Equisingularity and Simultaneous Resolution of Singularities

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Abstract

Zariski defined equisingularity on an n-dimensional hypersurface V via stratification by "dimensionality type," an integer associated to a point by means of a generic local projection to affine n-space. A possibly more intuitive concept of equisingularity can be based on stratification by simultaneous resolvability of singularities. The two approaches are known to be equivalent for families of plane curve singularities. In higher dimension we ask whether constancy of dimensionality type along a smooth subvariety W of V implies the existence of a simultaneous resolution of the singularities of V along W. (The converse is false.)

The underlying idea is to follow the classical inductive strategy of Jung—begin by desingularizing the discriminant of a generic projection—to reduce to asking if there is a canonical resolution process which when applied to quasi-ordinary singularities depends only on their characteristic monomials. This appears to be so in dimension 2. In higher dimensions the question is quite open.

1 Introduction—equisingular stratifications

The purpose, mainly expository and speculative, of this paper—an outgrowth of a survey lecture at the September 1997 Obergurgl working week—is to indicate some (not all) of the efforts that have been made to interpret *equisingularity,* and connections among them; and to suggest directions for further exploration.

The term "equisingularity" has various connotations. Common to these is the idea of stratifying an algebraic or analytic $\mathbb{C}$-variety V in such a way that along each stratum the points are, as singularities of V, equivalent in some pleasing sense, and somehow get worse as one passes from a stratum to its boundary. A stratification of V is among other things a partition into a locally finite family of submanifolds, the strata (see §2), so that whatever "equivalence" is taken to mean, there should be, locally on V, only finitely many equivalence classes of singularities.

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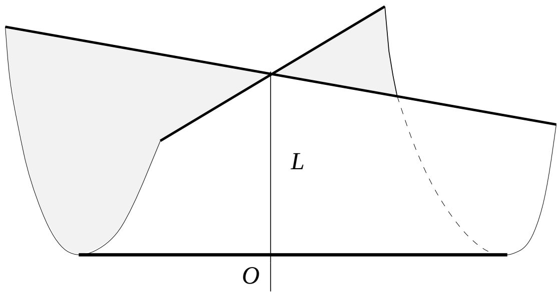
Locally along any stratum V gives rise to families of singularities, realized as the fibers of a retraction onto the stratum. Accordingly one also thinks about equisingularity in terms of some kind of equivalence of the individual singularities in a family. Equivalence may be specified, for example, by equality of some numerical invariants.

Here are some simple suggestive examples, and a little more background.

Examples. (a) The surface V in $\mathbb{C}^3$ given by the equation $X^3 = TY^2$ has singular locus $L\colon X = Y = 0$ (the T-axis), along which it has multiplicity 2 except at the origin O, where it has multiplicity 3. So there is something quite special about O. Interpreting V along L as a family of curve germs in the (X, Y)-plane, with parameter T, makes it plausible that the equisingular strata ought to be $V - L$, $L - \{O\}$, and $\{O\}$. Changing coordinates changes the family, but not the generic member, and not the fact that something special happens at the origin.

(b) Replacing X^3 by X^2, we get multiplicity 2 everywhere along L, including O. The corresponding family of plane curve germs consists of a pair of intersecting lines degenerating to a double line; so there is still something markedly special about O—a feeling reinforced by the following picture.

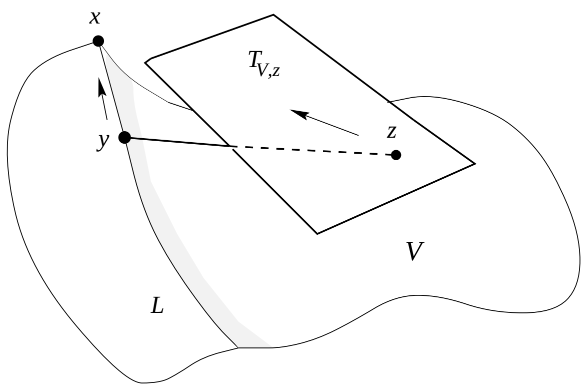


This indicates that equimultiplicity along L, while presumably *necessary*, is *not sufficient* for equisingularity.

(c) The surface $V\colon X(X+Y)(X-Y)(TX+Y) = 0$, along its singular locus $L\colon X = Y = 0$, can be regarded as a family of plane-curve germs all of which look topologically the same, namely four distinct lines through a point. Thus we have *topological equisingularity* along L. (In this example the homeomorphisms involved can be made piecewise linear, hence *bi-Lipschitz;* see e.g., [T2, pp. 349–361], [Z3, Introduction, §2, pp. 4–7], and [P] for more on Lipschitz equisingularity.) From this point of view, the equisingular strata should be $V - L$ and L. However, from the analytic—or even differentiable—point of view, all these germs are distinct, since the cross-ratio of the four lines varies with T. Thus differential isomorphism is too stringent a condition for equisingularity—there are too many equivalence classes.

