Basic facts about continuity

• The inverse function of a strictly increasing function is continuous.
• The composition of two continuous functions is continuous.
• The $i$-th projection $\mathbb{R}^n \rightarrow \mathbb{R}$, sending $(x_1, x_2, \ldots, x_n)$ to $x_i$, is continuous.
• A map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ sending $x = (x_1, x_2, \ldots, x_n)$ to

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

is continuous if (and, by the preceding two facts, only if) every one of the coordinate functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous.

• Any function given by a power series is continuous, as are its derivatives of any order.
• If $f$ is a continuous function of one variable, defined on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t) dt \quad (a \leq x \leq b)$$

is continuous.

Continuity of algebraic operations:

• Constant functions are continuous.
• The one-variable function taking $x$ to $1/x$ is continuous except at $x = 0$.
• The addition function $\mathbb{R}^2 \rightarrow \mathbb{R}$ taking $(x, y)$ to $x + y$ is continuous.
• The multiplication function $\mathbb{R}^2 \rightarrow \mathbb{R}$ taking $(x, y)$ to $xy$ is continuous.

Exercise. Use the above facts to prove Theorem 1 in §14.2.

For example, if $a$ is a fixed real number then the function $x^a = e^{a \cdot \ln(x)} \quad (0 < x < \infty)$ is continuous. For, $\ln(y)$, being defined by an integral, is continuous, and therefore so is its inverse $e^z$; and the function $x^a$ is constructed by a succession of steps (i.e., composition)

$$x \mapsto \ln(x) \mapsto (\ln(x), a) \mapsto a \cdot \ln(x) \mapsto e^{a \cdot \ln(x)},$$

each of which, by one of the above statements, is obtained by applying a continuous function.