About the fundamental class of a flat scheme-map

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Outline

1. Introduction: duality, abstract and concrete.
2. Fundamental class
3. Fundamental class and traces of differential forms.
4. Pseudofunctoriality of the fundamental class.

1. Introduction.

The fundamental class of an essentially-finite-type, separated, flat map of noetherian schemes $f\colon X \to Y$ with diagonal $\delta\colon X \to X \times_Y X$ will be explicated below as a $\mathsf{D}(X)$(:= derived-category)-map

$$\mathcal{H}_f := \mathsf{L}\delta^*\delta_*\mathcal{O}_X \to f^!\mathcal{O}_Y$$

where $\mathcal{H}_f$ is the Hochschild complex of f and $f^!\mathcal{O}_Y$ (with $f^!$ as in Grothendieck duality theory) is the relative dualizing complex.

It is a multifaceted intermediary—via canonical (up to sign) $\mathcal{O}_X$-maps

$$\Omega^i_f \to H^{-i}\mathsf{L}\delta^*\delta_*\mathcal{O}_X \qquad (i \in \mathbb{Z}) \tag{1}$$

from differential i-forms to Hochschild homology—between concrete aspects of differentials (residues, traces...) and abstract duality theory.

We'll go gradually toward stating its definition and basic properties, beginning with some historical and motivational background.

(1864) **Roch**'s piece of Riemann-Roch (jazzed up):
Let V be a smooth projective curve over $\mathbb{C}$, with sheaf of holomorphic differentials Ω, and F an invertible $\mathcal{O}_V$-module. The finite-dimensional $\mathbb{C}$-vector spaces $\mathrm{H}^1(V, F)$ and $\mathrm{Hom}_V(F, \Omega)$ are dual.

(1931) **Schmidt** (in connection with zeta-functions):
Same with any perfect field in place of $\mathbb{C}$.

(1950s):

Rosenlicht: Same for *any* curve over a perfect field, with Ω replaced by a certain sheaf of meromorphic differentials.

Serre: If V is a normal d-dimensional projective variety over a perfect field k, the reflexive hull of the sheaf $\Omega^d_{V/k}$ of degree-d Kähler differentials represents the functor $\mathrm{Hom}_k(\mathrm{H}^d(V, F), k)$ of coherent $\mathcal{O}_V$-modules F.

Grothendieck: For *any* d-dimensional projective variety V/k (k a field), the functor $\mathrm{Hom}_k(\mathrm{H}^d(V, F), k)$ of coherent $\mathcal{O}_V$-modules F is representable.

Consolidation

Theorem 1

For any d-dimensional variety proper over a perfect field k, the functor $\mathrm{Hom}_k(\mathrm{H}^d(V, F), k)$ of quasi-coherent $\mathcal{O}_V$-modules F has a canonical representing pair $(\omega_{V/k}, \int_{V/k})$, where the canonical module $\omega := \omega_{V/k}$—the sheaf of "regular differential d-forms"—is an $\mathcal{O}_V$-submodule of the constant sheaf $\Omega^d_{k(V)/k}$, containing the image of the natural map $\Omega^d_{V/k} \to \Omega^d_{k(V)/k}$, with equality over the smooth locus, and $\int_{V/k}\colon \mathrm{H}^d(V, \omega) \to k$ is the unique map t such that for each closed $v \in V$, the composition $\mathrm{H}^d_v(\hat{\omega}_v) \overset{\text{nat'l}}{\twoheadrightarrow} \mathrm{H}^d(V, \omega) \xrightarrow{t} k$ $[(\hat{\ })_v :=$ *completion at* $v]$ is the locally describable, canonical residue map res_v such that $(\hat{\omega}_v, \mathrm{res}_v)$ *represents the functor* $\mathrm{Hom}_k(\mathrm{H}^d_v(G), k)$ *of* $\hat{\mathcal{O}}_{V,v}$*-modules* G.

Thus, differentials and residues underlie a *canonical* realization of, and compatibility between, global and local duality.

Avatars of the fundamental class

This explicit version of duality was worked out for projective varieties by **Kunz** in the mid 1970s, and for arbitrary proper varieties by me in 1984 (Astérisque 117). The latter has a full treatment of the fundamental class and its relation to traces, residues and local duality—serving as a model for further developments. For instance, the main results were generalized to equidimensional generically smooth maps of noetherian schemes by **Hübl** and **Sastry** in *Amer. J. Math.* 115 (1993), 749–787.

In these works a low-tech version of duality, not using the derived category, suffices. But the complete arguments are quite lengthy.

Question

Can the concrete results of Hübl and Sastry be deduced quickly from abstract duality theory? And if so, in what generality?

The intent in what follows is to dig deeper, in search of a more wide-ranging approach.