(d) The notion of *Whitney-equisingularity* (alias *differential equisingularity*) of V along L at a smooth point x of L is associated with the Whitney conditions

$\mathfrak{W}(V, L)$ holding at x. These conditions signify that if $y \in L$ and $z \in V - \operatorname{Sing}(V)$ (where $\operatorname{Sing}(V)$ is the singular locus of V) approach x in such a way that the tangent space $T_{V,z}$ and the line joining y to z both have limiting positions, then the limit of the tangent spaces contains the limit of the lines. (The conditions can be described intrinsically, i.e., as a condition on the prime ideal of L in the local ring $\mathcal{O}_{V,x}$, see proof of Theorem 3.2 below.)



The conditions fail in example (b) above for $y = (X_y, Y_y, T_y) = (0, 0, t^{1/2})$ and $z = (X_z, Y_z, T_z) := (t^2, t, t^2)$, the limiting tangent space as $t \to 0$ being the plane $T = 2X$, while the limiting line is just $L\colon X = Y = 0$.

The conditions $\mathfrak{W}(V, L)$ by themselves are not enough: on the surface in $\mathbb{C}^3$ given by $X^2 = T^2Y^3$, $\mathfrak{W}(V, L_i)$ holds for $L_1 : X = T = 0$ and $L_2 : X = Y = 0$, but topological equisingularity fails at the origin along either of these components of the singular locus [Z1, p. 488]. One should require that a neighborhood of x in L be contained in a stratum of some Whitney stratification—a stratification such that $\mathfrak{W}(\overline{S_1}, S_2)$ holds everywhere along S_2 for any strata S_1, S_2 with $\overline{S_1} \supset S_2$.

Indeed, Thom and Mather showed that a Whitney stratification of V is topologically equisingular in the following sense: for any point x of a stratum S, the germ (V, x) together with its induced stratification is topologically the product of (S, x) with a stratified germ $(V_0, x) \subset (V, x)$ in which x is a stratum by itself (see [M1, p. 202, (2.7), and p. 220, Corollary (8.4)]).

(e) The obvious formulation of a converse to the Thom-Mather theorem is false: in [BS], Briançon and Speder showed that the family of surface germs

$$Z^5 + TY^6Z + Y^7X + X^{15} = 0$$

(each member, for small T, having an isolated singularity at the origin) is topologically equisingular, but not differentially so. On the other hand, there is a beautiful converse, due to Lê and Teissier, equating topological and Whitney-equisingularity of the totality of members of the family $(V \cap H, L, x)$ as H ranges over general smooth germs containing L at x (see [LT, §5], [T2, p. 480, §4]).

Though not the principal focus of this paper, the approach to equisingularity via $\mathfrak{W}$ is the most extensively explored one. Whitney stratifications are of basic importance, for example, in the classification theory of differentiable maps ([M1], [M2], [GL]), in D-module theory and the solution of the Riemann-Hilbert problem ([LM]), and in the Goresky-Macpherson theory of intersection homology ([Mc], [GMc]). (See also [DM], for the existence of Whitney stratifications in a general class of geometric categories.)

Hironaka proved that a Whitney stratification is *equimultiple:* for any two strata S_1, S_2, the closure $\overline{S_1}$ has the same multiplicity (possibly 0) at every point of S_2 [H1, p. 137, 6.2]. Work of Lê and Teissier in the early 1980's led to characterizations of Whitney-equisingularity by the constancy of a finite sequence of "polar multiplicities" [T2], [T3]. In the 1990's, work by Gaffney and others in the context of subfamilies of families of complete intersection singularities has resulted in characterizations of Whitney-equisingularity in families via constancy of sequences of Milnor numbers of the fibers, or constancy of multiplicities of Jacobian modules. (For this, and much more, see, e.g., [GM], [K]).

It is therefore hard to envision an acceptable definition of equisingularity which does not at least *imply* Whitney-equisingularity. But one may still wish to think about conditions which reflect the analytic—not just the differential—structure of V. Indeed, §§3–5 of this paper are devoted to describing two plausible formulations of "analytic equisingularity" and to comparing them with each other and with Whitney-equisingularity.

In brief—and informally—we regard equisingularity theory as being *the study of conditions which give rise to a satisfying notion of "equisingular stratification," and of connections among them.*

In practice, such conditions on a variety-germ V often reflect local constancy of some numerical characters of singularities in a family whose total space is V.

2 Stratifying conditions

To introduce more precision into the preceding vague indications about equisingularity (see e.g., 2.8), we spend a few pages on basic remarks about stratifications. Knowledgeable readers may prefer going directly to §3.

