Math 181 Midterm Exam 2

Nov. 6, 2007

[**Bold numbers**] indicate points (**20** total).

1. [**4**] Find
$$\int \frac{4}{x(x+2)^2}\,dx.$$

Solution. Write
$$\frac{4}{x(x+2)^2} = \frac{A}{x} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}.$$

Multiply through by $x(x+2)^2$ to get
$$4 = A(x+2)^2 + Bx(x+2) + Cx.$$

Set $x = 0$ to get $A = 1$. Set $x = -2$ to get $C = -2$. Comparing coefficients of x^2 on both sides gives $0 = A + B$, and so $B = -1$.

The integral becomes
$$\int \frac{dx}{x} - \int \frac{1}{(x+2)}\,dx - \int \frac{2}{(x+2)^2}\,dx = \ln|x| - \ln|x+2| + \frac{2}{x+2} + C.$$

2. [**3**] Evaluate
$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}\,dx.$$

Solution. Put $u = \sqrt{x}$, so that $x = u^2$, $dx = 2u\,du$.
The integral becomes
$$\int_0^\infty \frac{2u\,du}{u(1+u^2)} = 2\int_0^\infty \frac{du}{1+u^2} = 2\tan^{-1}u\Big|_0^\infty = \pi.$$

3. [**3**] Does the following series converge? Say why (otherwise no credit).
$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right).$$

Solution. This is a series of positive terms, which can be written as
$$\sum_{n=1}^\infty \frac{\sin(1/n)}{\sqrt{n}\cos(1/n)}.$$

Now $\sin(x) \le x$ for all $x > 0$ (because $x - \sin x$ has a nonnegative derivative, so is an increasing function, which has value 0 when $x = 0$); and $\cos(1/n) \ge \cos(1)$. Hence
$$\sum_{n=1}^\infty \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right) \le \sum_{n=1}^\infty \frac{1}{n}\,\frac{1}{\sqrt{n}\cos(1)} = \frac{1}{\cos(1)}\sum_{n=1}^\infty \frac{1}{n^{3/2}} < \infty,$$
the last inequality by comparison with $\int_1^\infty \frac{dx}{x^{3/2}}$.

4. [5] (a) Prove that the power series $\sum_{n=1}^{\infty} x^n/n$ and $\sum_{n=1}^{\infty} x^n/n^2$ both have radius of convergence equal to 1.

(b) Suppose $0 \leq a_n \leq b_n$ for all $n \geq 0$. Explain why the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $\geq$ the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n x^n$.

(c) Assuming you know (a) and (b), or otherwise, find the radius of convergence R of the power series $\sum_{n=2}^{\infty} x^n/(n \ln n)$. (Explain your answer, else no credit.)

Solution. (a) For $r = 1$ or $r = 2$ (or for that matter, any $r > 0$),

$$\lim_{n\to\infty} \frac{n^r}{(n+1)^r} = 1.$$

As shown in class, this limit is the radius of convergence.

(b) The statement means that the open interval of convergence of the first series contains that of the second, which holds because if x is in the open interval of convergence of the second series, then

$$\sum_{n=1}^{\infty} |a_n x^n| \leq \sum_{n=1}^{\infty} |b_n x^n| < \infty,$$

so x is inside, or on the boundary of, the interval of convergence of $\sum_{n=1}^{\infty} a_n x^n$. And it can't be on the boundary, because we can apply the same argument to $x + \epsilon$ when $\epsilon > 0$ is small enough that $x + \epsilon$ is still in the open interval of convergence of the second series.

Another way to argue, using one description of the radius of convergence given in class, is to say that if $|b_n x^n|$ is eventually < 1, then so is the smaller number $|a_n x^n| \ldots$

(c) By (a), and by (b) with $a_n = 1/(n \ln n)$ and $b_n = 1/n$, we get $R \geq 1$. Similarly, taking $a_n = 1/n^2$ and $b_n = 1/(n \ln n)$, we get $1 \geq R$. Thus $R = 1$.

5. **[5]** Prove that if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

$$\sum_{n=1}^{\infty}(a_n + b_n) = A + B.$$

Solution. Let $\epsilon > 0$. By assumption, there is an M_1 such that for all $N \geq M_1$,

$$\Big|A - \sum_{n=1}^{N} a_n\Big| < \epsilon/2.$$

Similarly, there is an M_2 such that for all $N \geq M_2$,

$$\Big|B - \sum_{n=1}^{N} b_n\Big| < \epsilon/2.$$

Let M be the larger of M_1 and M_2. Then for all $N \geq M$, we have

$$\begin{aligned}\Big|(A+B) - \sum_{n=1}^{N}(a_n + b_n)\Big| &= \Big|(A - \sum_{n=1}^{N} a_n) + (B - \sum_{n=1}^{N} b_n)\Big| \\ &\leq \Big|A - \sum_{n=1}^{N} a_n\Big| + \Big|B - \sum_{n=1}^{N} b_n\Big| < \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

This shows that

$$\lim_{N\to\infty} \sum_{n=1}^{N}(a_n + b_n) = A + B,$$

as desired.