Avatars of the fundamental class (ct'd)

To generalize to arbitrary smooth maps Grothendieck *defined* $f^!$ so that the fundamental class is the identity map; but to prove duality for proper such f, he still had to construct a counit $\mathrm{R}f_*f^! \to 1$.
For this he developed a rather sophisticated trace map between certain residual complexes—which were assumed to exist.
(See e.g., Conrad's SLN 1750, §3.4; and for a generalization to arbitrary Cousin complexes, the paper by **Sastry** in *Contemporary Math.* 375.)

An even more abstract such isomorphism, with Deligne's version of $(-)^!$, and *not requiring residual complexes,* was defined and applied for smooth maps by **Verdier** in:

Algebraic Geometry (Bombay, 1968). Oxford Univ. Press, 1969; 393–408.

Enter derived categories (Grothendieck duality)

Beginning in the late 1950s, Grothendieck formulated a vast generalization of then existing duality theory.

The main result in Hartshorne's Springer Lecture Notes (SLN) 20 (amended by Conrad, SLN 1750), exposing Grothendieck's ideas—via derived categories, à la Verdier—can be summarized as follows.

First, some notation:

• $[d]$ denotes "d-fold degree-shift."

• For a scheme S with bounded-below derived category $\mathrm{D}^+(S)$, $\mathrm{D}_c^+(S) \subset \mathrm{D}(S)$ is the full subcategory spanned by the $\mathcal{O}_X$-complexes with coherent homology sheaves; and similarly when S is replaced by a quasi-coherent $\mathcal{O}_S$-algebra.

• For finite $f\colon X \to Y$, with ringed-space factorization $X \to \overline{Y} := (Y, f_*\mathcal{O}_X)$, $(-)^\sim$ is the natural equivalence $\mathrm{D}_c^+(\overline{Y}) \xrightarrow{\approx} \mathrm{D}_c^+(X)$).

Theorem 2 (cf. SLN20, p. 383, Corollary 3.4)

In the presence of *residual complexes*, $\exists$ contravariant pseudofunctor $(-)^!$ over finite-type separated maps of noetherian schemes, with values in D_c^+,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \Longrightarrow \quad f^! \colon \mathrm{D}_\mathrm{c}^+(Y) \to \mathrm{D}_\mathrm{c}^+(X), \quad f^! g^! \xrightarrow{\sim} (gf)^!, \ \ldots$$

plus, for each proper f, a trace map $T_f \colon \mathrm{R}f_* f^! \to 1$, obtained by a (seemingly miraculous) gluing of the two pseudofunctors

(a) $\quad \Omega_f^d[d] \otimes_X f^* F \qquad$ (*f smooth, of rel. dim. d,* $F \in \mathrm{D}_\mathrm{c}^+(Y)$),

(b) $\quad \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F)^\sim \qquad$ (*f finite,* $F \in \mathrm{D}_\mathrm{c}^+(Y)$),

and their associated trace maps, such that:

(i) If f is étale then $f^!$ is the usual restriction functor f^*; and if, moreover, f is finite, then T_f is the usual trace $\mathrm{R}f_* f^! G \cong f_*\mathcal{O}_X \otimes G \to \mathcal{O}_Y \otimes G = G$.

(ii) (Duality) For proper f, T_f is the counit of an adjunction $\mathrm{R}f_* \dashv f^!$.

We won't dwell on what "residual complexes" are (see SLN 1750, §3.2). Commonly occurring noetherian schemes usually—not always—have them.

Remarks

- The proof in *loc. cit.* omits significant details, see top of p. 153, SLN1750.

- There is more below about (a), (b) and their trace maps.

- Grothendieck's earlier result for $f\colon V \to k$ and F a coherent $\mathcal{O}_V$-module (see above) results from Theorem 2 via natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{H}^d(V,F),k) &\cong \mathrm{Hom}_k(\mathrm{R}f_*F[d],k)\\ &\underset{\text{(ii)}}{\cong} \mathrm{Hom}_{\mathrm{D}(X)}(F[d],f^!k) \cong \mathrm{Hom}_{\mathcal{O}_X}(F,H^{-d}f^!k).\end{aligned}$$

- For simplicity, we haven't mentioned other basic properties of $(-)^!$, such as its interactions with derived $\otimes$ and $\mathcal{H}om$, with flat base change, etc.
In particular, there is a property, w.r.t. étale base change of proper maps that guarantees uniqueness (but not canonicity) of $(-)^!$ and T_f up to unique isomorphism.

Ideal theorem

In the introduction of SLN 20, there is envisioned an "Ideal Theorem," extending Theorem 2 to complexes with *quasi-coherent* homology (replace $\mathrm{D}_{\mathrm{c}}^{+}$ by $\mathrm{D}_{\mathrm{qc}}^{+}$), even when there are no residual complexes.

In the appendix to SLN 20, **Deligne** describes a non-constructive proof of the existence, under these relaxed conditions, of an *abstract* $(-)^!$ satisfying (i) and (ii),[1] hence, by the *concrete* duality theorems in SLN20 for smooth and for finite maps, and by the uniqueness of adjoints, restricting (up to isomorphism) to (a) and (b) on the categories of *proper* smooth maps of finite-dimensional noetherian schemes and of finite maps, respectively.

