META MATH!

THE QUEST FOR OMEGA

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THE LABYRINTH OF THE CONTINUUM

The "labyrinth of the continuum" is how Leibniz referred to the philosophical problems associated with real numbers, which we shall discuss in this chapter. So the emphasis here will be on philosophy and mathematics, rather than on physics as in the last chapter.

What is a real number? Well, in geometry it's the length of a line, measured exactly, with infinite precision, for example 1.2749591 … , which doesn't sound too problematical, at least at first. And in analytic geometry you need **two** real numbers to locate a point (in two dimensions), its distance from the x axis, and its distance from the y axis. **One** real will locate a point on a line, and the line that we will normally consider will be the so-called "unit interval" consisting of all the real numbers from zero to one. Mathematicians write this interval as [0, 1), to indicate that 0 is included but 1 is not, so that all the real numbers corresponding to these points have no integer part, only a decimal fraction. Actually, [0, 1] works too, as long as you write one as 0.99999 … instead of as 1.00000 … But not to worry, we're going to ignore all these subtle details. You get the general idea, and that's enough for reading this chapter.

[By the way, why is it called "real"? To distinguish it from so-called "imaginary" numbers like $\sqrt{-1}$. Imaginary numbers are neither more nor less imaginary than real numbers, but there was initially, several centuries ago, including at the time of Leibniz, much resistance to placing them on an equal footing with real numbers. In a letter to Huygens, Leibniz points out that calculations that temporarily traverse this imaginary world can in fact start and end with real numbers. The usefulness of such a procedure was, he argued, an argument in favor of such numbers. By the time of Euler, imaginaries were extremely useful. For example, Euler's famous result that

$$e^{ix} = \cos x + i \sin x$$

totally tamed trigonometry. And the statement (Gauss) that an algebraic equation of degree n has exactly n roots only works with the aid of imaginaries. Furthermore, the theory of functions of a complex variable (Cauchy) shows that the calculus and in particular so-called power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

make much more sense with imaginaries than without. The final argument in their favor, if any was needed, was provided by Schrödinger's equation in quantum mechanics, in which imaginaries are absolutely essential, since quantum probabilities (so-called "probability amplitudes") have to have direction as well as magnitude.]

As is discussed in Burbage and Chouchan, *Leibniz et l'infini,* PUF, 1993, Leibniz referred to what we call the infinitesimal calculus as "the calculus of transcendentals." And he called curves "transcendental" if they cannot be obtained via an algebraic equation, the way that the circles, ellipses, parabolas and hyperbolas of analytic geometry most certainly can.

Leibniz was extremely proud of his quadrature of the circle, a problem that had eluded the ancient Greeks, but that he could solve with *transcendental* methods:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots$$

What is the quadrature of the circle? The problem is to geometrically construct a square having the same area as a given circle, that is, to determine the area of the circle. Well, that's πr^2, r being the radius of the circle, which converts the problem into determining π, precisely what Leibniz accomplished so elegantly with the infinite series displayed above.

Leibniz could not have failed to be aware that in using this term he was evoking the notion of God's transcendence of all things human, of human limitations, of human finiteness.

As often happens, history has thrown away the philosophical ideas that inspired the creators and kept only a dry technical husk of what they thought that they had achieved. What remains of Leibniz's idea of transcendental methods is merely the distinction between algebraic numbers and transcendental numbers. A real number x is algebraic if it is the solution of an equation of the form

$$ax^n + bx^{n-1} + \ldots + px + q = 0$$

where the constants a, b, ... are all integers. Otherwise x is said to be transcendental. The history of proofs of the existence of transcendental numbers is rich in intellectual drama, and is one of the themes of this chapter.

