section, we consider the stability estimate of the source term $\boldsymbol{J}$. Since the temporal function g is given, we assume that there exists a subset $I \subset \mathbb{R}$ and constants $M, K > 0$ such that

$$\sup\{|\kappa|, \kappa \in I\} < K, \quad |\hat{g}(\kappa)| \geq M, \quad \forall \kappa \in I. \tag{2.20}$$

In many practical applications, the Gaussian type excitation signals always appear. For instance, $g(t)$ can be taken as a Gaussian-modulated sinusoidal pulse of the form

(2.21)

$$g(t) = \chi(t; \omega, \sigma, \tau) := \begin{cases} \sin(\omega t)\exp(-\sigma(t-\tau)^2), & 0 \leq t \leq 2m\pi/\omega, \\ 0, & t < 0 \quad \text{or} \quad t > 2m\pi/\omega, \end{cases}$$

for some $m \in \mathbb{N}$, where $\omega > 0$ is the center frequency, $\sigma > 0$ is the frequency bandwidth parameter, and $\tau > 0$ is a time-shift parameter related to the pulse peak time. In this case, the interval I can be chosen as $I = (\omega - \sigma, \omega + \sigma)$; see e.g., Figure 1. In general, g can be a linear combination of such pulse functions, i.e.,

$$g(t) = \sum_{j=1}^{N} \chi(t; \omega_j, \sigma_j, \tau_j).$$

To get a stability estimate, we restrict our discussions to an admissible set of spatial functions. For $\eta > 0$, we define

(2.22)
$$\mathcal{A}_\eta := \left\{ \boldsymbol{J} \in L^2(B_{\hat{R}})^3,\ \mathrm{supp}(\boldsymbol{J}) \subset B_{\hat{R}},\ \int_I |\boldsymbol{\xi}|^2 \int_{\mathbb{S}^2} |\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2 \mathrm{d}S\,\mathrm{d}|\boldsymbol{\xi}| \geq \eta\, ||\hat{\boldsymbol{J}}||^2_{L^2(\mathbb{R}^3)^3} \right\}.$$

Note that if $||\boldsymbol{J}||_{L^2(B_{\hat{R}})^3} \neq 0$, we can always claim that $\boldsymbol{J} \in \mathcal{A}_\eta$ for some $\eta > 0$ depending on I and the regularity of $\boldsymbol{J}$.

Lemma 2.7. *Let $\boldsymbol{E}(x,t)$ be the solution to the initial value problem (1.2)–(1.3) with $\boldsymbol{J} \in \mathcal{A}_\eta$ for some $\eta > 0$. Then*

$$\|\boldsymbol{J}\|^2_{L^2(B_R)^3} \leq \frac{C}{M^2\eta^2} \int_I (1+\kappa^2)\, \|\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}\|^2_{H^1(\Gamma_R)^3} \mathrm{d}\kappa,$$

where C is a constant depending on R and K.

Proof. It follows from (2.19) and the proof of Theorem 2.4 that we have

$$\mathrm{i}\kappa\hat{g}(\kappa)\hat{\boldsymbol{J}}(\boldsymbol{\xi}) = -\int_{\Gamma_R} \Big((\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu} \cdot \hat{\boldsymbol{E}}^{\mathrm{inc}} + (\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}) \cdot (\nabla \times \hat{\boldsymbol{E}}^{\mathrm{inc}}) \Big) \mathrm{d}S,$$

which implies that for each $\kappa \in I$

$$|\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2 \leq \frac{4\pi R^2}{\kappa^2|\hat{g}(\kappa)|^2} \int_{\Gamma_R} \left| (\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu} \cdot \hat{\boldsymbol{E}}^{\mathrm{inc}} + (\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}) \cdot (\nabla \times \hat{\boldsymbol{E}}^{\mathrm{inc}}) \right|^2 \mathrm{d}S.$$

Integrating over $\Sigma := \mathbb{S}^2 \times I$ by spherical coordinates and using the Cauchy–Schwarz inequality give

$$\int_I \kappa^2 \int_{\mathbb{S}^2} |\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2 \mathrm{d}S\,\mathrm{d}\kappa \leq \int_I \frac{8\pi R^2}{\hat{g}^2(\kappa)} \int_{\Gamma_R} |(\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu}|^2 + \kappa^2 |\boldsymbol{\nu} \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)|^2 \mathrm{d}S\mathrm{d}\kappa.$$

