Math 553 Homework 6

Due Wed. Oct. 19, 2011

1. In each case, decide whether the given structure forms a ring. If it is not a ring, determine which of the ring axioms hold and which fail:
   (a) U is an arbitrary set, and R is the set of subsets of U. Addition and multiplication of elements of R are defined by the rules $A + B = A \cup B$ and $A \cdot B = A \cap B$.
   (b) U is an arbitrary set, and R is the set of subsets of U. Addition and multiplication of elements of R are defined by the rules $A + B = (A \cup B) - (A \cap B)$ and $A \cdot B = A \cap B$.
   (c) R is the set of continuous functions $\mathbb{R} \to \mathbb{R}$. Addition and multiplication are defined by the rules $[f + g](x) = f(x) + g(x)$ and $[f \circ g](x) = f(g(x))$.
2. Let $\zeta = e^{2\pi i/3}$, so that $\zeta^3 = 1$.
   (a) Show that

$$\mathbb{Z}[\zeta] := \{ a + b\zeta \mid a, b \in \mathbb{Z} \}$$

is a subring of the field $\mathbb{C}$ of complex numbers.

(b) Let the *norm* of $s \in \mathbb{Z}[\zeta]$ be defined to be $N(s) := s\bar{s}$, where $\bar{s}$ is the complex conjugate of s. Show that for $s, t \in [\zeta]$,

$$N(st) = N(s)N(t).$$

   (c) Show that $s \in \mathbb{Z}[\zeta] \implies \bar{s} \in \mathbb{Z}[\zeta]$.
   (d) Show that s is a unit in $\mathbb{Z}[\zeta]$ if and only if $N(s) = 1$.
   (e) Show that the group of units in $\mathbb{Z}[\zeta]$ consists of all 6-th roots of unity in $\mathbb{C}$.
3. An element x of a ring R is called *nilpotent* if some power of x is zero. Prove that if x is nilpotent, then $1 + x$ is a unit in R.
4. Prove or disprove: If an ideal I contains a unit, then it is the unit ideal.
5. Prove that if two rings R, R' are isomorphic, then so are the polynomial rings $R[x]$ and $R'[x]$.
6. Let R be a ring, and let $f(y) \in R[y]$ be a polynomial in one variable with coefficients in R. Prove that the map $R[x, y] \to R[x, y]$ defined by

$$x \rightsquigarrow x + f(y), \quad y \rightsquigarrow y$$

   is an automorphism of $R[x, y]$.

   If you don't know what "automorphism" means, look in the index of D&F.
7. Determine all automorphisms of the polynomial ring $\mathbb{Z}[x]$.
8. Find a simpler description for each of the following rings.
   (a) $\mathbb{Z}[x]/(x^2 - 3,\ 2x + 4)$. (b) $\mathbb{Z}[i]/(2 + i) \quad (i^2 = -1)$.