Similarly, it was Cantor's obsession with God's infiniteness and transcendence that led him to create his spectacularly successful but extremely controversial theory of infinite sets and infinite numbers. What began, at least in Cantor's mind, as a kind of madness, as a kind of mathematical theology full—necessarily full—of paradoxes, such as the one discovered by Bertrand Russell, since any attempt by a finite mind to apprehend God is inherently paradoxical, has now been condensed and desiccated into an extremely technical and untheological field of math, modern axiomatic set theory.

Nevertheless, the intellectual history of the proofs of the existence of transcendental numbers is quite fascinating. New ideas totally transformed our way of viewing the problem, not once, but in fact **five** times! Here is an outline of these developments:

- Liouville, Hermite and Lindemann, with great effort, were the first to exhibit individual real numbers that could be proved to be transcendental. Summary: **individual transcendentals.**
- Then Cantor's theory of infinite sets revealed that the transcendental reals had the same cardinality as the set of all reals, while the algebraic reals were merely as numerous as the integers, a smaller infinity. Summary: **most reals are transcendental.**
- Next Turing pointed out that all algebraic reals are computable, but again, the uncomputable reals are as numerous as the set of all reals, while the computable reals are only as numerous as the integers. The existence of transcendentals is an immediate corollary. Summary: **most reals are uncomputable and *therefore* transcendental.**
- The next great leap forward involves probabilistic ideas: the set of random reals was defined, and it turns out that with probability one, a real number is random and therefore necessarily uncomputable and transcendental. Non-random, computable and algebraic reals all have probability zero. So now you can get a transcendental real merely by picking a real number at random with an infinitely sharp pin, or, alternatively, by using independent tosses of a fair coin to get its binary expansion. Summary: **reals are transcendental/uncomputable/random with probability one.** And in the next chapter we'll exhibit a natural construction that picks out an individual random real, namely the halting probability Ω, without the need for an infinitely sharp pin.
- Finally, and perhaps even more devastatingly, it turns out that the set of all reals that can be individually named or specified or even defined or referred to—constructively or not—within a formal language or within an individual FAS, has probability zero. Summary: **reals are un-nameable with probability one.**

So the set of real numbers, while natural—indeed, immediately given—geometrically, nevertheless remains quite elusive:

Why should I believe in a real number if I can't calculate it, if I can't prove what its bits are, and if I can't even refer to it? And each of these things happens with probability one! The real line from 0 to 1 looks more and more like a Swiss cheese, more and more like a stunningly black high-mountain sky studded with pin-pricks of light.

Let's now set to work to explore these ideas in more detail.

THE "UNUTTERABLE" AND THE PYTHAGOREAN SCHOOL

This intellectual journey actually begins, as is often the case, with the ancient Greeks. Pythagoras is credited with naming both mathematics and philosophy. And the Pythagoreans believed that number—whole numbers—rule the universe, and that God is a mathematician, a point of view largely vindicated by modern science, especially quantum mechanics, in which the hydrogen atom is modeled as a musical instrument that produces a discrete scale of notes. Although, as we saw in Chapter Three, perhaps God is actually a computer programmer!

Be that as it may, these early efforts to understand the universe suffered a serious setback when the Pythagoreans discovered geometrical lengths that cannot be expressed as the ratio of two whole numbers. Such lengths are called irrational or incommensurable. In other words, they discovered real numbers that cannot be expressed as a ratio of two whole numbers.

How did this happen?

The Pythagoreans considered the unit square, a square one unit in length on each side, and they discovered that the size of both of the two diagonals, $\sqrt{2}$, isn't a rational number n/m. That is to say, it cannot be expressed as the ratio of two integers. In other words, there are no integers n and m such that

$$\left(\frac{n}{m}\right)^2 = 2 \qquad \text{or} \qquad n^2 = 2m^2$$

An elementary proof of this from first principles is given in Hardy's well-known *A Mathematician's Apology.* He presents it there because he believes that it's a mathematical argument whose beauty anyone should be able to appreciate. However, the proof that Hardy gives, which is actually from Euclid's *Elements,* does not give as much insight as a more advanced proof using unique factorization into primes. I **did not** prove unique factorization in Chapter Two. Nevertheless, I'll use it here. It is relevant because the two sides of the equation

$$n^2 = 2m^2$$

would give us **two different** factorizations of the same number. How?

