Math 182 Midterm Exam 2

March 25, 2008

NAME: ______________________________

[**Bold numbers**] indicate points (**45** total).

For visibility, please put a box around your final anwer to each question.

I. [**4**] Find the mass of a thin plate with density function $1/r$, covering the region outside the circle $r = 3$ and inside the circle $r = 6\sin\theta$.

Solution. $r = 6\sin\theta$, or $x^2 + y^2 = r^2 = 6r\sin\theta = 6y$, is the circle $x^2 + (y-3)^2 = 9$ of radius 3, centered at $(0,3)$. Sketch the two circles. Where they intersect, $3 = 6\sin\theta$, i.e., $\sin\theta = 1/2$, i.e., $\theta = \pi/6$ or $5\pi/6$. The mass is then

$$\int_{\pi/6}^{5\pi/6}\int_{3}^{6\sin\theta}(1/r)r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}(6\sin\theta - 3)d\theta = -6\cos\theta - 3\theta\Big|_{\pi/6}^{5\pi/6}$$

$$= 12(\sqrt{3}/2) - 12\pi/6 = \boxed{6\sqrt{3} - 2\pi}.$$

II. Let $\mathbf{F}$ be the vector-field $(y\sin z, x\sin z, 1 + xy\cos z)$ on $\mathbf{R}^3$.

(a) [**3**] Find all potential functions for $\mathbf{F}$.

(b) [**3**] Calculate the work done by $\mathbf{F}$ along the curve

$$(x,y,z) = \big(\arccos(t/\pi), \arcsin(t/\pi), \sqrt[7]{\sin t}\,\big) \qquad (0 \le t \le \pi).$$

Solution. (a) If $f(x,y,z)$ is a potential, then $f_x = y\sin z$, so

$$f(x,y,z) = xy\sin z + g(y,z).$$

Then $f_y = x\sin z$ gives $g_y = 0$, i.e.,

$$g(y,z) = h(z).$$

Then $f_z = 1 + xy\cos z$ gives $h_z = 1$, i.e., $h(z) = z + C$ (C constant), and so

$$\boxed{f(x,y,z) = xy\sin z + z + C}.$$

(b) The work done is the difference between the values of any potential at the terminal and initial points, that is,

$$(xy\sin z + z)\Big|_{(\pi/2,0,0)}^{(0,\pi/2,0)} = \boxed{0}.$$

III. **[20]** Set up **but do not evaluate** integrals for the following:

(a) The average value of the function xyz over the region bounded by the coordinate planes and the plane $x + y + z = 2$.

(b) The volume of the region bounded underneath by the hemisphere $\rho = 1$, $z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$.

(c) The centroid of the region outside the cylinder $x^2 + y^2 = 1$ and bounded above by the paraboloid $z = 9 - x^2 - y^2$ and bounded below by the plane $z = 0$.

(d) The moment of inertia around its axis of a right circular cone of uniform density δ, with base radius a and height h.

Hint. Put the cone's vertex at the origin and its axis along the z-axis.

Solution.

(a) $$\boxed{\int_0^2 \int_0^{2-z} \int_0^{2-y-z} xyz\, dxdydz \Big/ \int_0^2 \int_0^{2-z} \int_0^{2-y-z} dxdydz}.$$

(b) Using spherical coordinates, sketch (neatly) what happens on any plane $\theta =$ constant, as ϕ goes from 0 to $\pi/2$. That will lead to

$$\boxed{\int_0^{2\pi} \int_0^{\pi/2} \int_1^{1+\cos\phi} \rho^2 \sin\phi\, drd\phi d\theta}.$$

(c) The cylinder meets the paraboloid in the circle $x^2 + y^2 = 1,\ z = 8$. Sketch as in (b), to get the limits of integration. By symmetry, the centroid is on the z-axis, at height $M_{xy}/M =$ (in cylindrical coordinates)

$$\boxed{\int_0^{2\pi} \int_0^8 \int_1^{\sqrt{9-z}} zr\, drdzd\theta \Big/ \int_0^{2\pi} \int_0^8 \int_1^{\sqrt{9-z}} r\, drdzd\theta}.$$

(d) In cylindrical coordinates,

$$\iiint r^2 \delta dV = \boxed{\delta \int_0^{2\pi} \int_0^h \int_0^{az/h} r^3\, drdzd\theta}.$$

IV. [**7**] Find the circulation around, and the total flux across, the ellipse $(\cos t, 4\sin t)$, $0 \le t \le 2\pi$, for the vector field (x, y).

Solution. Circulation. You can calculate the appropriate integral, but it's easier to notice that the field is conservative ($(x^2+y^2)/2$ is a potential), so the circulation is $\boxed{0}$.

Flux. At the point $(\cos t, 4\sin t)$, $\mathbf{F} = (\cos t, 4\sin t)$, the velocity vector is $\mathbf{v} = (-\sin t, 4\cos t)$, the unit tangent vector is $\mathbf{T} = (-\sin t, 4\cos t)/|\mathbf{v}|$, and the "right side" unit normal vector is $\mathbf{n} = \mathbf{T} \times \mathbf{k} = (4\cos t, \sin t)|\mathbf{v}|$.

The flux is

$$\int_0^{2\pi} (\mathbf{F}\cdot\mathbf{n})ds = \int_0^{2\pi} (\mathbf{F}\cdot\mathbf{n})|\mathbf{v}|dt = \int_0^{2\pi} (4\cos^2 + 4\sin^2 t)dt = 4\int_0^{2\pi} dt = \boxed{8\pi}.$$

Note. You can use $M\,dy - N\,dx$ instead of $(\mathbf{F}\cdot\mathbf{n})|\mathbf{v}|dt$.

V. [**8**] Let R be the region bounded by the four lines $y = x/2$, $y = 2x$, $y = 2x - 2$, and $x + y + 2 = 0$. Let $u = x - 2y$, $v = 2x - y$.

Fill in the boxes to make the following equality true:

$$\iint_R xy\,dxdy = \int_{\square}^{\square}\int_{\square}^{\square} \boxed{}\; dudv.$$

Solution. First solve the equations to get

$$x = (2v - u)/3, \;\; y = (v - 2u)/3.$$

So $xy = (2v^2 - 5vu + 2u^2)/9$, and in the (v, u)-plane, R becomes the parallelogram R' bounded by the lines $u = 0$, $v = 0$, $v = 2$, $u = v + 2$. Also, the Jacobian $\partial(x, y)/\partial(v, u) = 1/3$ (easy calculation).

Sketching R' to help get a handle on the limits of integration, you find:

$$\iint_R xy\,dxdy = \boxed{\int_0^2\int_0^{v+2} \frac{1}{27}(2u^2 - 5uv + 2v^2)\,dudv}.$$