In view of (2.20) and (2.22), we obtain

$$\int_{\mathbb{R}^3} |\hat{\boldsymbol{J}}(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi} \leq \frac{8\pi R^2}{M^2\eta^2} \int_I \int_{\Gamma_R} |(\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu}|^2 + \kappa^2 |\boldsymbol{\nu} \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)|^2 \mathrm{d}S\mathrm{d}\kappa.$$

Applying the Plancherel theorem yields

$$\|\boldsymbol{J}\|^2_{L^2(B_R)^3} \leq \frac{8\pi R^2}{M^2\eta^2} \int_I \int_{\Gamma_R} |(\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu}|^2 + \kappa^2 |\boldsymbol{\nu} \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)|^2 \mathrm{d}S\mathrm{d}\kappa. \tag{2.23}$$

Recalling that, for a tangential vector $\boldsymbol{f}$ defined on Γ_R, we have (see e.g., [29])

$$\|\boldsymbol{f}\|_{L^2(\Gamma_R)^3} \leq ||\boldsymbol{f}||_{H^{-1/2}(\mathrm{div},\Gamma_R)}, \quad ||\boldsymbol{f}||_{H^{-1/2}(\mathrm{div},\Gamma_R)} \leq ||\boldsymbol{f}||_{H^1(\Gamma_R)^3}.$$

On the other hand, it follows from the boundedness of the capacity operator (2.15) (see the appendix in Section 3.2 for details) that we get

$$\|(\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu}\|_{H^{-1/2}(\mathrm{div},\Gamma_R)} \leq C\, \|\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}\|_{H^{-1/2}(\mathrm{div},\Gamma_R)},$$

where the positive constant C depends on κ and R. Moreover, the constant $C = C(K,R)$ can be chosen to be uniform for all $\kappa \in I$. The previous two relations imply that

$$\|(\nabla \times \hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)) \times \boldsymbol{\nu}\|_{L^2(\Gamma_R)^3} \leq C\, \|\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}\|^2_{H^1(\Gamma_R)^3}.$$

Combing the above estimate with (2.23), we obtain

$$\|\boldsymbol{J}\|^2_{L^2(B_R)^3} \leq \frac{8\pi R^2 \max\{1, C\}}{M^2\eta^2} \int_I (1+\kappa^2)\, \|\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa) \times \boldsymbol{\nu}\|^2_{H^1(\Gamma_R)^3} \mathrm{d}\kappa,$$

which completes the proof. □

Below we state stability estimate of the source $\boldsymbol{J}$ in terms of the tangential components of $\boldsymbol{E}(\boldsymbol{x},t)$ on $\Gamma_R\times(0,T)$.

Theorem 2.8. *Assume $\boldsymbol{J}\in H^p(\mathbb{R}^3)^3\cap\mathcal{A}_\eta$ for some $p>3/2$ and $\eta>0$. Then there exists a constant $C=C(R,K)>0$ such that*

$$||\boldsymbol{J}||_{L^2(B_R)^3}\le\frac{C}{M^2\eta^2}\,||\boldsymbol{E}\times\boldsymbol{\nu}||_{H^1(0,T;H^1(\Gamma_R))^3}.$$

Proof. It follows from Lemma 2.3 and the Sobolev trace theorem that we have $\boldsymbol{E}\in H^1(0,T;H^1(\Gamma_R))$ if $\boldsymbol{J}\in H^p(\mathbb{R}^3)^3,\ p>3/2$, which implies that $\boldsymbol{E}\times\boldsymbol{\nu}\in H^1(\Gamma_R)^3$. Moreover, we obtain from the Plancherel theorem and Lemma 2.1 that

$$\begin{aligned}
&||\boldsymbol{E}\times\boldsymbol{\nu}||^2_{H^1(0,T;H^1(\Gamma_R))}\\
=&\int_0^T||\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}+||\boldsymbol{E}'(\boldsymbol{x},t)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}\,\mathrm{d}t\\
=&\int_{\mathbb{R}}||\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}+||\boldsymbol{E}'(\boldsymbol{x},t)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}\,\mathrm{d}t\\
=&\int_{\mathbb{R}}||\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}+\kappa^2\,||\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}\,\mathrm{d}\kappa\\
=&\int_{\mathbb{R}}(1+\kappa^2)\,||\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}||^2_{H^1(\Gamma_R)^3}\,\mathrm{d}\kappa,
\end{aligned}$$

