A SOURCE BOOK in MATHEMATICS

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CARDAN

Solution of the Cubic Equation

(Translated from the Latin by Professor R. B. McClenon, Grinnell College, Grinnell, Iowa.)

In his *Ars Magna* (Nürnberg, 1545) Girolamo Cardano (Hieronymus Cardanus, 1501–1576) states that Scipio del Ferro discovered the method of solving an equation of the type $x^3 + px = q$ about the year 1515. Nicolo Tartaglia (in the Latin texts, Tartalea) agrees to this but claims for himself the method of solving the type $x^3 + px^2 = q$ and also the independent discovery already made by Scipio del Ferro. Cardan secured the solution from Tartaglia and published it in his work above mentioned. The merits of the discoveries and the ethics involved in the publication may be found discussed in any of the histories of mathematics.

The selection here made is from Chapter XI of the *Ars Magna*, "De cubo & rebus æqualibus numero," the first edition, the type considered being $x^3 + px = q$, the particular equation being cub^e *p*; 6 reb^s æq̄lis 20; that is, $x^3 + 6x = 20$. The edition of 1570 differs considerably in the text. A facsimile of the two pages is given in Smith's *History of Mathematics*, vol. II, pp. 462, 463.

The translation can be more easily followed by considering the general plan as set forth in modern symbols.

Given

$$x^3 + 6x = 20.$$

Let

$$u^3 - v^3 = 20 \text{ and } u^3v^3 = (\tfrac{1}{3} \times 6)^3 = 8.$$

Then

$$(u - v)^3 + 6(u - v) = u^3 - v^3,$$

for

$$u^3 - 3u^2v + 3uv^2 - v^3 + 6u - 6v = u^3 - v^3,$$

whence

$$3uv(v - u) = 6(v - u)$$

and

$$uv = 2.$$

Hence

$$x = u - v.$$

But

$$u^3 = 20 + v^3 = 20 + \frac{8}{u^3},$$

whence

$$u^6 = 20u^3 + 8,$$

which is a quadratic in u^3. Hence u^3 can be found, and therefore v^3, and therefore $u - v$.

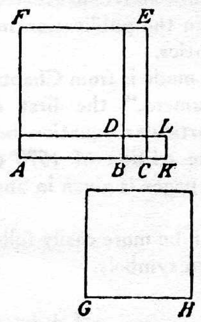
Concerning a Cube and "Things"[1] Equal to a Number

Chapter XI

Scipio del Ferro of Bologna about thirty years ago invented [the method set forth in] this chapter, [and] communicated it to Antonio Maria Florido of Venice, who when he once engaged in a contest with Nicolo Tartalea of Brescia announced that Nicolo also invented it; and he [Nicolo] communicated it to us when we asked for it, but suppressed the demonstration. With this aid we sought the demonstration, and found it, though with great difficulty, in the manner which we set out in the following.

Demonstration

For example, let the cube of *GH* and six times the side *GH* be equal[2] to 20. I take two cubes *AE* and *CL* whose difference shall



be 20, so that the product of the side *AC* by the side *CK* shall be 2,—*i.e.*, a third of the number of "things;"[3] and I lay off *CB* equal to *CK*, then I say that if it is done thus, the remaining line *AB* is equal to *GH* and therefore to the value of the "thing," for it was supposed of *GH* that it was so [*i. e.*, equal to x], therefore I complete, after the manner of the first theorem of the 6th chapter of this book, the solids *DA*, *DC*, *DE*, *DF*, so that we understand by *DC* the cube of *BC*, by *DF* the cube of *AB*, by *DA* three times *CB* times the square of *AB*, by *DE* three times *AB*[4] times the square of *BC*. Since therefore from *AC* times *CK* the result is 2, from 3 times *AC* times *CK* will result 6, the number of "things;" and

[1] [We shall render by "thing" Cardan's *res* or *positio*, the two words he employs to designate the unknown quantity in an equation.]

[2] [This is, $x^3 + 6x = 20$.]

[3] [Here $AC = u$, $CK = v$, $uv = 2 = \frac{1}{3}$ of the coefficient of x.]

[4] [In modern form, we have $DC = v^3$, $DF = (u - v)^3 = x^3$, $DA = 3(u - v)^2v$, and $DE = 3(u - v)v^2$.]