Well, factor n into primes, and factor m into primes. By doubling the exponent of each prime in the factorizations of n and m,

$$2^\alpha\, 3^\beta\, 5^\gamma \ldots \longrightarrow 2^{2\alpha}\, 3^{2\beta}\, 5^{2\gamma} \ldots,$$

we get factorizations of n^2 and m^2. This gives us a factorization of n^2 in which the exponent of 2 is even, and a factorization of $2m^2$ in which the exponent of 2 is odd. So we have two different factorizations of the same number into primes, which is impossible.

According to Dantzig, *Number, The Language of Science,* the discovery of irrational or incommensurable numbers like $\sqrt{2}$

> caused great consternation in the ranks of the Pythagoreans. The very name given to these entities testifies to that. *Algon,* the *unutterable,* these incommensurables were called . . . How can number dominate the universe when it fails to account even for the most immediate aspect of the universe, namely *geometry?* So ended the first attempt to exhaust nature by number.

This intellectual history also left its traces in the English language: In English such irrationals are referred to as "surds," which comes from the French *sourd-muet,* meaning deaf-mute, one who cannot hear or speak. So the English word "surd" comes from the French word for "deaf-mute," and algon = mute. In Spanish it's *sordomudo,* deaf-mute.

In this chapter we'll retrace this history, and we'll see that real numbers not only confound the philosophy of Pythagoras, they confound as well Hilbert's belief in the notion of a FAS, and they provide us with many additional reasons for doubting their existence, and for remaining quite skeptical. To put it bluntly, our purpose here is to review and discuss the mathematical arguments **against real numbers.**

THE 1800S: INDIVIDUAL TRANSCENDENTALS (LIOUVILLE, HERMITE, LINDEMANN)

Although Leibniz was extremely proud of the fact that he had been able to square the circle using transcendental methods, the 1800s wanted to be **sure** that they were really required. In other words, they demanded proofs that π and other individual numbers defined via the sums of infinite series **were not** the solution of any algebraic equation.

Finding a natural specific example of a transcendental real turned out to be much harder than expected. It took great ingenuity and cleverness to exhibit provably transcendental numbers!

The first such number was found by Liouville:

$$\text{Liouville number} = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \dots + \frac{1}{10^{n!}} + \dots$$

He showed that algebraic numbers cannot be approximated that well by rational numbers. In other words, he showed that his number cannot be algebraic, because there are rational approximations that work too well for it: they can get too close, too fast. But Liouville's number isn't a natural example, because no one had ever been interested in this particular number before Liouville. It was constructed precisely so that Liouville could prove its transcendence. What about π and Euler's number e?

Euler's number

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots$$

was finally proved transcendental by Hermite. Here at last was a natural example! This was an important number that people really cared about!

But what about the number that Leibniz was so proud of conquering? He had squared the circle by transcendental methods:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

But can you prove that transcendental methods are really necessary? This question attracted a great deal of attention after Hermite's result, since π seemed to be the obvious next candidate for a transcendence proof. This feat was finally accomplished by Lindemann, provoking the famous remark by Kronecker that "Of what use is your beautiful proof, since π **does not exist!**" Kronecker was a follower of Pythagoras; Kronecker's best known statement is, "God created the integers; all the rest is the work of man!"

These were the first steps on the long road to understanding transcendence, but they were difficult complicated proofs that were specially tailored for each of these specific numbers, and gave no general insight into what was going on.

