Math 181 optional problem

In class we deduced the Heine-Borel theorem from the LUB axiom. In this exercise you are asked to do the converse: deduce the LUB axiom from the Heine-Borel theorem—thereby showing that the two statements are logically equivalent.

Actually what will be deduced from the Heine-Borel theorem is the Dedekind Cut axiom; and since, as seen in class, the LUB axiom can be deduced from the Cut axiom, combining the two deductions shows that the LUB axiom is a logical consequence of the Heine-Borel theorem.

So assume that the Heine-Borel theorem is true. To deduce the Cut axiom means to show that *if a left half-line* L *and a right half-line* R *have no number in common, then there is a number lying neither in* L *nor in* R.

(i) Explain why for each x in L, there is an open interval I_x containing x and such that $I_x \subset L$.

(i)$'$ Explain why for each y in R, there is an open interval I_y containing y and such that $I_y \subset R$.

(ii) Explain why there is a number a in L and a number b in R with $a < b$.

Now assume that

($*$) every number lies either in L or in R.

It will be shown that this assumption leads to a contradiction, so it must be false, which is exactly what needs to be proved.

If ($*$) holds then, since $z \in I_z$, if we let z run through all the numbers in the closed interval $[a, b]$ then the collection of all the intervals I_z covers $[a, b]$. By the Heine-Borel assumption, a finite subcollection $\mathcal{S}$ of those intervals will still cover $[a, b]$.

(iii) Explain why at least one interval in $\mathcal{S}$ is a subset of L (for example, any one which contains a.)

(iv) Explain why, if you look at the right-hand end points of all those intervals in $\mathcal{S}$ which are subsets of L, the largest of those (finitely many!) end points, call it c, doesn't lie in L.

(v) Explain why any interval in $\mathcal{S}$ containing c must be a subset of R, and must overlap the interval $I \subset L$ whose right-hand endpoint is c, contradicting disjointness of L and R.

COMMENT. Yes, this problem may look hard to a beginner, whom it forces to get around in unfamiliar territory, with an unfamiliar language. But it shouldn't be beyond any potential mathematician, computer scientist, electrical engineer, lawyer or philosopher.