which completes the proof after combining the above identities with Lemma 2.7. □

3. IP2: Determination of temporal functions.

In this section, we consider IP2 and determine $\boldsymbol{g}$ from an observation of the solution for the initial value problem:

$$\begin{cases}\partial_t^2\boldsymbol{E}(\boldsymbol{x},t)-\Delta\boldsymbol{E}(\boldsymbol{x},t)=J(\boldsymbol{x})\boldsymbol{g}'(t), & \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ \boldsymbol{E}(\boldsymbol{x},0)=\partial_t\boldsymbol{E}(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3,\end{cases}\tag{3.1}$$

at a fixed point $\boldsymbol{x}_0\in\mathrm{supp}(J)$ (i.e., an interior observation) or at the boundary Γ_R (i.e., boundary observation).

3.1. Uniqueness and stability with interior data.

Following similar arguments as those in Lemma 2.3, we have the regularity of the solution for the initial value problem (3.1).

Lemma 3.1. *Let $\boldsymbol{g}'\in L^2(0,T)^3$ and $J\in H^p(\mathbb{R}^3)$ ($p>0$) be supported in $B_{\hat{R}}$. Then the problem (3.1) admits a unique solution*

$$\boldsymbol{E}\in C(0,T;H^{p+1}(\mathbb{R}^3))^3\cap H^\tau(0,T;H^{p-\tau+1}(\mathbb{R}^3))^3$$

for $\tau=1,2$, which satisfies

$$\|\boldsymbol{E}\|_{C(0,T;H^{p+1}(\mathbb{R}^3))^3}+\|\boldsymbol{E}\|_{H^\tau(0,T;H^{p-\tau+1}(\mathbb{R}^3))^3}\le C\|\boldsymbol{g}'\|_{L^2(0,T)^3}\|J\|_{H^p(\mathbb{R}^3)},\tag{3.2}$$

where the constant $C>0$ depends on R.

In the remaining part of this paper, we assume that $J\in H^p(\mathbb{R}^3)$ with $p>5/2$. According to Lemma 3.1 and the Sobolev embedding theorem, we have $\boldsymbol{E}\in C(0,T;C^2(\mathbb{R}^3))^3\cap H^2(0,T;C(\mathbb{R}^3))^3$ and the trace $t\mapsto\boldsymbol{E}(\boldsymbol{x}_0,t)$ for some point $\boldsymbol{x}_0\in\mathbb{R}^3$, is well-defined as an element of $H^2([0,T])^3$. Below we consider the inverse problem of determining the evolution function $\boldsymbol{g}(t)$ from the interior observation of the wave field $\boldsymbol{E}(\boldsymbol{x}_0,t)$ for $t\in(0,T)$ and some point $\boldsymbol{x}_0\in\mathrm{supp}(J)$.

Theorem 3.2. *Let $\boldsymbol{x}_0 \in B_R$ and assume that the set*

$$\mathcal{A}_{\boldsymbol{x}_0,p,\delta,M} := \{h \in H^p(\mathbb{R}^3) : \|h\|_{H^p(\mathbb{R}^3)} \le A, |h(\boldsymbol{x}_0)| \ge \delta, supp(h) \subset B_R\}, \quad M, \delta > 0,$$

is not empty. Then, for $J \in \mathcal{A}_{\boldsymbol{x}_0,p,\delta,M}$, the following estimate holds

$$\|\boldsymbol{g}'\|_{L^2(0,T)^3} \le C\|\partial_t^2 \boldsymbol{E}(\boldsymbol{x}_0,\cdot)\|_{L^2(0,T)^3},$$

where C depends on $p, \boldsymbol{x}_0, A, R, \delta$ and T. In particular, this estimate implies that the data $\{\boldsymbol{E}(\boldsymbol{x}_0,t) : t \in (0,T)\}$ determines uniquely the temporal function $\boldsymbol{g}$.

