Math 553 Homework 5

Due Wed. Sept. 28, 2011

Work in a fixed commutative monoid M with cancellation.

You may quote, without proof, any result proved in class.

1. Prove that if $[a,b]$ exists then for all c,
(i) $[ac,bc]=[a,b]c$.
(ii) (a,b) exists, and $(ac,bc)=(a,b)c$.

2. Prove that $[a,b]$ exists $\iff$ (ac,bc) exists for all c.

3. Prove: if a is prime and a doesn't divide b then $[a^n,b]=a^nb$ for all n

4. Suppose a is a unit or a product of primes. Prove:
(i) (a,b) and $[a,b]$ exist for all b.
(ii) If (b,c) exists then $(ab,ac)=a(b,c)$.

5. Assume that (x,y) exists for all $x,y\in M$.
(i) Prove that if $(a,b)=1$ then $(a^i,b^j)=1$ for all (i,j).
(ii) Without assuming $(a,b)=1$, deduce from (i) that $(a,b)^n=(a^n,b^n)$ for all n.
(iii) Prove that if $(a,b)=1$ and $ab=c^n$ then $a\sim(a,c)^n$ and $b\sim(b,c)^n$.

6. For any nonempty $A\subset M$, $\gcd(A)$ denotes a common divisor of the elements of A, which is divisible by all other common divisors. An easy induction shows that if any two elements of M have a gcd, then $\gcd(A)$ exists for any *finite* nonempty A.

Prove that unique factorization holds in M if and only if any two elements have a gcd, and for any $A\subset M$ there is a finite subset $A_0\subset A$ such that $\gcd(A)=\gcd(A_0)$.