There is a good summary of the origins of stratification theory in pp. 33–44 of [GMc]. What follows consists mostly of variants of material in [T2, pp. 382–402], and is straightforward to verify. The main result, Proposition 2.7, generalizes [T2, pp. 478–480] (which treats the case of Whitney stratifications).

2.1. We work either in the category of reduced complex-analytic spaces or in its subcategory of reduced finite-type algebraic $\mathbb{C}$-schemes; in either case we refer to objects V simply as "varieties." For any such V the set $\mathrm{Sing}(V)$ of singular (non-smooth) points is a closed subvariety of V. A *locally closed subvariety of* V is a subset W each of whose points has an open neighborhood $U \subset V$ whose intersection with W is a closed subvariety of U. Such a W is open in its closure.

By a *partition* (analytic, resp. algebraic) of a variety V we mean a locally finite family (P_α) of non-empty subsets of V such that for each α both the closure $\overline{P_\alpha}$ and the boundary $\partial P_\alpha := \overline{P_\alpha} - P_\alpha$ are closed subvarieties of V (so that P_α is a locally closed subvariety of V), such that $P_{\alpha_1} \cap P_{\alpha_2} = \emptyset$ whenever $\alpha_1 \neq \alpha_2$, and such that $V = \cup_\alpha P_\alpha$. The P_α will be referred to as the *parts* of the partition.

For example, a partition may consist of the fibers of an upper-semicontinuous function μ from V into a well-ordered set I, where "upper-semicontinuous" means that for all $i \in I$, $\{\, x \in V \mid \mu(x) \geq i \,\}$ is a closed subvariety of V.

We say that a partition $\mathfrak{P}_1$ *refines* a partition $\mathfrak{P}_2$, or write $\mathfrak{P}_1 \prec \mathfrak{P}_2$, if, $\mathfrak{P}_1$ and $\mathfrak{P}_2$ being identified with equivalence relations on V—subsets of $V \times V$—we have $\mathfrak{P}_1 \subset \mathfrak{P}_2$. So $\mathfrak{P}_1 \prec \mathfrak{P}_2$ iff the following (equivalent) conditions hold:

(i) Every $\mathfrak{P}_1$-part is contained in a $\mathfrak{P}_2$-part.

(i)$'$ With $\mathfrak{P}_x$ denoting the $\mathfrak{P}$-part containing x, $\mathfrak{P}_{1,x} \subset \mathfrak{P}_{2,x}$ for all $x \in V$.

(ii) Every $\mathfrak{P}_2$-part is a union of $\mathfrak{P}_1$-parts.

Let $\mathfrak{P}$ be a partition of V, and let $W \subset V$ be an *irreducible* locally closed subvariety. For any $\mathfrak{P}$-part P, $P \cap W = \overline{P} \cap W - \partial P \cap W$ is the difference of two closed subvarieties of W, so its closure in W is a closed subvariety of W, either equal to W or of lower dimension than W; and W is the union of the locally finite family of all such closures, hence equal to one of them. Consequently there is a unique part—denoted $\mathfrak{P}_W$—whose intersection with W is dense in W. Moreover, $W - \mathfrak{P}_W = \partial\mathfrak{P}_W \cap W$ is a proper closed subvariety of W.

2.2. Condition (ii) in the following definition of stratification is non-standard—but suits a discussion of equisingularity (and may be necessary for Proposition 2.7).

Definition. A *stratification* (analytic, resp. algebraic) of V is a decomposition into a locally finite disjoint union of non-empty, connected, locally closed subvarieties, the *strata,* satisfying:

(i) For any stratum S, with closure $\overline{S}$, the boundary $\partial S := \overline{S} - S$ is a union of strata (and hence is a closed subvariety of V).

(ii) For any $\overline{S}$ as in (i), $\mathrm{Sing}(\overline{S})$ is a union of strata.

From (i) it follows that a stratification is a partition, whose parts are the strata. Noting that a subset $Z \subset V$ is a union of strata iff Z contains every stratum which it meets, and that $\mathrm{Sing}(\overline{S})$ is nowhere dense in $\overline{S}$, we deduce:

(iii) Every stratum is smooth.

The strata of codimension i are called i-strata.

For stratifications $\mathfrak{S}_1, \mathfrak{S}_2$, we have $\mathfrak{S}_1 \prec \mathfrak{S}_2 \Leftrightarrow$ the closure of any $\mathfrak{S}_2$-stratum is the closure of a $\mathfrak{S}_1$-stratum.

2.3. One way to give a stratification is via a filtration

$$V = V_0 \supset V_1 \supset V_2 \supset \dots$$

by closed subvarieties such that:

(1) For all $i \geq 0$, V_{i+1} contains $\mathrm{Sing}(V_i)$, but contains no component of V_i. In other words, $V_i - V_{i+1}$ is a dense submanifold of V_i.

