Évariste Galois
and
Solvable Permutation Groups

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- He introduced the concept of group.
- He created Galois theory.
- He defined finite fields.

This Talk

We will learn some of the amazing other things Galois did, especially his work on solvable permutation groups.
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Outline

1. Lagrange
   - Three Snapshots

2. Galois
   - Four Snapshots

3. Solvable Permutation Groups
   - Primitive Equations
   - The Affine Linear Group
   - Finite Fields
   - The Main Theorem
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Three Snapshots from Lagrange

In 1770, Lagrange wrote the wonderful paper:

Réflexions sur la résolution des équations

which introduced many of the key players of group theory and Galois theory.

We will give three excerpts from this paper to show how Lagrange laid the foundation for what Galois did.
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In 1770, Lagrange wrote the wonderful paper: 

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A function $y$ of the roots of a polynomial gives values $y', y''$, ... when the roots are permuted. Lagrange says:

*il s’agit ici uniquement de la forme de valeurs et non de leur quantité absolue.*

Since $y = \frac{A(\alpha_1, \ldots, \alpha_n)}{B(\alpha_1, \ldots, \alpha_n)}$ has the form $\frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)}$, we see that Lagrange regarded the roots as variables!
Let $\sigma_i$ be the $i$th elementary symmetric polynomial. Consider

$$K = k(\sigma_1, \ldots, \sigma_n) \subseteq L = k(x_1, \ldots, x_n).$$

Lagrange’s Theorem

If $t, y \in L$ and $t$ is fixed by all permutations that fix $y$, then $t$ is a rational expression in $y, \sigma_1, \ldots, \sigma_n$.

$$K \subseteq K(y) \subseteq L$$

gives $\text{Gal}(L/K(y)) = \{\text{permutations fixing } y\}$. So

$$t \in \text{fixed field of } \text{Gal}(L/K(y)) \iff t \in K(y).$$

Hence $K(y)$ is the fixed field of $\text{Gal}(L/K(y))$.

Lagrange knew the Galois correspondence!
The Universal Extension

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Rational functions \( t, y \in L \) are **similar** ("semblable") when a permutation fixes \( t \) if and only if it fixes \( y \).

**Corollary of Lagrange’s Theorem**

Functions \( t, y \in L \) are similar if and only if

\[
K(t) = K(y).
\]

For us, the Galois correspondence of \( K \subseteq L \) is a bijection

intermediate fields \( \leftrightarrow \) subgroups of \( S_n \).

Lagrange worked with individual rational functions. The notion of “similar function” is his attempt at an intrinsic formulation.

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Four Snapshots from Galois

Évariste Galois was born October 25, 1811 and died May 31, 1832. We are celebrating the 200th anniversary of his birth.

In January 1831, he wrote the amazing paper:

*Mémoire sur les conditions de résolubilité des équations par radicaux*

I will give four excerpts from this memoir to give you a sense of how Galois thought about Galois theory.
PROPOSITION I

THÉORÈME. Soit une équation donnée, dont a, b, c, . . . sont les m racines. Il y aura toujours un groupe de permutations des lettres a, b, c, . . . qui jouira des propriété suivant:

1° que toute fonction des racines, invariable** par les substitutions de ce groupe, soit rationallement connue;

2° réciproquement, que toute fonction des racines déterminable rationallement, soit invariable par ces substitutions*.

The asterisks ** and * indicate marginal notes in Galois’s manuscript.
Marginal Note for “invariable**”

Nous appelons ici invariable non seulement une fonction dont la forme est invariable par les substitutions des racines entre elles, mais encore celle dont las valeur numérique ne varieriat pas par ces substitutions.

Galois is aware that his theory applies to the roots of any polynomial, not just the case when the roots are variables.

Galois has gone beyond Lagrange!
Quelle que soit l’équation donnée, on pourra trouver une fonction $V$ des racines telle que toutes les racines soient fonctions rationnelles des $V$. Cela posé, considérons l’équation irréductible donc $V$ est racine (lemmes III et IV). Soient $V, V', V'', \ldots, V^{(n-1)}$ les racines de cette équation.

