Math 181 Practice Exam 2

Nov. 1, 2007

1. Find
$$\int (\sin^{-1} x)^2\, dx.$$

Solution.
Let $u = \sin^{-1} x$, so that $x = \sin u$, $dx = \cos u\, du$. The integral becomes
$$\int u^2 \cos u\, du.$$
Integrate by parts to get
$$\int u^2 \cos u\, du = u^2 \sin u - 2\int u \sin u\, du.$$
Integrate by parts again to get
$$\int u^2 \cos u\, du = u^2 \sin u - 2(-u\cos u + \int \cos u) = u^2 \sin u + 2u\cos u - 2\sin u.$$
Using $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-x^2}$, conclude that
$$\boxed{\int (\sin^{-1} x)^2\, dx = x\big(\sin^{-1} x\big)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x\sqrt{1-x^2} - 2x + C}$$

2. Evaluate
$$\int_1^\infty \frac{3v-1}{4v^3-v^2}\, dv.$$

Solution. Write
$$\frac{3v-1}{4v^3-v^2} = \frac{A+Bv}{v^2} + \frac{C}{4v-1}.$$
Multiply through by $4v^3 - v^2$ to get
$$3v - 1 = (A+Bv)(4v-1) + Cv^2 = (4B+C)v^2 + (4A-B)v - A,$$
so
$$4B + C = 0, \quad 4A - B = 3, \quad -A = -1;$$
and these equations are easily solved to give $A = 1,\ B = 1,\ C = -4.$

The integral becomes:

$$\int_1^\infty \frac{dv}{v^2} + \int_1^\infty \frac{vdv}{v^2} - 4\int_1^\infty \frac{dv}{4v-1} = \Big(\frac{-1}{v} + \ln v - \ln(4v-1)\Big)\Big|_1^\infty$$
$$= \Big(\frac{-1}{v} + \ln\frac{v}{4v-1}\Big)\Big|_1^\infty$$
$$= \Big(\frac{-1}{v} + \ln\frac{1}{1-1/4v}\Big)\Big|_1^\infty$$
$$= \boxed{1 - \ln(4/3) \quad \text{or} \quad \ln(3e/4)}.$$

3. Does the following series converge? Say why (otherwise no credit).

$$\sum_{n=1}^\infty \frac{1}{(3n-2)^{n+\frac{1}{2}}}.$$

Solution. It *converges,* because $(3n-2)^{n+\frac{1}{2}} \geq n^n \geq n^2$, so

$$\sum_{n=1}^\infty \frac{1}{(3n-2)^{n+\frac{1}{2}}} \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty$$

(the last inequality by comparison with $\int_1^\infty dx/x^2$).

4. For which values of x does the power series $\sum_{n=1}^\infty (\operatorname{csch} n)x^n$ converge absolutely; and for which, conditionally?

Solution. We have

$$\frac{\operatorname{csch} n}{\operatorname{csch}(n+1)} = \frac{e^{n+1} - e^{-n-1}}{e^n - e^{-n}} = \frac{e - e^{-2n-1}}{1 - e^{-2n}},$$

whose limit as $n \to \infty$ is e. So the interval of convergence is $(-e, e)$: the series converges absolutely for any x in this interval, and diverges for x outside the interval.

At the endpoints $x = \pm e$ we have the series

$$\sum_{n=1}^\infty (\operatorname{csch} n)x^n = \sum_{n=1}^\infty \frac{2}{e^n - e^{-n}}(\pm e)^n = \sum_{n=1}^\infty \frac{2}{1 - e^{-2n}}(\pm 1)^n.$$

These series diverge, because the terms do not go to 0 as $n \to \infty$.

There are no values of x remaining for the series to converge conditionally at.