CANTOR: THE NUMBER OF TRANSCENDENTALS IS A HIGHER ORDER INFINITY THAN THE NUMBER OF ALGEBRAIC REALS

As I've said, a real number is one that can be determined with arbitrary precision, such as $\pi = 3.1415926 \dots$ Nevertheless, in the late 1800s two mathematicians, Cantor and Dedekind, were moved to come up with much more careful definitions of a real number. Dedekind did it via "cuts," thinking of an irrational real r as a way to partition all the rational numbers n/m into those less than r and those greater than r. In Cantor's case a real was defined as an infinite sequence of rational numbers n/m that approach r more and more closely.[7]

History did not take any more kindly to their work than it has to any other attempt at a "final solution."

But first, let me tell you about Cantor's theory of infinite sets and his invention of new, infinite numbers for the purpose of measuring the sizes of all infinite sets. A very bold theory, indeed!

Cantor's starting point is his notion of comparing two sets, finite or infinite, by asking whether or not there is a one-to-one correspondence, a pairing between the elements of the two sets that exhausts both sets and leaves no element of either set unpaired, and no element of one of the sets paired with more than one partner in the other set. If this can be done, then Cantor declares that the two sets are equally big.

Actually, Galileo had mentioned this idea in one of his dialogues, the one published at the end of his life when he was under house arrest. Galileo points out that there are precisely as many positive integers 1, 2, 3, 4, 5, ... as there are square numbers 1, 4, 9, 16, 25, ... Up to that point, history has decided that Galileo was right on target.

However, he then declares that the fact that the squares are just a

[7]Earlier versions of the work of Dedekind and of Cantor on the reals are due to Eudoxus and to Cauchy, respectively. History repeats itself, even in mathematics.

tiny fraction of all the positive integers contradicts his previous observation that they are equally numerous, and that this paradox precludes making any sense of the notion of the size of an infinite set.

The paradox of the whole being equivalent to one of its parts may have deterred Galileo, but Cantor and Dedekind took it entirely in stride. It did not deter them at all. In fact, Dedekind even put it to work for him, he used it. Dedekind **defined** an infinite set to be one having the property that a proper subset of it is just as numerous as it is! In other words, according to Dedekind, a set is infinite if and only if it can be put in a one-to-one correspondence with a part of itself, one that excludes some of the elements of the original set!

Meanwhile, Dedekind's friend Cantor was starting to apply this new way of comparing the size of two infinite sets to common everyday mathematical objects: integers, rational numbers, algebraic numbers, reals, points on a line, points in the plane, etc.

Most of the well-known mathematical objects broke into two classes: 1) sets like the algebraic real numbers and the rational numbers, which were exactly as numerous as the positive integers, and are therefore called "countable" or "denumerable" infinities, and 2) sets like the points in a finite or infinite line or in the plane or in space, which turned out all to be exactly as numerous as each other, and which are said to "have the power of the continuum." And this gave rise to two new infinite numbers, $\aleph_0$ (aleph-nought) and c, both invented by Cantor, that are, respectively, the size (or as Cantor called it, the "power" or the "cardinality") of the positive integers and of the continuum of real numbers.

Comparing Infinities!

$$\#\{\text{reals}\} = \#\{\text{points in line}\} = \#\{\text{points in plane}\} = c$$

$$\#\{\text{positive integers}\} = \#\{\text{rational numbers}\}$$
$$= \#\{\text{algebraic real numbers}\} = \aleph_0$$

Regarding his proof that there were precisely as many points in a plane as there are in a solid or in a line, Cantor remarked in a letter to Dedekind, "Je le vois, mais je ne le crois pas!," which means "I see it, but I don't believe it!," and which happens to have a pleasant melody in French.

And then Cantor was able to prove the extremely important and basic theorem that c is larger than $\aleph_0$, that is to say, that the continuum is a nondenumerable infinity, an uncountable infinity, in other words, that there are more real numbers than there are positive integers, infinitely more. This he did by using Cantor's well-known diagonal method, explained in Wallace, *Everything and More,* which is all about Cantor and his theory.