Proof. Clearly, the solution $\boldsymbol{E}$ of (3.1) is given by

$$\boldsymbol{E}(\boldsymbol{x},t) = (2\pi)^{-3}\int_{\mathbb{R}^3}\Big(\int_0^t \hat{J}(\boldsymbol{\xi})|\boldsymbol{\xi}|^{-1}\sin((t-s)|\boldsymbol{\xi}|)\boldsymbol{g}'(s)\mathrm{d}s\Big)e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{\xi}$$

where $(\boldsymbol{x},t) \in \mathbb{R}^3 \times [0,T]$. Applying Fubini's theorem yields

$$\boldsymbol{E}(\boldsymbol{x},t) = (2\pi)^{-3}\int_0^t \Big(\int_{\mathbb{R}^3} \hat{J}(\boldsymbol{\xi})|\boldsymbol{\xi}|^{-1}\sin((t-s)|\boldsymbol{\xi}|)e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{\xi}\Big)\boldsymbol{g}'(s)\mathrm{d}s$$

where $(\boldsymbol{x},t) \in \mathbb{R}^3 \times [0,T]$. In particular, in view of Lemma 3.1, we have

$$\boldsymbol{E} \in C(0,T;H^{p+1}(\mathbb{R}^3))^3 \cap H^2(0,T;H^{p-1}(\mathbb{R}^3))^3$$

which satisfies (3.2). Furthermore, direct calculations show that

$$-\Delta\boldsymbol{E}(\boldsymbol{x},t) = (2\pi)^{-3}\int_0^t \Big(\int_{\mathbb{R}^3} \hat{J}(\boldsymbol{\xi})|\boldsymbol{\xi}|\sin((t-s)|\boldsymbol{\xi}|)e^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{\xi}\Big)\boldsymbol{g}'(s)\mathrm{d}s$$

where $(\boldsymbol{x},t) \in \mathbb{R}^3 \times [0,T]$. Moreover, we have

$$\begin{aligned}
|-\Delta\boldsymbol{E}(\boldsymbol{x},t)| &\le (2\pi)^{-3}\int_0^t |\boldsymbol{g}'(s)|\mathrm{d}s \int_{\mathbb{R}^3} |\hat{J}(\boldsymbol{\xi})||\boldsymbol{\xi}|\mathrm{d}\boldsymbol{\xi} \\
&\le (2\pi)^{-3}\int_0^t |\boldsymbol{g}'(s)|\mathrm{d}s\ \|\hat{J}(\boldsymbol{\xi})(1+|\boldsymbol{\xi}|^2)^{p/2}\|_{L^2(\mathbb{R}^3)} \\
&\quad \|(1+|\boldsymbol{\xi}|^2)^{(1-p)/2}\|_{L^2(\mathbb{R}^3)} \\
&\le A_0\|J\|_{H^p(\mathbb{R}^3)}\int_0^t |\boldsymbol{g}'(s)|\mathrm{d}s, \qquad (3.3)
\end{aligned}$$

where $A_0 = (2\pi)^{-3}\|(1+|\boldsymbol{\xi}|^2)^{(1-p)/2}\|_{L^2(\mathbb{R}^3)} < \infty$. Since $|J(\boldsymbol{x}_0)| \ge \delta$, we derive from (3.3) and the governing equation of $\boldsymbol{E}$ in (3.1) that

$$\begin{aligned}
|\boldsymbol{g}'(t)| &= \frac{1}{J(\boldsymbol{x}_0)}|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,t) - \Delta\boldsymbol{E}| \\
&\le A_1|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,t)| + A_2\int_0^t |\boldsymbol{g}'(s)|\mathrm{d}s
\end{aligned}$$

for all $t \in (0,T)$ where $A_1 = 1/\delta, A_2 = A_0A/\delta$. Applying the Grownwall inequality, we get

$$\begin{aligned}
|\boldsymbol{g}'(t)| &\le A_1|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,t)| + A_1A_2\int_0^t |\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,s)|e^{A_2(t-s)}\mathrm{d}s \\
&\le A_1|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,t)| + A_1A_2T^{\frac{1}{2}}e^{A_2T}\|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,\cdot)\|_{L^2(0,T)^3}.
\end{aligned}$$

Therefore, taking the norm $L^2(0,T)$ on both sides of the above inequality, we obtain

$$\|\boldsymbol{g}'\|_{L^2(0,T)^3} \le (A_1 + A_1A_2Te^{A_2T})\|\partial_t^2\boldsymbol{E}(\boldsymbol{x}_0,\cdot)\|_{L^2(0,T)^3},$$

which completes the proof. □

3.2. **Uniqueness with boundary measurement data.** To state the uniqueness result with boundary measurement data, we need the concept of non-radiating source.

