# Math 181 Midterm Exam 1a

**Oct. 12, 2007**

[**Bold numbers**] indicate points (**15** total).

*For each answer, show how you got it. This may get you partial credit, even when the final answer is incorrect. No points for mere guessing.*

I. [**3**] You start with 100 grams of a radioactive substance, and 2 days later, 80 grams are left. How many grams will be left after 4 more days?

A. 40 B. 45.8 C. 51.2 D. 56.2 E. 64

*Solution.* The amount after $t$ days total is $100e^{-tk}$ for some constant $k$; and

$$100e^{-6k} = \left(100e^{-2k}\right)^3 = 100(80/100)^3 = 51.2.$$

II. [**4**] Use the substitution $x = 2\tan^{-1}(t)$ to find

$$\int \frac{dx}{1-\sin(x)}.$$

You can use, without proof, the fact—shown in class—that

$$t = \tan(x/2) \implies \sin(x) = 2t/(1+t^2).$$

A. $\frac{2}{1+\cos(x)} + C$ B. $\frac{2}{1-\tan(x/2)} + C$ C. $-\frac{\ln(1-\sin(x))}{\cos(x)}$

D. $\frac{2}{1-\cot(x/2)} + C$ E. $\frac{1+t^2}{1+t^2-2t} + C$

*Solution.* We have $dx = 2dt/(1+t^2)$, and the integral becomes

$$\int \frac{2dt}{(1+t^2)(1-\frac{2t}{1+t^2})} = \int \frac{2dt}{1+t^2-2t} = \int \frac{2dt}{(t-1)^2}$$

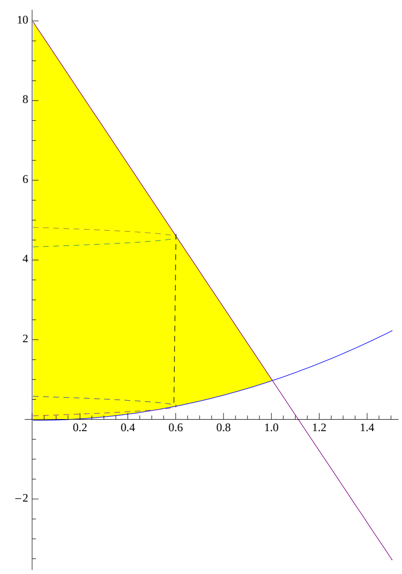
$$= \frac{2}{1-t} + C = \frac{2}{1-\tan(x/2)} + C.$$

*Remark.* Compare this answer with no. 73 in the table of integrals at the back of the textbook. Explain the apparent discrepancy.

III. **[4]** Use the shell method to find the volume of the solid generated by revolving, around the $y$-axis, the region in the right half-plane (that is, where $x \geq 0$) bounded by the three curves

$$y = x^2, \quad y = 10 - 9x, \quad \text{and} \quad x = 0.$$

A. 32 B. $5\pi$ C. $10\pi/3$ D. $5\pi/2$ E. $7\pi/2$




*Solution.* The sharp corner of the region, on the right, is where the two curves intersect, so at that point, $y = x^2 = 10 - 9x$. The roots of $x^2 + 9x - 10$ are 1 and $-10$, and so the corner is at $x = 1$, $y = 1$. "Adding up" the volumes of all cylindrical shells—like the indicated one, having radius $x$, height $(10 - 9x) - x^2$ and "width" $dx$—gives the total volume to be

$$\int_0^1 2\pi x(10 - 9x - x^2)\, dx = 2\pi(10/2 - 9/3 - 1/4) = 7\pi/2.$$

IV. [**4**] Find the area of the surface generated by revolving the curve

$$x = \frac{1}{4}(e^{3t} - 3t), \quad y = e^{3t/2} \qquad (0 \le t \le 1)$$

around the $x$-axis.

A. $\frac{\pi}{3}(e^{9/2} + 3e^{3/2} - 4)$ B. $\frac{\pi}{4}(e^3 - 3)e^{3/2}$ C. $\pi\big((\frac{1}{4}(e^3 - 3))^2 + e^3\big)$

D. $\frac{\pi}{3}(e^{9/2} + 3e^{3/2})$ E. 6

*Solution.*

$$\begin{aligned}
2\pi \int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt &= 2\pi \int_0^1 e^{3t/2}\sqrt{\frac{9}{16}(e^{3t} - 1)^2 + \frac{9}{4}e^{3t}}\,dt \\
&= 2\pi \int_0^1 \frac{3}{4}e^{3t/2}\sqrt{(e^{3t} - 1)^2 + 4e^{3t}}\,dt \\
&= \frac{3}{2}\pi \int_0^1 e^{3t/2}\sqrt{e^{6t} + 2e^{3t} + 1}\,dt \\
&= \frac{3}{2}\pi \int_0^1 e^{3t/2}(e^{3t} + 1)\,dt \\
&= \frac{3}{2}\pi\big(\frac{2}{9}(e^{9/2} - 1) + \frac{2}{3}(e^{3/2} - 1)\big) \\
&= \frac{\pi}{3}(e^{9/2} + 3e^{3/2} - 4).
\end{aligned}$$