[1] For an account of Deligne's arguments, see SLN 1960, §4.1. A more conceptual proof, based on an analog of Brown representability, was given by **Neeman** (1996).

Questions and comments

• Does the Ideal Theorem hold for this abstract $(-)^!$?

A path to proving this was opened up by **Verdier** in "Base change for twisted inverse images...", *Algebraic Geometry* (Bombay, 1968). Oxford Univ. Press, 1969; 393–408. but I'm not aware of any complete, detailed exposition.

Such a proof would just be part of a project to

translate abstract Grothendieck duality theory (as initially formulated by **Verdier** and **Deligne**, and exposed in detail in SLN 1960[2]) into concrete terms (as in SLN 20 and SLN 1750).

In particular, Theorem 1, and the complexity of its proof as well as that of some constructions in SLN20, lead one to ask:

• Can the concrete representations (a) and (b) of abstract $(-)^!$ be deduced from (i) and (ii), without the use of residual complexes?

• Can this be done *canonically* (not just up to unique isomorphism), in such a way that Theorem 1 falls out?

[2] see also **Neeman's** paper, `arXiv:1406.7599`.

Canonical concrete $(-)^!$ for finite maps

The case of a finite map $f\colon X \to Y$ is relatively straightforward.
Though proving concrete duality as in SLN20 for such f is not so hard, one can also *deduce* it from abstract duality by showing that the sheafified duality isomorphism

$$f_* f^! F = f_* \mathrm{R}\mathcal{H}om_X(\mathcal{O}_X, f^! F) \xrightarrow{\ \sim\ } \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \qquad (F \in \mathrm{D_{qc}}(Y))$$

(right-conjugate to the *projection isomorphism* $f_* f^* G \xleftarrow{\ \sim\ } G \otimes^{\mathsf{L}}_Y f_*\mathcal{O}_X$)
is actually a $\mathrm{D}(f_*\mathcal{O}_X)$-map, hence is f_* of a unique $\mathrm{D}(X)$-isomorphism (which, one checks, is pseudofunctorial),

$$\gamma_f(F)\colon f^! F \xrightarrow{\ \sim\ } \bar{f}^* \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) = \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F)^{\sim}$$

such that the following $\mathrm{D}(Y)$-diagram commutes:

$$\begin{array}{ccc} f_* f^! F & \xrightarrow[f_*\gamma_f]{\ \sim\ } & \mathrm{R}\mathcal{H}om_Y(f_*\mathcal{O}_X, F) \\ {\scriptstyle T_f}\big\downarrow & & \big\downarrow{\scriptstyle \text{natural}} \\ F & =\!=\!= & \mathrm{R}\mathcal{H}om_Y(\mathcal{O}_Y, F) \end{array}$$

Canonical concrete $(-)^!$ for smooth maps

For smooth $f\colon X \to Y$ of relative dimension d, and a given abstract $(-)^!$, we seek a canonical pseudofunctorial isomorphism

$$v_f(-)\colon f^!(-) \xrightarrow{\ \sim\ } \Omega_f^d[d] \otimes_X f^*(-)$$

where Ω_f^d is the invertible $\mathcal{O}_X$-module of relative Kähler d-forms.

In Theorem 3 of his 1968 paper on flat base change for $(-)^!$, **Verdier** produced[3] an efficient construction of a *functorial* such v_f. But he made no explicit mention of *pseudofunctoriality*—a nontrivial matter, discussed below for the more general fundamental class map.[4]

[3] more or less; a more detailed description of v_f is given in §3 of my paper with Neeman, Alg. Geom. (2018), 131–159.

[4] Recently, **Nayak** and **Sastry** proved pseudofunctoriality of v_f in the context of *formal schemes*, see Prop. 7.2.4 of `arXiv:1903.01779v3`.

Concrete v_f = identity

For an example, consider *concrete* duality, as described in SLN 20, p. 383, Corollary 3.4. There one *identifies* $f^!G$ pseudofunctorially with

$$f^{\#}(G) := \Omega_f^d[d] \otimes_X f^*(G) \qquad (G \in \mathrm{D}(Y)),$$

so that v_f becomes a functorial automorphism of $f^{\#}$.

The claim is that v_f is the identity automorphism.

For this, it suffices that $v_f(\mathcal{O}_Y)$ be the identity (V paper, 3.4.5).

To check that $v_f(\mathcal{O}_Y)$ is the identity, let $\delta\colon X \to Z := X \times_Y X$ be the diagonal, and let $p_i\colon Z \to X$ $(i = 1, 2)$ be the (smooth) canonical projections.
Let $\omega \in \mathrm{D}(X)$ be the invertible $\mathcal{O}_X$-complex

$$\omega := \Omega_f^d[d].$$

For an $\mathcal{O}_X$-complex E, let $E^\vee$ be the $\mathcal{O}_X$-complex $\mathcal{H}om_X(E, \mathcal{O}_X)$.
One has the isomorphism $\omega^\vee \xrightarrow{\sim} (\Omega_f^d)^\vee[-d]$ given by multiplication in degree d by $(-1)^{d(d+1)/2}$.