Definition 3.3. The compactly supported function J is called a non-radiating source at the frequency $\kappa \in \mathbb{R}^+$ of the Maxwell equations if there exists a vector $\boldsymbol{P} \in \mathbb{C}^3$ such that the unique radiating solution to the inhomogeneous Maxwell system

$$\nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x})) + \kappa^2 \boldsymbol{E}(\boldsymbol{x}) = J(\boldsymbol{x})\boldsymbol{P} \tag{3.4}$$

vanishes identically in $\mathbb{R}^3 \backslash \overline{\mathrm{supp}(J)}$. The source J is not a non-radiating source at the frequency $\kappa \in \mathbb{R}^+$ if the unique solution to (3.4) does not vanish for all $\boldsymbol{P} \in \mathbb{C}^3$.

Theorem 3.4. *Suppose that $J \in L^2(B_R)$ is a compacted supported function over B_R and that J is not a non-radiating source for all $\kappa \in \mathbb{R}^+$. Then the temporal function $\boldsymbol{g} \in C_0([0,T])^3$ can be uniquely determined by the boundary measurement data $\{\boldsymbol{E} \times \boldsymbol{\nu} : \boldsymbol{x} \in \Gamma_R, t \in (0,T)\}$.*

Proof. Denote by $\boldsymbol{e}_j$ $(j = 1, 2, 3)$ the unit vectors in Cartesian coordinate system. Let $\boldsymbol{w}_j = \boldsymbol{w}_j(\boldsymbol{x}, \kappa)$ be the unique radiating solution to the inhomogeneous equations

$$\nabla \times (\nabla \times \boldsymbol{w}_j(\boldsymbol{x}, \kappa)) - \kappa^2 \boldsymbol{w}_j(\boldsymbol{x}, \kappa) = J(\boldsymbol{x})\boldsymbol{e}_j, \quad j = 1, 2, 3,$$

which does not vanish identically in $|\boldsymbol{x}| \geq R$ by our assumption. Let the matrix $\mathbf{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3) \in \mathbb{C}^{3\times 3}$ be the unique radiating solution to the matrix equation

$$\nabla \times \Big(\nabla \times \mathbf{W}(\boldsymbol{x}, \kappa)\Big) - \kappa^2 \mathbf{W}(\boldsymbol{x}, \kappa) = J(\boldsymbol{x})\mathbb{I}, \quad \boldsymbol{x} \in \mathbb{R}^3 \times (0, \infty),$$

which gives that

$$\mathbf{W}(\boldsymbol{x}, \kappa) = -\frac{1}{\mathrm{i}\kappa} \int_{\mathbb{R}^3} \hat{\mathbb{G}}(\boldsymbol{x} - \boldsymbol{y}, \kappa) J(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Here $\hat{\mathbb{G}}$ is the Green tensor to the time-harmonic Maxwell equations and is given in (2.3). In view of (1.2), the Fourier transform $\hat{\boldsymbol{E}}(\boldsymbol{x}; \kappa)$ of $\boldsymbol{E}(\boldsymbol{x}, \kappa)$ can be written as

$$\hat{\boldsymbol{E}}(\boldsymbol{x}, \kappa) = \mathrm{i}\kappa \mathbf{W}(\boldsymbol{x}, \kappa) \hat{\boldsymbol{g}}(\kappa), \quad \forall \kappa \in \mathbb{R}^+,\ |\boldsymbol{x}| = R. \tag{3.5}$$

We claim that for each $\kappa_0 \in \mathbb{R}^+$, there always exists $\boldsymbol{x}_0 \in \Gamma_R$ such that

$$\det(\mathbf{W}(\boldsymbol{x}_0, \kappa_0)) \neq 0.$$

Suppose on the contrary that $\det(\mathbf{W}(\boldsymbol{x}, \kappa_0)) = 0$ for all $\boldsymbol{x} \in \Gamma_R$. This implies that there exist $c_j \in \mathbb{C},\ j = 1, 2, 3$ which are not all equal to zero such that

$$\boldsymbol{V}(\boldsymbol{x}) := c_1 \boldsymbol{w}_1(\boldsymbol{x}, \kappa_0) + c_2 \boldsymbol{w}_2(\boldsymbol{x}, \kappa_0) + c_3 \boldsymbol{w}_3(\boldsymbol{x}, \kappa_0) = 0, \quad \boldsymbol{x} \in \Gamma_R.$$