In fact, it turns out that the infinity of transcendental reals is exactly as large as the infinity of all reals, and the smaller infinity of algebraic reals is exactly as large as the infinity of positive integers. Immediate corollary: most reals are transcendental, not algebraic, infinitely more so.

Well, this is like stealing candy from a baby! It's much less work than struggling with individual real numbers and trying to prove that they are transcendental! Cantor gives us a much more general perspective from which to view this particular problem. And it's much easier to see that **most** reals are transcendental than to decide if a **particular** real number happens to be transcendental!

So that's the first of what I would call the "philosophical" proofs that transcendentals exist. Philosophical as opposed to highly technical, like flying by helicopter to the top of the Eiger instead of reaching the summit by climbing up its infamous snow-covered north face.

Is it really that easy? Yes, but this set-theoretic approach created as many problems as it solved. The most famous is called Cantor's continuum problem.

What is Cantor's continuum problem?

Well, it's the question of whether or not there happens to be any set that has more elements than there are positive integers, and that has fewer elements than there are real numbers. In other words, is there an infinite set whose cardinality or power is bigger than $\aleph_0$ and smaller than c? In other words, is c the next infinite number after $\aleph_0$, which has the name $\aleph_1$ (aleph-one) reserved for it in Cantor's theory, or are there a lot of other aleph numbers in between?

Cantor's Continuum Problem

Is there a set S such that $\aleph_0 < \#S < c$?

In other words, is $c = \aleph_1$, which is the first cardinal number after $\aleph_0$?

A century of work has not sufficed to solve this problem!

An important milestone was the proof by the combined efforts of Gödel and Paul Cohen that the usual axioms of axiomatic set theory (as opposed to the "naive" paradoxical original Cantorian set theory) do not suffice to decide one way or another. You can add a new axiom asserting there is a set with intermediate power, or that there is no such set, and the resulting system of axioms will not lead to a contradiction (unless there was already one there, without even having to use this new axiom, which everyone fervently hopes is not the case).

Since then there has been a great deal of work to see if there might be new axioms that set theorists can agree on that might enable them to settle Cantor's continuum problem. And indeed, something called the axiom of projective determinacy has become quite popular among set theorists, since it permits them to solve many open prob-

lems that interest them. However, it doesn't suffice to settle the continuum problem!

So you see, the continuum refuses to be tamed!

And now we'll see how the real numbers, annoyed at being "defined" by Cantor and Dedekind, got their revenge in the century after Cantor, the 20th century.

BOREL'S AMAZING KNOW-IT-ALL REAL NUMBER

The first intimation that there might be something wrong, something terribly wrong, with the notion of a real number comes from a small paper published by Émile Borel in 1927.

Borel pointed out that if you really believe in the notion of a real number as an infinite sequence of digits 3.1415926 … , then you could put all of human knowledge into a single real number. Well, that's not too difficult to do, that's only a finite amount of information. You just take your favorite encyclopedia, for example, the *Encyclopaedia Britannica,* which I used to use when I was in high-school—we had a nice library at the Bronx High School of Science—and you digitize it, you convert it into binary, and you use that binary as the base-two expansion of a real number in the unit interval between zero and one!

So that's pretty straightforward, especially now that most information, including books, is prepared in digital form before being printed.

But what's more amazing is that there's nothing to stop us from putting an infinite amount of information into a real number. In fact, there's a single real number, I'll call it Borel's number, since he imagined it, in 1927, that can serve as an oracle and answer any yes/no question that we could ever pose to it. How? Well, you just number all the possible questions, and then the Nth digit or Nth bit of Borel's number tells you whether the answer is yes or no!

If you could come up with a list of all possible yes/no questions and only valid yes/no questions, then Borel's number could give us the answer in its binary digits. But it's hard to do that. It's much easier to simply list all possible texts in the English language (and Borel did it using the French language), all possible finite strings of characters that you can form using the English alphabet, including a blank for use between words. You start with all the one-character strings, then all the two-character strings, etc. And you number them all like that . . .