Concrete $v_f =$ identity, ct'd

The definition in (V paper, 3.1.7), gives that $v_f(\mathcal{O}_Y)$ is the composite map

$$\mathcal{O}_X \otimes_x \omega \xrightarrow[\text{via } \alpha]{\sim} (\omega^\vee \otimes_x \omega) \otimes_x \omega \xrightarrow[\text{via } \beta]{\sim} (\delta^! \mathcal{O}_Z \otimes_x \omega) \otimes_x \omega \xrightarrow[\text{via } \gamma]{\sim} \mathcal{O}_X \otimes_x \omega,$$

where the isomorphism α is the natural one, β comes from *ibid.* (3.1.6), and γ from *ibid.* (3.1.3). That this is the identity means that α^{-1} factors as

$$\omega^\vee \otimes_x \omega \xrightarrow[\beta\otimes 1]{\sim} \delta^! \mathcal{O}_Z \otimes_x \omega \xrightarrow[\gamma]{\sim} \mathcal{O}_X,$$

i.e., the natural diagram (with $\otimes := \otimes_x$ and ψ the isomorphism $\psi_{\delta, p_1}(\mathcal{O}_X)$ in SLN 1750, p. 77, (2.7.2))

$$\begin{array}{ccccc}
\omega^\vee \otimes \omega & \xleftarrow{\ \alpha\ } & \mathcal{O}_X & \xrightarrow{\ \psi\ } & \delta^! p_1^\# \mathcal{O}_X \\
{\scriptstyle \beta \otimes 1}\downarrow & & & & \uparrow \\
\delta^! \mathcal{O}_Z \otimes \omega & \longrightarrow & \delta^! \mathcal{O}_Z \otimes \delta^* p_2^* \omega & \longrightarrow & \delta^! p_2^* \omega
\end{array}$$

commutes. But this commutativity is essentially the definition of ψ.

Desirable properties of v_f: Residue Theorem

In view of the relation between duality and residues given by Theorem1, showing that this composite does indeed correspond to the trace map (say, for simplicity, when $Y = Spec(k)$ (k a perfect field)) involves proving a Residue Theorem, asserting that for each closed point $v \in X$, the natural composite map

$$\mathrm{H}^d_v(\hat{\Omega}^d_{f,v}) \to \mathrm{H}^d(X, \Omega^d_f) \xrightarrow{\text{via } c_f} \mathrm{H}^{-d}(X, f^!k) \xrightarrow{\text{via } T_f} k$$

is the residue map res_v.

In other words, v_f globalizes the residue map.

Desirable properties of v_f: $(-)^!$ and regular differentials

When V is an arbitrary k-variety, the map (**??**) localizes over the smooth locus to an isomorphism, and the $\mathcal{O}_V$-module $H^{-d}f^!k$ is torsion-free.

So with $i\colon \operatorname{Spec}(k(V)) \to V$ the natural map, one gets an *injection*

$$H^{-d}f^!k \hookrightarrow i_*i^*H^{-d}f^!k \xrightarrow{\sim} i_*\Omega^d_{k(V)/k}$$

whose image, one shows, is the above canonical sheaf ω of regular d-forms.

2. Fundamental class

More generally:

For essentially-finite-type flat separated equid'l $f\colon X \to Y$, rel. dim. d, specify and study a canonical pseudofunctorial fundamental class map

$$c_f(F)\colon \Omega_f^d[d] \otimes_X f^*F \longrightarrow f^!F \qquad (F \in \mathrm{D}^+_{\mathrm{qc}}(Y))$$

which is just $v_f(F)^{-1}$ when f is smooth.

The term "fundamental class" reflects an interpretation of such a $c_f(\mathcal{O}_Y)$ when X is a codimension-e cycle in a smooth Y-scheme Z, as an element of $H^e_X(Z, \Omega^e_{Z/Y})$, see e.g., Grothendieck's Séminaire Bourbaki talk (no. 149), May 1957; or Angéniol's SLN 896 (dealing with Chow schemes).

Fundamental class and $\int$

For *proper* f, such a $c_f(F)$ would be dual to a canonical $\mathrm{D}(Y)$-map

$$\int_f(F)\colon \mathrm{R}f_*(\Omega_f^d[d]\otimes_X f^*F) \underset{\text{projn}}{\xrightarrow{\sim}} \mathrm{R}f_*\Omega_f^d[d]\otimes^{\mathsf{L}}_Y F \longrightarrow F = \mathcal{O}_Y\otimes^{\mathsf{L}}_Y F.$$

Thus it would suffice to specify a suitable family of canonical $\mathrm{D}(Y)$-maps

$$\int_f\colon \mathrm{R}f_*\Omega_f^d[d] \longrightarrow \mathcal{O}_Y$$

or, equivalently, $\mathcal{O}_Y$-maps

$$R^d f_*\Omega_f^d := H^d\mathrm{R}f_*\Omega_f^d \longrightarrow \mathcal{O}_Y.$$

The canonical $\int_f$ we have in mind turns out to be closely tied to traces and residues of differentials. So this is ultimately about a canonical framework for relating concrete algebraic phenomena involving differentials—for instance, local duality—to (global) abstract duality.

