Math 553 Homework 4

Due Wed. Sept. 19, 2012

1. Let H be a normal subgroup of a finite group G, and let $N \triangleleft H$ be a Sylow subgroup of H. Prove that $N \triangleleft G$.

2. Let G be a finite group. Let N be a normal subgroup such that G/N is nilpotent. Suppose that for every Sylow subgroup P of G, PN is nilpotent. Prove that G is nilpotent.

Hint. Use 1.

3. DF, p. 187, `#24`. (The hard part is "if." Begin with the case $r \le 2$, then do induction on $n = |G|$. Using 1 above, and DF, p. 149, `#56` (which you may assume—though you might enjoy actually working it out, via `#53`), show that there exists a nontrivial $N \triangleleft G$ as in 2 above such that $|N|$ divides p^2 for some prime p. Deduce that G is abelian.)

4. Describe explicitly some groups of order 1,163,225 such that *every* group of that order is isomorphic to precisely one of those you've described. (Justify.)

5. (a) Show that a simple group which has a subgroup of index $n > 2$ is isomorphic to a subgroup of the alternating group A_n.

(b) What is the smallest index $[A_n : G]$ occurring for a subgroup $G \subsetneq A_n$?

(Explain your answer. You may use Theorem 24 on page 149 in D&F.)

(c) Show that there is no simple group of order 112.

(d) Show that there is no simple group of order 120.

Hint: Consider the normalizer of a Sylow 5-subgroup.

(e) Is every group of order 120 solvable?

6. Fill in the blanks with positive integers in all possible ways which make the resulting statement true. (Justify your answer.)

There are exactly ___ distinct abelian groups of order 2800 having exactly ___ elements of order 28.