By uniqueness of the exterior Dirichlet boundary value problem, we conclude that $\boldsymbol{V}(\boldsymbol{x}) = 0$ in $|\boldsymbol{x}| > R$, and by unique continuation it holds that $\boldsymbol{V}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ lying outside of the support of J. On the other hand, it is easy to observe that $\boldsymbol{V}$ satisfies the inhomogeneous equation

$$\nabla \times \Big(\nabla \times \boldsymbol{V}(\boldsymbol{x})\Big) - \kappa_0^2 \boldsymbol{V}(\boldsymbol{x}) = J(\boldsymbol{x})\boldsymbol{P},$$

where $\boldsymbol{P} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3$, which contradicts the fact that J is not a non-radiating source. This proves the existence of $\boldsymbol{x}_0 \in \Gamma_R$ such that $\det(\mathbf{W}(\boldsymbol{x}_0, \kappa_0)) \neq 0$.

Therefore, we get from (3.5) that

$$\mathrm{i}\kappa \hat{\boldsymbol{g}}(\kappa_0) = \Big[\mathbf{W}(\boldsymbol{x}_0, \kappa_0)\Big]^{-1} \hat{\boldsymbol{E}}(\boldsymbol{x}_0, \kappa_0) \in \mathbb{C}^{3\times 1} \quad \text{for some} \quad \boldsymbol{x}_0 \in \Gamma_R. \tag{3.6}$$

Note that κ_0 is arbitrary and the point $\boldsymbol{x}_0$ depends on κ_0. Hence, if $\boldsymbol{E}\times\boldsymbol{\nu}=0$ for all $\boldsymbol{x}\in\Gamma_R$ and $t\in(0,T)$, then $\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}=0$ for all $\boldsymbol{x}\in\Gamma_R$ and $\kappa\in\mathbb{R}^+$. From the uniqueness of the Dirichlet boundary value problem for the Maxwell equations, we get $\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)=0$ for all $\boldsymbol{x}\in\Gamma_R$ and $\kappa\in\mathbb{R}^+$. This together with (3.6) implies that $\hat{\boldsymbol{g}}=0$ for all $\kappa\in\mathbb{R}$ and thus $\boldsymbol{g}\equiv 0$. □

Remark 3.5. Theorem 3.4 remains true if the measurement surface Γ_R is replaced by an arbitrary open subset $\Lambda_R\subset\Gamma_R$ with positive Lebesgue measure, because in the frequency domain the knowledge of $\nu\times\hat{\boldsymbol{E}}$ on Λ_R uniquely determines $\nu\times\hat{\boldsymbol{E}}|_{\Gamma_R}$ by analyticity.

Appendix. Transparent boundary conditions. In this section, we present the transparent boundary condition for time-harmonic Maxwell equations (2.12) at a fixed frequency, following the arguments of [12]. From the numerical point of view, this boundary condition can be used to transform the boundary value problem of the Maxwell equation from an unbounded domain with radiation condition to a bounded computational domain . For the sake of convenience, we drop the hat in $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{H}}$. Introduce the tangential trace spaces:

$$\begin{aligned}\boldsymbol{L}_t^2(\Gamma_R)&=\{\boldsymbol{u}\in L^2(\Gamma_R)^3:\boldsymbol{u}\cdot\hat{\boldsymbol{x}}=0\}\\ \boldsymbol{H}_t^s(\Gamma_R)&=\{\boldsymbol{u}\in\boldsymbol{H}^s(\Gamma_R),\boldsymbol{u}\cdot\hat{\boldsymbol{x}}=0\},\\ \boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R)&=\{\boldsymbol{u}\in\boldsymbol{H}_t^{-1/2}(\Gamma_R),\mathrm{curl}_{\Gamma_R}\boldsymbol{u}\in H^{-1/2}(\Gamma_R)\},\\ \boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R)&=\{\boldsymbol{u}\in\boldsymbol{H}_t^{-1/2}(\Gamma_R),\mathrm{div}_{\Gamma_R}\boldsymbol{u}\in H^{-1/2}(\Gamma_R)\},\end{aligned}$$

where $\hat{\boldsymbol{x}}=\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$. It is known that for $\boldsymbol{E}\in H(\mathrm{curl},B_R)$, the tangential component $(\hat{\boldsymbol{x}}\times\boldsymbol{E})\times\hat{\boldsymbol{x}}|_{\Gamma_R}$ belongs to $\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R)$ which is the dual space of $\boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R)$, with respect to the scalar product in $\boldsymbol{L}_t^2(\Gamma_R)$.