Note: *non-proper* f are in play too, substantially complicating matters.

Avatars of the fundamental class

As a model, for varieties over perfect fields, *Astèrisque 117* (1984) has a full treatment of the fundamental class and its relation to traces, residues and local duality. This is expanded to more general bases by **Hübl** and **Sastry** in *Amer. J. Math.* 115 (1993), 749–787.

In these works a toned-down version of duality, not using the derived category, suffices. But the complete arguments are quite lengthy.

Question

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Avatars of the fundamental class (ct'd)

To generalize to arbitrary smooth maps Grothendieck *defined* $f^!$ so that the fundamental class is the identity map; but to prove duality for proper such f, he still had to construct a counit $\mathrm{R}f_*f^! \to 1$. For this he developed a rather sophisticated trace map between certain residual complexes—which were assumed to exist. (See e.g., Conrad's SLN 1750, §3.4; and for a generalization to arbitrary Cousin complexes, the paper by **Sastry** in *Contemporary Math.* 375.)

An even more abstract such isomorphism, with Deligne's version of $(-)^!$, and *not requiring residual complexes,* was defined and applied for smooth maps by **Verdier** in:

Algebraic Geometry (Bombay, 1968). Oxford Univ. Press, 1969; 393–408.

Fundamental class via Hochschild homology

Recall the Hochschild complex $\mathcal{H}_f := \mathsf{L}\delta^*\delta_*\mathcal{O}_X$ of a flat scheme-map $f\colon X \to Y$ with diagonal $\delta\colon X \to X \times_Y X$.

With I the kernel of $\mathcal{O}_{X\times_Y X} \to \delta_*\mathcal{O}_X$, there is a natural (up to sign) isomorphism $\Omega_f = I/I^2 \xrightarrow{\sim} H^{-1}\mathcal{H}_f$, that extends to a map of alternating graded algebras $\oplus_i \Omega^i_f \to \oplus_i H^{-i}\mathcal{H}_f$—an isomorphism when f is smooth.

With $p_j\colon X \times_Y X \to X$ ($j = 1, 2$) the projections, consider the following composite $\mathsf{D}(X)$-map from $\mathcal{H}_f$ to the relative dualizing complex $f^!\mathcal{O}_Y$:

$$C_f\colon \mathsf{L}\delta^*\delta_*\mathcal{O}_X \xrightarrow{\sim} \mathsf{L}\delta^*\delta_*\delta^! p_2^!\mathcal{O}_X \longrightarrow \mathsf{L}\delta^* p_2^!\mathcal{O}_X \xrightarrow{\sim} \mathsf{L}\delta^* p_1^* f^!\mathcal{O}_Y \xrightarrow{\sim} f^!\mathcal{O}_Y.$$

The maps here are the natural ones, the third being (inverse to) the flat base-change isomorphism. (This is why f needs to be flat.)

If f is equidimensional, rel. dim. d, then $H^{-e}f^!\mathcal{O}_Y = 0$ for all $e > d$, so there is a natural $\mathsf{D}(X)$-map $H^{-d}f^!\mathcal{O}_Y \to f^!\mathcal{O}_Y$.
Thus we have a composite $\mathsf{D}(X)$-map

$$c_f\colon \Omega^d_f[d] \to H^{-d}\mathcal{H}_f \xrightarrow{H^{-d}C_f} H^{-d}f^!\mathcal{O}_Y \to f^!\mathcal{O}_Y.$$

Fundamental class via Hochschild (ct'd)

Examples

1. The maps C_f and c_f "commute" with localization on both X and Y.

2. The map c_f extends Verdier's isomorphism for $F = \mathcal{O}_Y$ to arbitrary flat equidimensional maps: for smooth f, the two maps coincide.
(Proof not trivial—see paper by me and **Neeman** in
Algebraic Geometry 5 (2018), 131–159.)
In particular, c_f is an isomorphism when f is smooth.

3. In the affine case, say $f = \mathrm{Spec}(g)$ where $g\colon A \to B$ is a ring-map, the map $\delta_*\mathcal{O}_X \to p_2^!\mathcal{O}_X$ underlying the first two arrows in the definition of c_f sheafifies the natural map $B \otimes_{B\otimes_A B} \bar{\mu}$ with $\bar{\mu}$ the natural composite

$$B \xrightarrow{\mu} \mathrm{Hom}_A(B, B) \to \mathrm{RHom}_A(B, B)$$

where μ takes $b \in B$ to "multiplication by b."
(Proof not trivial—see paper by **Iyengar**, me and **Neeman** in
Compositio Math. 151 (2015), 735–764.)

