Math 181 Midterm Exam 1

Oct. 2, 2007

[**Bold numbers**] indicate points (**15** total).

I. [**2**] Circle the value of

$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\frac{n\pi}{n}\right).$$

<u>Hint</u>. Think about Riemann sums.

A. 1 B. 2 C. $\frac{\pi}{2}$ D. $\frac{2}{\pi}$ E. ∞

Solution. These Riemann sums converge to

$$\int_0^1 \sin(\pi x)\,dx = -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 = \frac{2}{\pi}\,.$$

II. (a) [**2**] For $u > 1$, set $g(u) = \frac{1}{2}\ln\big((u+1)/(u-1)\big)$. Show that the derivative $g'(u) = 1/(1-u^2)$; and deduce that $\sqrt{1+e^{2x}} - g\big(\sqrt{1+e^{2x}}\,\big)$ is an antiderivative of $\sqrt{1+e^{2x}}$.

Solution.

$$g'(u) = \frac{1}{2}\frac{d}{du}\big(\ln(u+1) - \ln(u-1)\big) = \frac{1}{2}\left(\frac{1}{u+1} - \frac{1}{u-1}\right) = \frac{1}{1-u^2}\,.$$

Now use the chain rule to get

$$\frac{d}{dx}\big(\sqrt{1+e^{2x}} - g(\sqrt{1+e^{2x}}\,)\big) = \frac{2e^{2x}}{2\sqrt{1+e^{2x}}} - \frac{2e^{2x}}{2\sqrt{1+e^{2x}}}\,\frac{1}{1-(1+e^{2x})}$$

$$= \frac{1}{\sqrt{1+e^{2x}}}\big(e^{2x}+1\big) = \sqrt{1+e^{2x}}.$$

Now let C be the curve which is the graph of $y = e^x$ $(0 \le x \le \ln\sqrt{2}\,)$.

(b) [**2**] Show that the length of C is $\sqrt{3} - \sqrt{2} - g(\sqrt{3}\,) + g(\sqrt{2}\,)$.

Solution. The length is given by

$$\int_0^{\ln\sqrt{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\ln\sqrt{2}} \sqrt{1 + e^{2x}}\, dx$$

$$= \left(\sqrt{1 + e^{2x}} \;-\; g(\sqrt{1 + e^{2x}}\,)\right)\Big|_0^{\ln\sqrt{2}}$$

$$= \sqrt{3} - \sqrt{2} - g(\sqrt{3}\,) + g(\sqrt{2}\,)$$

(c) [**1**] Set up an integral for the area of the surface obtained by rotating C around the y-axis. Do not evaluate the integral.

Solution.

$$2\pi \int_0^{\ln\sqrt{2}} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^{\ln\sqrt{2}} x\, \sqrt{1 + e^{2x}}\, dx.$$

(d) [**1**] Set up an integral for the area of the surface obtained by rotating C around the x-axis. Do not evaluate the integral.

$$2\pi \int_1^{\sqrt{2}} y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_1^{\sqrt{2}} y \sqrt{1 + \frac{1}{y}}\, dy = 2\pi \int_1^{\sqrt{2}} \sqrt{y^2 + y}\, dy.$$

III. [**3**] (Work can be fun.) A container has the shape obtained by rotating the graph of $y = 14x - 17.5$ ($1.25 \le x \le 1.75$) around the y-axis, and sealing the bottom with a circular disc. It is filled with strawberry soda weighing 0.58 oz/in^3, and a straw is put in, sticking out one inch above the top. How much work, in units of ounce-inches, does it take to suck up all the soda through the straw? (Neglect friction, neglect the width of the straw, and assume that the straw always stays perpendicular, with negligible distance between it and the bottom of the container.) Just set up an appropriate integral, but don't evaluate it.

Solution. Note that the container is 7 inches tall, and the top of the straw is 8 inches above the bottom of the container. The integral is:

$$.58\pi \int_0^7 \big((y + 17.5)/14\big)^2 (8 - y)\,dy.$$

IV. [**2**] Find all antiderivatives of $e^{3x}/(e^x + 5)$.

Solution. Substitute $y = e^x + 5$, so $dy = e^x dx$. Then

$$\begin{aligned}\int \frac{e^{3x}}{(e^x+5)}\,dx = \int \frac{(y-5)^2}{y}\,dy &= \int \frac{y^2 - 10y + 25}{y}\,dy \\ &= y^2/2 - 10y + 25\ln y + C \\ &= \frac{(e^x+5)^2}{2} - 10(e^x+5) + 25\ln(e^x+5) + C \\ &= \frac{e^{2x}}{2} - 5e^x + 25\ln(e^x+5) + C.\end{aligned}$$

V. [**2**] In a room at temperature 70 degrees, a cup of coffee cools down from 170 degrees to 151 degrees in 2 minutes. What will the temperature of the coffee be, in degrees, after 1 more minute has passed? You shouldn't need a calculator.

A. 140.9 B. 141.5 C. 142.9 D. 145 E. 146.3

Solution. Newton's law of cooling gives that the temperature at time t is $70 + (170-70)e^{-kt}$ for some k. So $151 = 70 + 100e^{-2k}$, that is, $e^{-2k} = 81/100$, that is, $e^{-k} = 9/10$. Therefore $e^{-3k} = (9/10)^3 = 729/1000$, and the answer is $70 + 100(729/1000) = 142.9$.