Math 553 Homework 7

Due Fri. Oct. 19, 2012

1. Show that a monoid M is a commutative monoid with cancelation if and only if there exists an injective monoid homomorphism from M into an abelian group.

2. In this problem you may assume the universal property of monoid algebras, as presented in class.

Let R be a commutative ring and let M and N be commutative monoids. With coordinatewise multiplication, $M \times N$ is then also a commutative monoid.

(a) Consider the category $\mathbf{T}$ of triples (S, μ, ν) such that S is a commutative R-algebra and $\mu\colon M \to S$ and $\nu\colon N \to S$ are monoid homomorphisms, maps between such triples being defined in the obvious way.

Find monoid homomorphisms $\mu_1\colon M \to (R[M])[N]$ and $\nu_1\colon N \to (R[M])[N]$, $\mu_2\colon M \to R[M \times N]$ and $\nu_2\colon N \to R[M \times N]$, such that both $((R[M])[N], \mu_1, \nu_1)$ and $(R[M \times N], \mu_2, \nu_2)$ are initial objects in $\mathbf{T}$; and deduce that there is an R-algebra isomorphism

$$\alpha\colon (R[M])[N] \xrightarrow{\sim} R[M \times N]$$

such that

$$\alpha\Big(\sum_{n\in N} \big(\sum_{m\in M} r_{mn} m\big) n\Big) = \sum_{(m,n)\in M\times N} r_{mn}(m, n).$$

(b) Explain carefully how the isomorphism α specializes to give isomorphisms of polynomial rings, such as

$$R[W, X][Y, Z] \xrightarrow{\sim} R[W, X, Y, Z].$$

(c) Let $\phi\colon M \to N$ be a monoid homomorphism, and let $\theta\colon R[M] \to R[N]$ be the corresponding R-algebra homomorphism (given by the universal property of $R[M]$, so that $\theta(\sum r_m m) = \sum r_m \phi(m)$). Show that the kernel of θ is generated by the set of elements of the form $1.m - 1.m'$ with $\phi(m) = \phi(m')$.

3. Let p be a prime ideal in an integral domain R, and let M consist of all elements in R lying outside p (so that M is a multiplicative submonoid of R). In this case it is customary to denote the ring of fractions R_M by R_p.

(a) Show that if q is any prime ideal in R_p, then $q \cap R$ is a prime ideal in R, contained in p; and that one obtains in this way a one-one correspondence between all prime ideals in R_p and those prime ideals in R which are contained in p.

(b) Show that R_p is a local ring (see D&F, p. 259, #37).