Examples

4. If in 2., B is *finite* (and flat) over A, then c_f can be identified with the sheafification of the natural composite

$$B \xrightarrow{\mu} \mathrm{Hom}_A(B, B) \to \mathrm{Hom}_A(B, B) \otimes_{B\otimes_A B} B \cong \mathrm{Hom}_A(B, A),$$

which is the B-homomorphism taking $1 \in B$ to the trace map.

Thus for any quasi-finite f, the map c_f globalizes the usual trace map.

Consequence: if f is any Cohen-Macaulay map, then c_f is an isomorphism *only if* f is smooth. (First proved algebraically by **Kunz** and **Waldi**.)

5. If $d = 0$ (i.e., $f\colon X \to Y$ is quasi-finite), then c_f maps $\mathcal{O}_X$ to $f^!\mathcal{O}_Y$.

Chatzistamatiou and Rülling, in a recent Compositio paper, used a subtle variation on this theme—one that can be viewed as a globalization of SLN20's "fundamental local isomorphism"—as a basic tool for resolving outstanding questions about *birational invariance of the cohomology of structure sheaves of excellent regular schemes.*

Their result is generalized by **Kovács** in arXiv:1703.02269, Thm. 8.6.

Remarks. 1. The map C_f can be viewed as an orientation in a bivariant theory of Hochschild homology.

2. There is an involutive ambiguity in the definition of C_f, arising from the fact that if σ is the symmetry automorphism of $X \times_Y X$ then $\sigma_* C_f = \sigma^* C_f \neq C_f$.

There is another ambiguity in the choice of sign of the map $I/I^2 \to H^{-1}\mathcal{H}_f$ entering into the definition of c_f.

These annoying sign problems will be ignored in what follows; but they must eventually be dealt with.

For such questions, the following theorem of **Angéniol** and **El Zein**, a concrete characterization, via standard traces, of the fundamental class (in extended circumstances), might well be useful.

3. Fundamental class and traces of differential forms

Theorem (Mémoires de la S. M. F., 58 (1978), p. 81)

Given $f\colon X \to S$ *equidimensional, rel. dim.* d*, and of finite tor-dim., suppose either that* f *is Gorenstein or that* S *is a* $\mathbb{Q}$*-scheme. There is a unique* $\mathrm{D}(X)$*-map* $c_f\colon \Omega_f^d[d] \to f^!\mathcal{O}_S$ *having the localized trace property: for any commutative*

$$\begin{array}{ccc} U & \xrightarrow{\ \ i\ \ } & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{\ \ g\ \ } & S \end{array}$$

with i *étale,* h *finite and* g *smooth of rel. dim.* d*, the map dual to the natural composite*

$$h_*h^*\Omega_g^d[d] \cong h_*\mathcal{O}_U \otimes \Omega_g^d[d] \xrightarrow{\text{trace}\otimes 1} \mathcal{O}_Y \otimes \Omega_g^d[d] = \Omega_g^d[d]$$

is the natural composite

$$h^*\Omega_g^d[d] \to \Omega_{gh}^d[d] \cong i^*\Omega_f^d[d] \xrightarrow{i^*c_f} i^*f^!\mathcal{O}_S \cong (fi)^!\mathcal{O}_S \cong h^!g^!\mathcal{O}_S = h^!\Omega_g^d[d].$$

Remarks.

1. Application of the homology functor H^{-d} gives a *group isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(X)}(\Omega_f^d[d], f^!\mathcal{O}_S) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\Omega_f^d, H^{-d}f^!\mathcal{O}_S).$$

So c_f can be viewed as a canonical map from $\Omega_f^d[d]$ to $H^{-d}f^!\mathcal{O}_S$, the canonical (up to isomorphism) module of f; and this points to the existence of a canonical (absolutely) module consisting of certain meromorphic differential forms—"regular" differentials, see below.

2. trace: $h_*\mathcal{O}_U \to \mathcal{O}_Y$ exists since finite tor-dimensionality of *fi* and smoothness of g force $h_*\mathcal{O}_U \in \mathrm{D}(Y)$ to be perfect, and for perfect complexes such traces can be defined.

Remarks (ct'd)

4. The restriction on the characteristic is due to their using in the *existence* proof a theorem of Bott about Grassmannians.
The *uniqueness* proof doesn't need this restriction. CHECK!

5. For *flat* f we will indicate how the previously described "fundamental class via Hochschild" provides a characteristic-free generalization (not involving Grassmannians or G's trace) of A-E's map.

Fundamental class and traces (ct'd)

If f itself factors as $X \xrightarrow{h} Y \xrightarrow{g} S$ (h finite, g smooth) then c_f (which is independent of h) is dual to an $\mathcal{O}_Y$-map

$$h_*\Omega^d_f[d] \to \Omega^d_g[d]$$

called the trace map for differential forms.

As often happens when abstract duality is interpreted in concrete terms, finding *explicit algebraic formulae* for this map is not easy when h is not generically étale. (The "Cartier operator" in positive characteristic is a special case.) This is not the place for details. But the general idea can be gleaned from Prop. 6.3.1 in **Angéniol** and **Lejeune-Jalabert**'s *Calcul différentiel et class caractéristiques...*, Herrmann, Paris, (1989).