The strata are the connected components of $V_i - V_{i+1}$ $(i \geq 0)$, and their closures are the irreducible components of the V_i. (Note that (1) forces the germs of the V_i at any $x \in V$ to have strictly decreasing dimension, whence $\cap V_i$ is empty.) It follows that a closed subset Z of V is a union of strata $\Leftrightarrow$ for any component W of any V_i, if $W \not\subset Z$ then $W \cap Z \subset V_{i+1}$. So for (i) and (ii) above to hold we need:

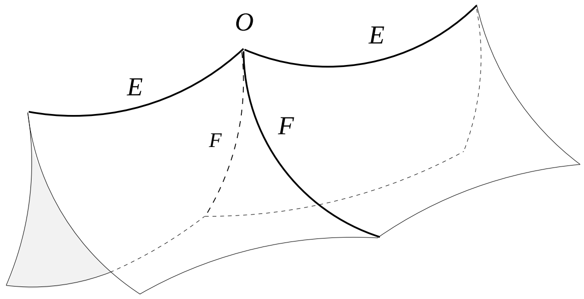
(2) If W, W' are irreducible components of V_i, V_j respectively and $W \not\subset W'$ then $W \cap W' \subset V_{i+1}$, and if $W \not\subset \mathrm{Sing}(W')$ then $W \cap \mathrm{Sing}(W') \subset V_{i+1}$.

For any stratification, let $0 = n_0 < n_1 < n_2 < \dots$ be the integers occurring as codimensions of strata. Redefine $V_i :=$ union of all strata of codimension $\geq n_i$, thereby obtaining a filtration which gives back the stratification and which satisfies:

(3) There is a sequence of integers $0 = n_0 < n_1 < n_2 < \dots$ such that V_i has pure codimension n_i in V.

Thus there is a one-one correspondence between the set of stratifications and the set of filtrations which satisfy (1), (2) and (3).

In the following illustration, a tent which should be imagined to be bottomless and also to stretch out infinitely in both horizontal directions, V_1 can be taken to be the union of the ridges (labeled E, F), and V_2 to be the vertex O.



To say that an irreducible subvariety W is not contained in a closed subvariety Z is to say that $W \cap Z$ is nowhere dense in W. It follows that the conditions (1), (2) and (3) are local: they hold for a filtration if and only if they hold for the germ of that filtration at every point $x \in V$—and indeed, they can be checked inside the complete local ring $\widehat{\mathcal{O}_{V,x}}$. Moreover if a filtration $\mathfrak{F}_x$ of a germ (V,x) satisfies these conditions, then there is such a filtration $\mathfrak{F}_N$ of an open neighborhood N of x in V, whose germ at x equals $\mathfrak{F}_x$. Thus there is a *sheaf of stratifications* whose stalk at x can be identified with the set of all filtrations $\mathfrak{F}$ of (V,x) satisfying (1), (2) and (3).

2.4. Stratifications can be determined by *local stratifying conditions,* as follows. We consider conditions $C = C(W_1, W_2, x)$ defined for all $x \in V$ and all pairs $(W_1, x) \supset (W_2, x)$ of subgerms of (V,x) with (W_1, x) *equidimensional* (all components of W_1 containing x have the same dimension) and (W_2, x) *smooth.* Such a pair can be thought of as two radical ideals $p_1 \subset p_2$ in the local ring $R := \mathcal{O}_{V,x}$, with R/p_1 equidimensional and R/p_2 regular. For example, $C(W_1, W_2, x)$ might be defined to mean that W_1 is equimultiple along W_2 at x—i.e., the local rings R/p_1 and $(R/p_1)_{p_2}$ have the same multiplicity. Or, $C(W_1, W_2, x)$ could signify that the Whitney conditions $\mathfrak{W}(W_1, W_2)$ hold at x, these conditions being expressible within $\mathcal{O}_{V,x}$ (see §1, Example (d)).

(Recall that there is a natural equivalence between the category of local analytic $\mathbb{C}$-algebras and the category of complex analytic germs—the category of pointed spaces localized with respect to open immersions, i.e., enlarged by the adjunction of formal inverses for all open immersions. A similar equivalence holds in the algebraic context.)

For such a C and for any subvarieties W_1, W_2 of V with W_1 closed and locally equidimensional, and W_2 locally closed, set

$$\begin{aligned} C(W_1, W_2) &:= \big\{\, x \in W_2 \,\big|\, W_2 \text{ smooth at } x, \text{ and if } x \in W_1 \text{ then} \\ &\qquad\qquad \big((W_1, x) \supset (W_2, x) \text{ and } C(W_1, W_2, x)\big) \,\big\}. \\ \not{C}(W_1, W_2) &:= W_2 - C(W_1, W_2). \end{aligned}$$

For example, if W_1 contains no component of W_2 then

$$\not{C}(W_1, W_2) = \mathrm{Sing}(W_2) \cup (W_1 \cap W_2).$$

The condition C is called *stratifying* if for any such W_1 and W_2, $\not{C}(W_1, W_2)$ is contained in a nowhere dense closed subvariety of W_2. (It suffices that this be so whenever W_2 is smooth, connected, and contained in W_1.)

