MA553 HW4, #3

Let $n = p_1^{e_1} \cdots p_r^{e_r}$, where the p_i are distinct primes, $1 \le e_i \le 2$, and for $i \ne j$, $p_i \not| (p_j^2 - 1)$. It is to be shown that every group G of order n is abelian. In fact it suffices to show that G is nilpotent, i.e., G is the product of its Sylow subgroups—which have either prime or prime-squared order, and so are abelian.

When $r = 1$, G is obviously nilpotent. When $r = 2$ the number $1 + kp_1$ of Sylow p_1-subgroups divides $p_2^{e_2}$, so is either 1, p_2 or p_2^2; and since the last two possibilities contradict the assumption on n, therefore $k = 0$, i.e., G has a normal Sylow p_1-subgroup, and similarly, G has a normal Sylow p_2-subgroup. Thus G is nilpotent.

Suppose then that $r > 2$, and proceed by induction on n. That means one may assume that if the prime-power factorization of $n_1 < n$ satisfies the above conditions then any group of order n_1 is abelian. In particular, any proper subgroup or quotient group of G may be assumed abelian.

By D&F, p. 149, #56, then, G is solvable. So G has a normal subgroup H of prime index, say p_i. Then $|H|$ is divisible by any p_j with $j \ne i$; and H being abelian, problem 1 shows that the Sylow p_j-subgroup N of H is normal in G.

As above, G/N is abelian; and if P is any Sylow subgroup of G then PN is a subgroup of G whose order—which divides $|P||N|$—has at most two distinct prime factors, so that PN is a proper, hence abelian, subgroup of G. By problem 2, G is nilpotent. QED