Fundamental class and traces (ct'd)

For varieties over any perfect field, there is a treatment in *Astèrisque 117* of the fundamental class vis-à-vis such traces, that does not depend on the derived-category version of duality. This is done for more general bases by **Sastry** and **Hübl** in *Amer. J. Math.* 115 (1993), 749–787.
In these situations, one can stick to generically étale h.

An extensive *purely algebraic* study of traces of differentials appears in **Kunz** and **Waldi**'s *Regular differential forms*, *Contemp. Math.* 79, (1988). This is motivated by, but has little technical help from, duality theory.

Regular differentials–canonical dualizing sheaf

If the flat equidimensional map $f : X \to S$ is generically smooth, then c_f (which commutes with localization) becomes an isomorphism over a dense open subset, say $u : U \hookrightarrow X$. The canonical (up to isomorphism) module $\omega_f := H^{-d} f^! \mathcal{O}_S$, being isomorphic to some $\mathcal{H}om$, is torsion-free, and so is isomorphic to its natural image in $u_* u^* \omega_f = u_* u^* \Omega_f^d$. This image is a *truly canonical* coherent sheaf $\tilde{\omega}_f$ of meromorphic d-forms, the sheaf of regular differentials.

The sheaf $\tilde{\omega}_f$ represents the functor $\mathrm{Hom}_S(R^d f_* M, \mathcal{O}_S)$ of coherent (or even quasi-coherent) $\mathcal{O}_X$-modules M, since the isomorphic sheaf ω_f does so.

Examples

1. For any fixed finite S-map $h\colon X \to Y$ with Y smooth over S, one finds that: a meromorphic differential ν is regular iff

$$GENERIC\,\mathrm{trace}\,h_*(\mathcal{O}_X\nu) \subset \mathcal{O}_Y.$$

For example, if the nonsmooth locus of f has *depth* ≥ 2 in X then the sheaf of regular differentials is Ω_f^{d**} where $d = rel.\ dim.\ f$ and $*$ is the functor $\mathcal{H}om(-, \mathcal{O}_X)$.

2. When $Y = S$, then $d = 0$ and $\tilde{\omega}_f$ is just the *Dedekind different*.

Residue Theorem–vague remarks

In a related vein, there is a close connection between fundamental classes and residues.

This was hinted at by Grothendieck, in his 1958 Edinburgh talk, where "residual complexes" first appeared.

Residues also appear in the work of Angéniol and El Zein.

As mentioned before, *Astérisque 117*, and more generally Hübl and Sastry in Amer. J. Math. 115 (1993), treat, concretely, the case of varieties over a perfect field. FIX The main result there, the Residue Theorem reifies c_f as a globalization of the local residue maps at the points of X, leading to explicit versions of local and global duality and their relation.

The relation between the fundamental class and residues becomes clearer, and more general, over formal schemes, where local and global duality merge into a single theory with fundamental classes and residues conjoined. (A complete exposition has yet to appear.)

Trace via Hochschild

In §4.5 of *Contemporary Math.* **61**, there is defined (in essence), relative to a pair of flat maps

$$\mathrm{Spec}(S) \xrightarrow{h} \mathrm{Spec}(R) \xrightarrow{g} \mathrm{Spec}(A)$$

with h finite and g equidimensional, of rel. dim. d,
a $\mathrm{D}(\mathrm{Spec}(R))$-map, the Hochschild trace,

$$\tau\colon h_*\mathcal{H}_{gh} \to \mathcal{H}_g.$$

As discussed there, in a number of cases—for instance, if g is smooth—there results a natural commutative diagram

$$\begin{array}{ccc} h_*\Omega^d_{gh}[d] & \xrightarrow{\text{trace}} & \Omega_g[d] \\ \downarrow & & \downarrow \\ h_*\mathcal{H}_{gh} & \xrightarrow[\tau]{} & \mathcal{H}_g \end{array}$$

Trace via Hochschild (ct'd)

These *local considerations can be globalized,* via simplicial arguments applied to an affine open covering, such as were used by **Swan** in his 1997 paper on Hochschild homology of schemes.

(The difficulty to overcome is that, while the Hochschild homology sheaves are quasicoherent, the sheaf of *bar resolutions* used to define τ is not, so that standard pasting methods aren't sufficient.)

Recall that the Hochschild fundamental class of $f\colon X \to Y$ is compatible with étale localization on both X and Y.

Hochschild trace theorem

This all suggests the following assertion (proof in progress!), stated, for simplicity, for finite-type maps, but which can be extended to yield a characteristic-free generalization of the Angéniol-El Zein theorem.

