Math 553 Homework 3

Due Wed. Sept. 14, 2011

1. Let p be a prime number. For any finite group H, $\mathcal{P}_H$ denotes the set of all Sylow p-subgroups of H. (If p doesn't divide the order $|H|$, define the Sylow p-subgroup to be the subgroup consisting of the identity element alone). Let N be a normal subgroup of the finite group G. Prove that

$$\mathcal{P}_N = \{\, P \cap N \mid P \in \mathcal{P}_G \,\}$$

and that

$$\mathcal{P}_{G/N} = \{\, PN/N \mid P \in \mathcal{P}_G \,\}.$$

2. Let G be a finite group, $p > 0$ a prime number, and H a normal p-subgroup of G. Prove the following assertions.

(a) H is contained in each Sylow p-subgroup of G.

(b) If K is any normal p-subgroup of G, then HK is a normal p-subgroup of G.

(c) The subgroup $O_p(G)$ generated by all normal p-subgroups of G is equal to the intersection of all the Sylow p-subgroups of G.

(d) $O_p(G)$ is the unique largest normal p-subgroup of G.

(e) $O_p(\overline{G}) = \{1\}$ where $\overline{G} = G/O_p(G)$.

Before doing problems 3–6, read pp. 181–184 in D&F.

3. (a) Show that the center of the group of transformations

$$x \mapsto ax + b \qquad (a, b \in \mathbb{Z}/5\mathbb{Z},\ a \neq 0)$$

is trivial (i.e., consists of the identity alone).

(b) Let $\zeta = e^{\frac{2\pi i}{5}}$, a fifth root of unity. Is the group (of order 20) generated by the complex matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad (i^2 = -1).$$

isomorphic to the group in (a)?

(c) Show that the Sylow 5-subgroup of a group of order 20 is normal.

(d) Show that there are exactly five distinct groups of order 20.

(Note: saying that there are exactly n distinct groups having a certain property means, more formally, that n is the maximum possible cardinality of a set of such groups with no two members isomorphic.)

(OVER)

4. Groups of order 30 are classified in the first example on p. 182 in D&F. Show that the four different groups of order 30 have 1, 3, 5, and 15 elements of order 2, respectively.

5. Groups of order 12 are classified in the second example on p. 182 in D&F. For each of the following groups, determine which of the groups in that example it's isomorphic to.

(a) The multiplicative group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad (a, b, c \in \mathbb{Z}_3,\ ac \neq 0).$$

(b) The multiplicative group generated by the complex matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad (i^2 = -1,\ \omega^3 = 1,\ \omega \neq 1).$$

(c) The transformations of the form $x \mapsto ax + b\ \ (a \neq 0)$ of the field $\mathbb{F}_4$ into itself.

($\mathbb{F}_4$ is a field with four elements, whose existence and uniqueness is discussed in the example in D&F, p. 549. We'll come to that later; for now you just have to know what a field is, and you may assume that there is one with four elements, and that in it, 1+1=0— or, if you're ambitious, construct $\mathbb{F}_4$ by defining a suitable multiplication on the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

(d) The dihedral group D_{12}.

(e) A non-abelian semidirect product of a group of order 4 by a group of order 3.

6. Let p be an odd prime. Consider the group, of order p^3, consisting of all matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with a, b, and c in the field $\mathbb{Z}/p\mathbb{Z}$. (The group operation is the usual multiplication of matrices.) To which of the two non-abelian groups discussed in the Example in D&F, p. 183 is this group isomorphic? (Why?)