For any $S \subset V$, denote by S^{Zar} the Zariski closure of S, i.e., the smallest closed subvariety of V which contains S. If S is a union of $\mathfrak{S}$-strata for some stratification $\mathfrak{S}$, then S^{Zar} is just the topological closure of S.

A stratification $\mathfrak{S}$ is a *C-stratification* if for any strata S_1, S_2 with $S_2 \subset \overline{S_1}$ it holds that $\not{C}(\overline{S_1}, \overline{S_2})^{\mathrm{Zar}}$ is a union of strata. For *any* two strata S_1, S_2 of a C-stratification, $\not{C}(\overline{S_1}, \overline{S_2})^{\mathrm{Zar}}$ is a union of strata, since

$$\overline{S_1} \not\supset S_2 \implies \not{C}(\overline{S_1}, \overline{S_2}) = \mathrm{Sing}(\overline{S_2}) \cup (\overline{S_1} \cap \overline{S_2});$$

and if moreover C is stratifying then $C(\overline{S_1}, S_2) = S_2$.

For given stratifying C, the condition "$C(\overline{S_1}, S_2) = S_2$ for all $\mathfrak{S}$-strata S_1, S_2" might not by itself ensure that $\mathfrak{S}$ is a C-stratification. If every $\mathfrak{S}$ for which it holds *is* a C-stratification then we'll say that C is a ***good*** stratifying condition. (See, for instance, Example (d) below.) If C is good and if C' is any stratifying condition which implies C, then every C' stratification is a C-stratification.

We will show in 2.6 below that for each stratifying condition C there exists a C-stratification. In fact there exists a *coarsest* C-stratification—one which is refined by all others (see 2.7).

2.5. Examples. (a) If C is the empty condition (i.e., $C(W_1, W_2, x)$ holds for all pairs $(W_1, x) \supset (W_2, x)$ as in §2.4) then C is a good stratifying condition, and every stratification is a C-stratification.

(b) The logical conjunction ("and") of a finite family of stratifying conditions, $\hat{C} := \wedge_{i=1}^{n} C_i$, is again stratifying. If a stratification $\mathfrak{S}$ is a C_i-stratification for all i then $\mathfrak{S}$ is a $\hat{C}$-stratification; but the converse does not always hold. It does hold if each C_i is good—in which case $\hat{C}$ is good too.

(c) Let $\mathfrak{P} = (P_\alpha)$ be a partition of V. For $x \in V$ let α_x be such that $x \in P_{\alpha_x}$. Define $C_{\mathfrak{P}}(W_1, W_2, x)$ to mean that $(W_2, x) \subset (P_{\alpha_x}, x)$. (In this example W_1 plays no role.) Then $C_{\mathfrak{P}}$ is stratifying: as in 2.1, for any irreducible locally closed subvariety $W_2 \subset V$ there is a unique α such that $W_2 - P_\alpha$ is a proper closed subvariety of W_2, and then for any closed locally equidimensional $W_1 \supset W_2$ we have $\not{C}_{\mathfrak{P}}(W_1, W_2) = (W_2 - P_\alpha) \cup \operatorname{Sing}(W_2)$, a nowhere dense closed subvariety of W_2. The condition $C_{\mathfrak{P}}(\overline{S_1}, S_2)$ on a pair of strata of a stratification $\mathfrak{S}$ means that the part $\mathfrak{P}_{S_2}$ of $\mathfrak{P}$ which meets S_2 in a dense subset actually contains S_2, and hence is the only part of $\mathfrak{P}$ meeting S_2. One deduces that *a $C_{\mathfrak{P}}$-stratification is just a stratification which refines $\mathfrak{P}$.*

Exercise. For two partitions $\mathfrak{P}$, $\mathfrak{P}'$, $\big(C_{\mathfrak{P}'} \Rightarrow C_{\mathfrak{P}}\big) \iff \mathfrak{P}' \prec \mathfrak{P}$.

(d) More generally, suppose given a partition $\mathfrak{P}_W$ of W for every equidimensional locally closed subvariety $W \subset V$. (For example, if (P_α) is a partition of V, one can set $\mathfrak{P}_W := (W \cap P_\alpha)$.) With $C_W := C_{\mathfrak{P}_W}$ as in Example (c), define $C(W_1, W_2, x)$ to mean $C_{W_1}(W_1, W_2, x)$. Arguing as before, one finds that C is a stratifying condition on V, and that a C-stratification is one such that, if S is any stratum then each part of $\mathfrak{P}_{\overline{S}}$ is a union of strata—a condition which holds if and only if for any two strata S_1, S_2, we have $C(\overline{S_1}, S_2) = S_2$. Thus this C is a good stratifying condition.