Soient $\varphi V, \varphi_1 V, \varphi_2 V, \ldots, \varphi_{m-1} V$ les racines de la proposée. Écrivons les $n$ permutations suivants des racines:

$$(V), \quad \varphi V, \quad \varphi_1 V, \quad \varphi_2 V, \quad \ldots, \quad \varphi_{n-1} V,$$

$$(V'), \quad \varphi V', \quad \varphi_1 V', \quad \varphi_2 V', \quad \ldots, \quad \ldots,$$

$$(V^{(m-1)}), \quad \varphi V^{(m-1)}, \quad \varphi_1 V^{(m-1)}, \quad \varphi_2 V^{(m-1)}, \quad \ldots, \quad \varphi_{n-1} V^{(m-1)}.$$

Je dis que ce groupe de permutations jouit de la propriété énoncée.
For Galois, permutations are bijections

\[ \{1, \ldots, m\} \longrightarrow \{\text{roots}\} \quad \leftarrow \text{call these arrangements} \]

while substitutions are bijections

\[ \{\text{roots}\} \longrightarrow \{\text{roots}\} \quad \leftarrow \text{call these substitutions}. \]

- Arrangements give a strong visual picture.
- Substitutions give a group.
- Arrangements give a principal homogeneous space for the substitutions.

Marginal note for “substitutions*”

*Mettre partout à la place du mot permutation le mot substitution*

But then he crosses this out!
For Galois, *permutations* are bijections

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Galois introduced normal subgroups as follows.

**PROPOSITION III**

**Théorème.** Si II’on adjoint á une équation toutes les racines d’une équation auxiliaire, les groupes dont il est question dans le théorème II jouiront de plus de cette propriété que les substitutions sont les mêmes dans chaque groupe.

If $G$ is the Galois group and $a$ is one arrangement, then $G \cdot a$ is Galois’s “groupe.” The cosets of $H \subseteq G$ give

$$G \cdot a = g_1 H \cdot a \cup g_2 H \cdot a \cup \cdots .$$

These are the “groupes” of PROPOSITION III. Key observation:

$$g(g_i H \cdot a) = g_i H \cdot a \iff g \in g_i Hg_i^{-1} .$$

So “sont les mêmes” means $H$ is normal!
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So “sont les mêmes” means $H$ is normal!
Galois had a wonderful result about the Galois group of a solvable polynomial of prime degree.

**PROPOSITION VII**

**PROBLÈME.** Quel est le groupe d’une équation irréductible d’un degré premier $n$, soluble par radicaux?

Faisons en général $x_n = x_0$, $x_{n+1} = x_1$, ...  

Donc, “si d’une équation irréductible de degré premier est soluble par radicaux, le groupe de cette équation ne saurait contenir que les substitutions de la forme

$$x_k \quad x_{ak+b}$$

$a$ et $b$ étant des constants.”
In modern terms, Galois’s Proposition VII says that the Galois group of an irreducible polynomial of prime $p$ is solvable by radicals if and only if the Galois group is isomorphic to a subgroup of the affine linear group

$$\text{AGL}(1, \mathbb{F}_p) = \{ x \mapsto ax + b \mid a, b \in \mathbb{F}_p, \ a \neq 0 \}.$$ 

There are two aspects of this result worth mentioning:

- Up to conjugacy, $\text{AGL}(1, \mathbb{F}_p)$ is the maximal solvable subgroup of the symmetric group $S_p$.
- This proposition implies that such a polynomial is solvable by radicals iff any two roots generate the splitting field.
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There is More!

The four snapshots from Galois:

- are well-known to most mathematicians, and
- illustrate nicely the power of the theory he developed.
- However, **Galois did a lot more!**
- In particular, we will explore:
  - what Galois knew about solvable permutation groups, and
  - why he invented finite fields.
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The Main Problem

Definition

On appelle équations non primitives les équations qui étant, par exemple, du degré $mn$, se décomposent en $m$ facteurs du degré $n$, au moyen d’une seule équation du degré $m$.

Example

$x^4 - 2$ has degree $4 = 2 \cdot 2$. Adjoining roots of $x^2 - 2$ gives

$$x^4 - 2 = (x^2 - \sqrt{2})(x^2 + \sqrt{2}).$$

Thus $x^4 - 2$ is *imprimitive* (“non primitive”).

Revenons maintenant à notre object, et cherchons en général dans quel cas une équation primitive est soluble par radicaux.
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The First Result

Galois’s Version

... pour qu’une équation primitive soit soluble par radicaux, it faut que son degré soit de la forme $p^\nu$, $p$ étant premier.

Definition

A subgroup $G \subseteq S_n$ is **imprimitive** if

$$\{1, \ldots, n\} = R_1 \cup \cdots \cup R_k, \ k > 1, \ |R_i| > 1$$

for some $i$ and elements of $G$ preserve the $R_i$. Then $G$ is **primitive** if it is not imprimitive.

Theorem

If $G \subseteq S_n$ is primitive and solvable, then $n = p^\nu$, $p$ prime.
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If $G \subseteq S_n$ is primitive and solvable, then $n = p^\nu$, $p$ prime.
Proof

Assume $G \subseteq S_n$ is primitive and solvable. To show: $n = p^\nu$.