Let $Y_n^m(\hat{\boldsymbol{x}}), m=-n,...,n, n=1,2,...$, be the spherical harmonics which satisfies

$$\triangle_{\partial B_1}Y_n^m(\hat{\boldsymbol{x}})+n(n+1)Y_n^m(\hat{\boldsymbol{x}})=0\quad\text{on }\partial B_1,$$

where

$$\triangle_{\partial B_1}=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Laplace-Beltrami operator for the surface of the unit sphere ∂B_1. Here $\hat{\boldsymbol{x}}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$. The set of all spherical harmonics $\{Y_n^m(\hat{\boldsymbol{x}}):m=-n,...,n,n=1,2,\cdots\}$ forms a complete orthonormal basis of $L^2(\partial B_1)$. Denote the vector spherical harmonics

$$\boldsymbol{U}_n^m=\frac{1}{\sqrt{n(n+1)}}\nabla_{\partial B_1}Y_n^m,\quad \boldsymbol{V}_n^m=\hat{\boldsymbol{x}}\times\boldsymbol{U}_n^m,$$

where

$$\nabla_{\partial B_1}Y_n^m=\frac{\partial Y_n^m}{\partial\theta}\boldsymbol{e}_\theta+\frac{1}{\sin\theta}\frac{\partial Y_n^m}{\partial\varphi}\boldsymbol{e}_\varphi,$$

and $\{\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_\varphi\}$ are the unit vectors of the spherical coordinates. The set of all vector spherical harmonics $\{\boldsymbol{U}_n^m,\boldsymbol{V}_n^m:m=-n,...,n,n=1,2,\cdots\}$ forms a complete orthonormal basis of $\boldsymbol{L}_t^2(\partial B_1):=\{\boldsymbol{u}\in L^2(\partial B_1)^3:\boldsymbol{u}\cdot\hat{\boldsymbol{x}}=0\text{ on }\partial B_1\}$.

Let $h_n^{(1)}(z)$ be the spherical Hankel function of the first kind of order n. We introduce the vector wave equations

$$\boldsymbol{M}_n^m(r,\hat{\boldsymbol{x}})=\nabla\times\{\boldsymbol{x}h_n^{(1)}(\kappa r)Y_n^m(\hat{\boldsymbol{x}})\},\quad \boldsymbol{N}_n^m=\frac{1}{\mathrm{i}\kappa}\nabla\times\boldsymbol{M}_n^m(r,\hat{\boldsymbol{x}}).$$

For the tangential trace $\hat{\boldsymbol{x}} \times \boldsymbol{E}|_{\Gamma_R} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{mn} \boldsymbol{U}_n^m(\hat{\boldsymbol{x}}) + b_{mn} \boldsymbol{U}_n^m(\hat{\boldsymbol{x}})$, we know that $\boldsymbol{E}$ in the domain $\mathbb{R}^3 \backslash \bar{B}_R$ can be written as (see [12])

$$\boldsymbol{E} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{a_{nm} \boldsymbol{M}_n^m(r, \hat{\boldsymbol{x}})}{h^{(1)}(\kappa R)\sqrt{n(n+1)}} + \frac{\mathrm{i}\kappa R b_{nm} \boldsymbol{M}_n^m(r, \hat{\boldsymbol{x}})}{z_n^{(1)}(\kappa R)\sqrt{n(n+1)}},$$

where $z_n^{(1)}(z) = h_n^{(1)}(z) + z h_n^{(1)'}(z)$. The capacity operator $T : \boldsymbol{H}^{-1/2}(\mathrm{div}, \Gamma_R) \to \boldsymbol{H}^{-1/2}(\mathrm{div}, \Gamma_R)$ is the transparent boundary condition defined by

$$T(\hat{\boldsymbol{x}} \times \boldsymbol{E}) = \frac{1}{\mathrm{i}\kappa} \hat{\boldsymbol{x}} \times (\nabla \times \boldsymbol{E}),$$

where

$$T(\hat{\boldsymbol{x}} \times \boldsymbol{E}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{-\mathrm{i}\kappa R b_{nm} h_n^{(1)}(\kappa R)}{z_n^{(1)}(\kappa R)} \boldsymbol{U}_n^m(\hat{\boldsymbol{x}}) + \frac{a_{nm} z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R h_n^{(1)}(\kappa R)} \boldsymbol{V}_n^m(\hat{\boldsymbol{x}}).$$

It is known (see [12]) that T is continuous.

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