(e) The Whitney conditions $\mathfrak{W}(W_1, W_2, x)$ (see §2.4) are stratifying. This was first shown, of course, by Whitney, [Wh, p. 540, Lemma 19.3]. It was shown much later, by Teissier, that $\not{\mathfrak{W}}(W_1, W_2)$ itself is analytic ([T2, p. 477, Prop. 2.1]. Indeed, the main result in [T2], Theorem 1.2 on p. 455, states in part that the stratifying condition $\mathfrak{W}$ is of the type described in example (d) above, the partition $\mathfrak{P}_W$ being given by the level sets of the polar multiplicity sequence on W.

Proposition-Definition 2.6. *For any stratifying condition C, the following inductively defined filtration of V gives rise, as in 2.3, to a C-stratification $\mathfrak{S}_C$: $V_0 = V$, and for $i > 0$, V_{i+1} is the union of all the $W^* \subset V_i$ such that*

(a) *W^* is a component either of* $\operatorname{Sing}(V_j)$ *for some $j \le i$ or of $\not{C}(W', W)^{\mathrm{Zar}}$ for some components W', W of V_j, V_k respectively, with $j \le k \le i$; and*

(b) *W^* is not a component of V_i.*

Proof. It is clear that V_{i+1} contains every component of $\operatorname{Sing}(V_i)$, but no component of V_i. If W and W' are components of V_i and V_j respectively, and if $j > i$, then $W \cap W' \subset V_{i+1}$; while if $j \le i$ and $W \not\subset W'$ then $W \cap W' \subset \not{C}(W', W) \subset V_{i+1}$, since by the definition of stratifying condition, no component of $\not{C}(W', W)^{\mathrm{Zar}}$ is a component of V_i. So, strata being connected components of $V_i - V_{i+1}$ $(i \ge 0)$, the closure of any stratum—i.e., any component of any V_i—is a union of strata (see 2.3). Now any component W^* of $\operatorname{Sing}(V_j)$ is a component of V_i for some $i > j$: otherwise, by the definition of V_i, $W^* \subset \cap V_i = \emptyset$ (see 2.3). Hence $\operatorname{Sing}(V_j)$ is a union of strata. Similarly, if $W' \supset W$ are closures of strata, and so components of V_j, V_k respectively, with $j \le k$, then $\operatorname{Sing}(W', W)^{\mathrm{Zar}}$ is a union of strata. Thus the filtration does indeed give a C-stratification. ○

Remarks. 1. If $C(W, W, x)$ holds for every smooth point x of every component W of V, then $V_1 = \operatorname{Sing}(V)$.

2. Any stratification $\mathfrak{S}$ is $\mathfrak{S}_C$ for a good C, viz. $C := C_{\mathfrak{S}}$ (Example 2.5(c)). This results at once from Proposition 2.7 below. Or, if $\mathfrak{V}$: $V = V_0^* \supset V_1^* \supset \dots$ is the filtration associated to $\mathfrak{S}$, satisfying conditions (1), (2) and (3) in 2.3, so that $V_i^* \subset V$ has pure codimension, say n_i, then one can check for irreducible components W, W' of V_i^*, V_j^* respectively, that

$$\not{C}(W', W) = \not{C}(W', W)^{\mathrm{Zar}} \subset W \cap V_{i+1}^* = \not{C}(W, W),$$

whence a component of $\not{C}(W', W)$ is a component of V_ℓ^* iff it is the closure of an n_ℓ-stratum on the boundary of W; and it follows that the filtration described in Proposition-Definition 2.6 is identical with $\mathfrak{V}$.

Proposition 2.7. *For any stratifying condition C, $\mathfrak{S}_C$ (defined in 2.6) is the coarsest C-stratification of V—every C-stratification refines $\mathfrak{S}_C$. In particular, if C is* good (§2.4) *then $\mathfrak{S}_C$ is the coarsest among all stratifications such that $C(\overline{S_1}, S_2) = S_2$ for any two strata S_1, S_2.*

Proof. Let $V = V_0 \supset V_1 \supset V_2 \supset \dots$ be as in 2.6, and let $\mathfrak{S}$ be a C-stratification of V. The assertion to be proved is: *If Z is an irreducible component of V_j $(j \ge 0)$, then $Z = \overline{\mathfrak{S}_Z}$, the closure of $\mathfrak{S}_Z$.* (Recall from 2.1 that $\mathfrak{S}_Z$ is the unique stratum containing a dense open subset of Z, and see the last assertion in 2.2.)