Let $N$ be a minimal normal subgroup of $G$. One can prove that there is a simple group $A$ such that

$$N \cong A^\nu$$

This is a standard fact about minimal normal subgroups. Then:

- $G$ primitive $\implies N$ transitive.
- $G$ solvable $\implies N \cong \mathbb{F}_p^\nu$.
- $N$ transitive and abelian $\implies$ its isotropy subgroups are all equal and hence trivial. Thus:

$$p^\nu = |N| = |\text{orbit}| \cdot |\text{isotropy subgroup}| = n \cdot 1 = n.$$  QED
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When $G$ is primitive and solvable, the proof just given shows that $N \cong \mathbb{F}_p^\nu$ is normal in $G$. One can show without difficulty that

$$G \subseteq \text{AGL}(\nu, \mathbb{F}_p) = \{x \mapsto Ax + b \mid A \in \text{GL}(\nu, \mathbb{F}_p), \ b \in \mathbb{F}_p^\nu\}.$$ 

Galois knew this!

**Letter to Chevalier, 29 May 1832**

Toute les permutations d’une équation primitive soluble par radicaux sont de la forme

$$x_{k.l.m...}/x_{ak+bl+cm+...+f.a_1k+b_1l+c_1m+...+g.\ ...}$$

$k, l, m, \ldots$ étant $\nu$ indices qui prenant chacun $p$ valeurs indiquent toutes les racines. Les indices sont pris suivant le module $p$, c’est-à-dire que la racines sera la même quand on adjoutera à l’un des indices un multiple de $p$. 
Finite Fields and the Affine Semilinear Group

When a primitive of degree $p^\nu$ is solvable by radicals, its Galois group lies in

$$\text{AGL}(\nu, \mathbb{F}_p) \subseteq S_{p^\nu}.$$ 

However, $\text{AGL}(\nu, \mathbb{F}_p)$ is not solvable when $\nu \geq 2$. What is its maximal primitive solvable subgroup? This is the question Galois wanted to answer.

Galois used the **Galois theory of finite fields** to create some primitive solvable permutation subgroups of $\text{AGL}(\nu, \mathbb{F}_p)$:

- $\text{AGL}(1, \mathbb{F}_p^\nu)$: The **affine linear group** over $\mathbb{F}_p^\nu$ is
  \[ \{ x \mapsto ax + b \mid a, b \in \mathbb{F}_p^\nu, a \neq 0 \}. \]

- $\text{AGL}(1, \mathbb{F}_p^\nu)$: The **affine semilinear group** over $\mathbb{F}_p^\nu$ is
  \[ \{ x \mapsto a\sigma(x) + b \mid a, b \in \mathbb{F}_p^\nu, \sigma \in \text{Gal}(\mathbb{F}_p^\nu/\mathbb{F}_p), a \neq 0 \}. \]
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What Galois Knew

Why Do We Need Finite Fields?

C’est surtout dans la théorie des permutations . . . que la considération des racines imaginaires des congruences paraît indispensable. Elle donne une moyen simple et facile de reconnaître dans quel cas une équation primitive soit soluble par radicaux . . .

Galois notes that if the Galois group of a primitive equation lies in $AΓL(1, \mathbb{F}_p^\nu )$, then the equation is solvable by radicals.

Galois Goes On To Say:

Cette remarque aurait peu d’importance, si je n’étais parvenu á démontrer que réciproquement une équation primitive ne saurait soluble par radicaux, á moins de satisfaire aux conditions que je viens d’enoncer.
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Galois Knew That There Were Exceptions:

J’excepte les équations du 9ᵉ et 25ᵉ degré.

- In 1868, Jordan classified primitive solvable subgroups of $S_p^2$. He used $AGL(1, \mathbb{F}_p^2)$ together with two other groups.
- Of Jordan’s groups, only $AGL(1, \mathbb{F}_p^2)$ is doubly-transitive.
- Doubly-transitive subgroups are automatically primitive, though the converse (as shown by Jordan) is false.

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The Main Theorem

Subsequent Developments

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Huppert’s Theorem

Here is the great theorem proved by Huppert in 1957.

**Theorem**

Assume $G \subseteq S_n$ is solvable and doubly-transitive. Then:

- $n = p^\nu$, $p$ prime.
- Furthermore, if

$$p^\nu \notin \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\},$$

then $G \subseteq \Gamma L(1, \mathbb{F}_{p^\nu})$ up to conjugacy.

Given that Galois invented groups in 1829, his level of insight is astonishing! So, in this 200th anniversary of Galois’s birth, we have more to celebrate than we thought!
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