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(Facing page 204.)

therefore from *AB* times 3 *AC* times *CK* there results 6 "things" *AB*, or 6 times *AB*, so that 3 times the product of *AB*, *BC*, and *AC* is 6 times *AB*. But the difference of the cube *AC* from the cube *CK*, and likewise from the cube *BC*, equal to it by hypothesis, is 20;[1] and from the first theorem of the 6th chapter, this is the sum of the solids *DA*, *DE*, and *DF*, so that these three solids make 20.[2] But taking *BC minus*, the cube of *AB* is equal to the cube of *AC* and 3 times *AC* into the square of *CB* and minus the cube of *BC* and minus 3 times *BC* into the square of *AC*.[3] By the demonstration, the difference between 3 times *CB* times the square of *AC*, and 3 times *AC* times the square of *BC*, is [3 times][4] the product of *AB*, *BC*, and *AC*.[5] Therefore since this, as has been shown, is equal to 6 times *AB*, adding 6 times *AB* to that which results from *AC* into 3 times the square of *BC* there results 3 times *BC* times the square of *AC*, since *BC* is minus.[6] Now it has been shown that the product of *CB* into 3 times the square of *AC* is minus; and the remainder which is equal to that is plus, hence 3 times *CB* into the square of *AC*[7] and 3 times *AC* into the square of *CB* and 6 times *AB* make nothing.[8] Accordingly, by common sense, the difference between the cubes *AC* and *BC* is as much as the totality of the cube of *AC*, and 3 times *AC* into the square of *CB*, and 3 times *CB* into the square of *AC* (minus), and the cube of *BC* (minus), and 6 times *AB*.[9] This therefore is 20, since the difference of the cubes *AC* and *CB* was 20.[10] Moreover, by the second theorem of the 6th chapter, putting *BC* minus, the cube of *AB* will be equal to the cube of *AC* and 3 times *AC* into the square of *BC* minus the cube of *BC* and minus 3 times *BC* into the square of *AC*.[11] Therefore the cube of *AB*, with 6 times *AB*, by common sense, since it is equal to the cube of *AC* and 3 times *AC* into the square of *CB*, and minus 3 times *CB* into the square of *AC*,[12] and

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[1] [That is, $u^3 - v^3 = 20$.]
[2] [That is, $(u - v)^3 + 3(u - v)^2v + 3(u - v)v^2 = 20$.]
[3] [That is, $(u - v)^3 = u^3 + 3uv^2 - v^3 - 3vu^2$.]
[4] [The original omits "triplum" here.]
[5] [That is, $3vu^2 - 3uv^2 = 3(u - v)uv$.]
[6] [That is, $6(u - v) + 3uv^2 = 3u^2v$.]
[7] [In the text this is *AB*.]
[8] [That is $-3vu^2 + 3uv^2 + 6(u - v) = 0$.]
[9] [That is, $u^3 - v^3 = u^3 + 3uv^2 - 3vu^2 - v^3 + 6(u - v) = 20$.]
[10] [That is, $u^3 - v^3 = 20$.]
[11] [That is, $(u - v)^3 = u^3 + 3uv^2 - v^3 - 3vu^2$.]
[12] [The text has *AB*.]

minus the cube of CB and 6 times AB, which is now equal to 20, as has been shown, will also be equal to 20.[1] Since therefore the cube of AB and 6 times AB will equal 20, and the cube of GH, together with 6 times GH, will equal 20, by common sense and from what has been said in the 35th and 31st of the 11th Book of the *Elements*,[2] GH will be equal to AB, therefore GH is the difference of AC and CB. But AC and CB, or AC and CK, are numbers or lines containing an area equal to a third part of the number of "things" whose cubes differ by the number in the equation, wherefore we have the

RULE

Cube the third part of the number of "things," to which you add the square of half the number of the equation,[3] and take the root of the whole, that is, the square root, which you will use, in the one case adding the half of the number which you just multiplied by itself,[4] in the other case subtracting the same half, and you will have a "binomial" and "apotome" respectively; then subtract the cube root of the apotome from the cube root of the binomial, and the remainder from this is the value of the "thing."[5] In the example, the cube and 6 "things"[6] equals 20; raise 2, the 3rd part of 6, to the cube, that makes 8; multiply 10, half the number, by itself, that makes 100; add 100 and 8, that makes 108; take the root, which is $\sqrt{108}$, and use this, in the first place adding 10, half the number, and in the second place subtracting the same amount, and you will have the binomial $\sqrt{108} + 10$, and the apotome $\sqrt{108} - 10$; take the cube root of these and subtract that of the apotome from that of the binomial, and you will have the value of the "thing," $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.

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[1] [That is, $x^3 + 6x = u^3 + 3uv^2 - 3vu^2 - v^3 + 6(u - v) = 20$.]

[2] [Evidently an incorrect reference to Euclid. It does not appear in the edition of 1570.]

[3] [That is, if the equation is $x^3 + px = q$, take $(\frac{1}{3}p)^3 + (\frac{1}{2}q)^2$.]

[4] [That is, adding $\frac{1}{2}q$.]

[5] [That is, $\sqrt[3]{\sqrt{(\frac{1}{3}p)^3 + (\frac{1}{2}q)^2} + \frac{1}{2}q} - \sqrt[3]{\sqrt{(\frac{1}{3}p)^3 + (\frac{1}{2}q)^2} - \frac{1}{2}q}$.]

[6] [$x^3 + 6x = 20$.]