Basic facts about continuity

- The inverse function of a strictly increasing function is continuous.
- The composition of two continuous functions is continuous.
- The *i-th projection* $\mathbf{R}^n \to \mathbf{R}$, sending $(x_1, x_2, \ldots, x_n)$ to x_i, is continuous.
- A map $f\colon \mathbf{R}^n \to \mathbf{R}^m$ sending $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to

$$f(\mathbf{x}) = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\big)$$

is continuous if (and, by the preceding two facts, only if) every one of the coordinate functions $f_i\colon \mathbf{R}^n \to \mathbf{R}$ is continuous.

- Any function given by a power series is continuous, as are its derivatives of any order.
- If f is a continuous function of one variable, defined on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)dt \quad (a \le x \le b)$$

is continuous.

Continuity of algebraic operations:

- Constant functions are continuous.
- The one-variable function taking x to $1/x$ is continuous except at $x = 0$.
- The addition function $\mathbf{R}^2 \to \mathbf{R}$ taking (x, y) to $x + y$ is continuous.
- The multiplication function $\mathbf{R}^2 \to \mathbf{R}$ taking (x, y) to xy is continuous.

Exercise. Use the above facts to prove Theorem 1 in §14.2.

For example, if a is a fixed real number then the function $x^a = e^{a \cdot ln(x)} \quad (0 < x < \infty)$ is continuous. For, $\ln(y)$, being defined by an integral, is continuous, and therefore so is its inverse e^z; and the function x^a is constructed by a succession of steps (i.e., composition)

$$x \mapsto \ln(x) \mapsto \big(\ln(x), a\big) \mapsto a \cdot \ln(x) \mapsto e^{a \cdot \ln(x)},$$

each of which, by one of the above statements, is obtained by applying a continuous function.