Then you can use the *N*th digit of Borel's number to tell you whether the *N*th string of characters is a valid text in English, then whether it's a yes/no question, then whether it has an answer, then whether the answer is yes or no. For example, "Is the answer to this question 'No'?" looks like a valid yes/no question, but in fact has no answer.

So we can use *N*th digit 0 to mean bad English, 1 to mean not a yes/no question, 2 to mean unanswerable, and 3 and 4 to mean "yes" and "no" are the answers, respectively. Then 0 will be the most common digit, then 1, then there'll be about as many 3's as 4's, and, I expect, a smattering of 2's.

Now Borel raises the extremely troubling question, "Why should we believe in this real number that answers every possible yes/no question?" And his answer is that he doesn't see any reason to believe in it, none at all! According to Borel, this number is merely a mathematical fantasy, a joke, a *reductio ad absurdum* of the concept of a real number!

You see, some mathematicians have what's called a "constructive" attitude. This means that they only believe in mathematical objects that can be constructed, that, given enough time, in theory one could actually calculate. They think that there ought to be some way to **calculate** a real number, to calculate it digit by digit, otherwise in what sense can it be said to have some kind of mathematical existence?

And this is precisely the question discussed by Alan Turing in his famous 1936 paper that invented the computer as a mathematical concept. He showed that there were lots and lots of computable real numbers. That's the positive part of his paper. The negative part is that he also showed that there were lots and lots of uncomputable real numbers. And that gives us another philosophical proof that there are

transcendental numbers, because it turns out that all algebraic reals are in fact computable.

TURING: UNCOMPUTABLE REALS ARE TRANSCENDENTAL

Turing's argument is very simple, very Cantorian in flavor. First he invents a computer (on paper, as a mathematical idea, a model computer). Then he points out that the set of all possible computer programs is a countable set, just like the set of all possible English texts. Therefore the set of all possible computable real numbers must also be countable. But the set of all reals is uncountable, it has the power of the continuum. Therefore the set of all uncomputable reals is also uncountable and has the power of the continuum. Therefore most reals are uncomputable, infinitely more than are computable.

That's remarkably simple, if you believe in the idea of a general-purpose digital computer. Now we are all very familiar with that idea. Turing's paper is long precisely because that was not at all the case in 1936. So he had to work out a simple computer on paper and argue that it could compute anything that can ever be computed, before giving the above argument that most real numbers will then be uncomputable, in the sense that there cannot be a program for computing them digit by digit forever.

The other difficult thing is to work out in detail precisely why algebraic reals can be computed digit by digit. Well, it's sort of intuitively obvious that this has to be the case; after all, what could possibly go wrong?! In fact, this is now well-known technology using something called Sturm sequences, that's the slickest way to do this; I'm sure that it comes built into *Mathematica* and *Maple*, two symbolic computing software packages. So you can use these software packages to calculate as many digits as you want. And you need to be able to calculate hundreds of digits in order to do research the way described by Jonathan Borwein and David Bailey in their book *Mathematics by Experiment*.

But in his 1936 paper Turing mentions a way to calculate algebraic reals that will work for a lot of them, and since it's a nice idea, I thought I'd tell you about it. It's a technique for root-solving by successive interval halving.

Let's write the algebraic equation that determines an individual algebraic real r that we are interested in as $\phi(x) = 0$; $\phi(x)$ is a polynomial in x. So $\phi(r) = 0$, and let's suppose we know two rational numbers α, β such that $\alpha < r < \beta$ and $\phi(\alpha) < \phi(r) < \phi(\beta)$ and we also know that there is no other root of the equation $\phi(x) = 0$ in that interval. So the signs of $\phi(\alpha)$ and $\phi(\beta)$ have to be different, neither of them is zero, and precisely one of them is greater than zero and one of them is less than zero, that's key. Because if ϕ changes from positive to negative it must pass through zero somewhere in between.