For $j = 0$, it is clear that $Z = \overline{\mathfrak{S}_Z}$. Assume the assertion for all $j \le i$. Let Z be a component of V_{i+1}. By 2.6, Z is a component either of (a): $\operatorname{Sing}(V_j)$ $(j \le i)$ or of (b): $\not{C}(W', W)^{\mathrm{Zar}}$ with W' a component of V_j and W a component of V_k $(j \le k \le i)$.

In case (a), the inductive hypothesis gives that every component W'' of V_j is the closure of a $\mathfrak{S}$-stratum, so both W'' and $\mathrm{Sing}(W'')$ are unions of $\mathfrak{S}$-strata (see 2.2); and it follows easily that $\mathrm{Sing}(V_j)$ is a union of $\mathfrak{S}$-strata, one of which must be $\mathfrak{S}_Z$ (because $\mathrm{Sing}(V_j)$ meets $\mathfrak{S}_Z$). Since $Z \subset \overline{\mathfrak{S}_Z} \subset \mathrm{Sing}(V_j)$ and Z is a component of $\mathrm{Sing}(V_j)$, therefore $Z = \overline{\mathfrak{S}_Z}$.

Similarly, in case (b) the inductive hypothesis gives that W' and W are both closures of $\mathfrak{S}$-strata, and so, $\mathfrak{S}$ being a C-stratification, $\not{C}(W', W)^{\mathrm{Zar}}$ is a union of $\mathfrak{S}$-strata, one of which must be $\mathfrak{S}_Z$, whence, as before $Z = \overline{\mathfrak{S}_Z}$. Thus the statement holds for $j = i + 1$, and the Proposition results, by induction. ○

Corollary. *If C and C' are stratifying conditions such that C is good and $C' \Rightarrow C$ then $\mathfrak{S}_{C'} \prec \mathfrak{S}_C$.*

Corollary. (Cf. [Wh, p. 536, Thm. 18.11, and p. 540, Thm. 19.2].) *If C is a good stratifying condition and $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_n$ are partitions of V, there is a coarsest C-stratification refining all the $\mathfrak{P}_i$.*

Proof. The stratifications $C_{\mathfrak{P}_i}$ of Example 2.5(c) are good (see Example 2.5(d)), and so in view of Example 2.5(b)), Proposition 2.7 shows that $\mathfrak{S}_{C \wedge C_{\mathfrak{P}_1} \wedge \cdots \wedge C_{\mathfrak{P}_n}}$ does the job. ○

2.8. For a stratifying condition C say that V is *C-equisingular* at a subgerm (W, x) if there is a C-stratification $\mathfrak{S}$ such that $x \in \mathfrak{S}_W$, the unique $\mathfrak{S}$-stratum containing a dense open subvariety of W—or equivalently, if $(W, x) \subset (\mathfrak{S}_{C,x}, x)$ where $\mathfrak{S}_{C,x}$ is the unique $\mathfrak{S}_C$-stratum containing x.

As a special case of the exercise in Example 2.5(c), it holds for any two stratifying conditions C and C' on V that

$$\begin{aligned}\big(\mathfrak{S}_{C'} \prec \mathfrak{S}_C\big) \iff \big(&\text{for all subgerms } (W,x) \text{ of } V,\\ &V \text{ is } C'\text{-equisingular at } (W,x) \Rightarrow V \text{ is } C\text{-equisingular at } (W,x)\big).\end{aligned}$$

In particular, by the first corollary of Proposition 2.7, if C is good and $C' \Rightarrow C$ then C'-equisingularity implies C-equisingularity.

3 The Zariski stratification

In the early 1960's, Zariski developed a comprehensive theory of equisingularity in codimension 1 (see [Z3]). Let x be a point on a hypersurface V, around which the singular locus W is a smooth manifold of codimension 1. The fibers of any local retraction of V onto W form a family of plane curve germs. Zariski showed that if the singularity type of the members of one such family is constant—where singularity type is determined in the classical sense via characteristic pairs, embedded topology, multiplicity sequence, etc.—then the *Whitney conditions* hold along W at x; and conversely, the Whitney conditions imply *constancy of singularity type* in any such family. In this case, moreover, the Whitney conditions are equivalent to *topological triviality* of V along W near x. Furthermore, if these conditions hold, the singularities of V along W near x can be resolved by blowing up W and its successive strict transforms (along which the conditions continue to hold), in a way corresponding exactly to the desingularization of any of the above plane curve germs by successive blowing up of infinitely near points—and conversely. This last condition can be thought of as *simultaneous desingularization* of any family of plane curve germs arising as the fibers of a retraction.