Theorem

Let $X \xrightarrow{h} Y \xrightarrow{g} Z$ be flat separated finite-type maps of noetherian schemes, equidimensional of rel. dim. 0 and d respectively. The following natural diagram commutes.

$$
\begin{array}{ccc}
h_*\mathcal{H}_{gh} & \xrightarrow{\;c_{gh}\text{ via Hochschild}\;} & h_*(gh)^!\mathcal{O}_Z \\
 & & \Vert \\
\Big\downarrow \tau & & h_*h^!g^!\mathcal{O}_Z \\
 & & \Big\downarrow \\
\mathcal{H}_g & \xrightarrow[\;c_{gh}\text{ via Hochschild}\;]{} & g^!\mathcal{O}_Z
\end{array}
$$

The *strategy of the proof* is to reduce to the case of affine schemes, where explicit descriptions of the maps in play are available (see previous remarks for c_f), at which point what remains is to prove commutativity of a concrete—though complicated—diagram in the derived category of a ring.

4. Pseudofunctoriality of the fundamental class

Pseudofunctoriality of the fundamental class can be explained as follows.

To begin, there is a natural bifunctorial map

$$\chi_f(E,F)\colon f^!E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F \longrightarrow f^!(E \otimes^{\mathsf{L}}_Y F) \qquad (E, F \in \mathsf{D}^+_{\mathrm{qc}}(Y)).$$

For *proper* f, this is dual to the composite map

$$\mathsf{R}f_*(f^!E \otimes^{\mathsf{L}}_X \mathsf{L}f^*F) \xrightarrow[\text{projection}]{\sim} \mathsf{R}f_*f^!E \otimes^{\mathsf{L}}_Y F \xrightarrow[\text{counit}\otimes 1]{} E \otimes^{\mathsf{L}}_Y F;$$

and it extends to general (essentially-)finite-type maps via (**Nayak**'s generalization of) Nagata's compactification theorem.

One verifies that this map is actually pseudofunctorial.

Also known (but not needed here):
$\chi_f(E,F)$ *is an isomorphism* $\Leftrightarrow$ f *has finite tor. dim.*

Pseudofunctoriality (ct'd)

Assume further that the fundamental class c_f satisfies a condition of compatibility with $\otimes$, namely that it be equal to the composite

$$\Omega_f^d[d] \otimes_X^{\mathsf{L}} \mathsf{L}f^* \xrightarrow{c_f(\mathcal{O}_Y)\otimes^{\mathsf{L}} 1} f^!\mathcal{O}_Y \otimes_X^{\mathsf{L}} \mathsf{L}f^* \xrightarrow{\chi_f} f^! \qquad (d = \text{rel. dim. } f)$$

This condition does hold for all the preceding avatars of c_f.

Thus for maps of finite tor. dim., can substitute $c_f(\mathcal{O}_Y) \otimes^{\mathsf{L}} 1$ for c_f.

Reduced then to showing, for equidimensional maps of finite tor. dim.:

$$\Omega_f^d[d] \otimes_X^{\mathsf{L}} \mathsf{L}f^*F \xrightarrow{c_f(\mathcal{O}_Y)\otimes^{\mathsf{L}} 1} f^!\mathcal{O}_Y \otimes_X^{\mathsf{L}} \mathsf{L}f^*F \qquad (d = \text{rel. dim.} f)$$

behaves pseudofunctorially w.r.t. composites $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$.

Pseudofunctoriality (ct'd)

Modest technical manipulations (omitted here) reduce the problem to: show commutativity of the next diagram, relating three maps,

$$
\begin{array}{ccc}
\Omega_f^d[d] \otimes_X \mathsf{L}f^*\Omega_g^e[e] & \xrightarrow{\text{via fund'l class}} & f^!\mathcal{O}_Y \otimes_X \mathsf{L}f^*g^!\mathcal{O}_Z \\
 & & \big\downarrow \chi_f \\
\text{natural}\ \Big\downarrow & & f^!g^!\mathcal{O}_Z \\
 & & \big\downarrow \simeq \\
\Omega_{gf}^{d+e}[d+e] & \xrightarrow[\text{fund'l class}]{} & (gf)^!\mathcal{O}_Z
\end{array}
$$

This was done for *proper* maps in joint work with **Sastry**, J. Alg. Geom. (1992), 101–130. The proof is nontrivial, making use of the connection between fundamental classes and residues, local duality, etc.

At the end of that paper, a sketch is given of how to extend to nonproper maps. But details have yet to appear.

Extensions

The theory of $(-)^!$ extends to maps of noetherian formal schemes, see Contemporary Math. 244 (1999);

and to essentially-finite-type separated maps of noetherian schemes (in particular, to Commutative Local Algebra), see **Nayak**'s paper Advances in Mathematics 222 (2009), 527–546;

and to numerous other contexts ...
(as part of Grothendieck's "six operations")

and, *conjecturally*, to derived algebraic geometry, which encodes the homotopical rather than the possibly less basic homological features of the theory. In that context, flatness is more-or-less built in, so is no longer needed in the definition of c_f.
Some steps in this direction have been taken by **Shaul** in his study of duality for differential graded rings, see
Advances in Mathematics 320 (2017), 279–328.