Then you just bisect this interval $[\alpha,\beta]$. You look at the midpoint $(\alpha + \beta)/2$, which is also a rational number, and you plug that into ϕ and you see whether or not $\phi((\alpha+\beta)/2)$ is equal to zero, less than zero, or greater than zero. It's easy to see which, since you're only dealing with rational numbers, not with real numbers, which have an infinite number of digits.

Then if ϕ of the midpoint gives zero, we have found r and we're finished. If not, we choose the left half or the right half of our original interval in such a way that the sign of ϕ at both ends is different, and this new interval replaces our original interval, r must be there, and we keep on going like that forever. And that gives us better and better approximations to the algebraic number r, which is what we wanted to show was possible, because at each stage the interval containing r is half the size it was before.

And this will work if r is what is called a "simple" root of its defining equation $\phi(r) = 0$, because in that case the curve for $\phi(x)$ will in fact cross zero at $x = r$. But if r is what is called a "multiple" root, then the curve may just graze zero, not cross it, and the Sturm sequence approach is the slickest way to proceed.

Now let's stand back and take a look at Turing's proof that there are transcendental reals. On the one hand, it's philosophical like Cantor's proof; on the other hand, it is some work to verify in detail that all algebraic reals are computable, although to me that seems obvious in some sense that I would be hard-pressed to justify/explain.

At any rate, now I'd like to take another big step, and show you

that there are uncomputable reals in a very different way from the way that Turing did it, which is very much in the spirit of Cantor. Instead I'd like to use probabilistic ideas, ideas from what's called measure theory, which was developed by Lebesgue, Borel, and Hausdorff, among others, and which immediately shows that there are uncomputable reals in a totally un-Cantorian manner.

REALS ARE UNCOMPUTABLE WITH PROBABILITY ONE!

I got this idea from reading Courant and Robbins, *What is Mathematics?*, where they give a measure-theoretic proof that the reals are non-denumerable (more numerous than the integers).

Let's look at all the reals in the unit interval between zero and one. The total length of that interval is of course exactly one. But it turns out that all of the computable reals in it can be covered with intervals having total length exactly ϵ, and we can make ϵ as small as we want. How can we do that?

Well, remember that Turing points out that all the possible computer programs can be put in a list and numbered one by one, so there's a first program, a second program, and so forth and so on . . . Some of these programs don't compute computable reals digit by digit; let's just forget about them and focus on the others. So there's a first computable real, a second computable real, etc. And you just take the first computable real and cover it with an interval of size $\epsilon/2$, and you take the second computable real and you cover it with an interval of size $\epsilon/4$, and you keep going that way, halving the size of the covering interval each time. So the total size of all the covering intervals is going to be exactly

$$\frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{16} + \frac{\epsilon}{32} + \dots = \epsilon$$

which can be made as small as you like.

And it doesn't matter if some of these covering intervals fall partially outside of the unit interval, that doesn't change anything.

So all the computable reals can be covered this way, using an arbitrarily small part ϵ of the unit interval, which has length exactly equal to one.

So if you close your eyes and pick a real number from the unit interval at random, in such a way that any one of them is equally likely, the probability is zero that you get a computable real. And that's also the case if you get the successive binary digits of your real number using independent tosses of a fair coin. It's possible that you get a computable real, but it's infinitely unlikely. So with probability one you get an uncomputable real, and that has also got to be a transcendental number, what do you think of that!

Liouville, Hermite and Lindemann worked so hard to exhibit individual transcendentals, and now we can do it, almost certainly, by just picking a real number out of a hat! That's progress for you!

So let's suppose that you do that and get a specific uncomputable real that I'm going to call $R*$. What if you try to prove what some of its bits are when you write $R*$ in base-two binary?

Well, we've got a problem if we try to do that . . .