Unfortunately, in higher codimensions no two of these characterizations of equisingularity in codimension 1 remain equivalent; and anyway there is as yet no definitive sense in which two hypersurface singularities can be said to have the same singularity type. Each of the characterizations suggests a higher-dimensional theory. (See in particular the remarks in §1, following Example (e), pointing to the work of Teissier, Gaffney, etc.) Exploration of the remaining connections among these characterizations leads to interesting open questions, a few of which are stated below.

After some tentative attempts at generalizing the notion of equisingularity to higher codimensions (see e.g., [Z1, p. 487, §4]), Zariski formulated (in essence) the following definition of the dimensionality type $\mathrm{dt}_{\mathrm{Z}}(V,x)$ of a hypersurface germ (V,x)—possibly empty—of dimension d, where $d=-1$ if (V,x) is empty, and otherwise x is a $\mathbb{C}$-rational point of the algebraic or analytic variety V. (One can regard such a (V,x) algebraically as being a local $\mathbb{C}$-algebra of the form $R/(f)$ where R is a formal power-series ring in $d+1$ variables over $\mathbb{C}$ and $0 \neq f \in R$.) Zariski-equisingularity is defined by local constancy of dt_{Z}.

The intuition which inspired Zariski's definition does not lie on the surface. It may have come out of his extensive work on ramification of algebraic functions and on fundamental groups of complements of projective hypersurfaces, or have been partly inspired by Jung's method of desingularization (which begins with a desingularization—assumed, through induction on dimension, to exist—of the discriminant of a general projection.) Anyway, here it is:

$$\begin{aligned} &\mathrm{dt}_{\mathrm{Z}}(V,x) = -1 && \text{if } (V,x) \text{ is empty,} \\ &\mathrm{dt}_{\mathrm{Z}}(V,x) = 1 + \mathrm{dt}_{\mathrm{Z}}(\Delta_\pi, 0) && \text{otherwise,} \end{aligned}$$

where $\pi\colon (V,x) \to (\mathbb{C}^d, 0)$ is a general finite map germ, with discriminant $(\Delta_\pi, 0)$ (the hypersurface subgerm of $(\mathbb{C}^d,0)$ consisting of points over which the fibers of π have less than maximal cardinality, i.e., at whose inverse image π is not étale). Here π is defined by its coordinate functions $\xi_1,\dots,\xi_d$, power series in $d+1$ variables, with linearly independent linear terms; and if (V,x) is then represented—via Weierstrass preparation—by an equation

$$f(Z) = Z^n + a_1(\xi_1,\dots,\xi_d)Z^{n-1} + \cdots + a_n(\xi_1,\dots,\xi_d) = 0 \qquad (a_i \in \mathbb{C}[[\xi_1,\dots,\xi_d]])$$

then Δ_π is given by the vanishing of the Z-discriminant of f, so that $(\Delta_\pi, 0)$ is a hypersurface germ of dimension $d-1$, whose dt_{Z} may be assumed, by induction on dimension, already to have been defined.

A property of finite map germs π holds for *almost all* π if there is a finite set of polynomials in the (infinitely many) coefficients of $\xi_1, \dots, \xi_d$ such that the property holds for all π for whose coefficients these polynomials do not simultaneously vanish. (The coefficients depend on a choice of generators for the maximal ideal of the local $\mathbb{C}$-algebra of (V,x)—i.e., of an embedding of (V,x) into $\mathbb{C}^{d+1}$—but the notion of "holding for almost all π" does not.) It follows from [Z2, p. 490, Proposition 5.3] that $\mathrm{dt}_{\mathrm{Z}}(\Delta_\pi, 0)$ has the same value for almost all π; and that enables the preceding inductive definition of dt_{Z} (which is a variant of the original definition in [Z2, §4]).

It is readily seen that (V,x) is smooth iff Δ_π is empty for almost all π, i.e., $\mathrm{dt}_{\mathrm{Z}}(V,x) = 0$. So $\mathrm{dt}_{\mathrm{Z}}(V,x) = 1$ means that almost all π have smooth discriminant, which amounts to Zariski's classical definition of codimension-1 equisingularity of (V,x) (see e.g., [Z3, pp. 20–21].

For another example, suppose the components $(V_i, 0)$ of $(V,0) \subset (\mathbb{C}^{d+1}, 0)$ are smooth, with *distinct* tangent hyperplanes $\sum_{j=1}^{d+1} a_{ij} X_j = 0$. Then a similar property holds for $(\Delta_\pi, 0) = (\cup_{i \ne i'} \pi(V_i \cap V_{i'}), 0)$ if π is general enough; and it follows by induction on d that $\mathrm{dt}_{\mathrm{Z}}(V,0)$ is one less than the rank of the matrix (a_{ij}).

Hironaka proved, in [H2], that on a d-dimensional *algebraic* $\mathbb{C}$-variety V which is locally embeddable in $\mathbb{C}^{d+1}$, *the function* dt_{Z} *is upper-semicontinuous.* More precisely, it follows from the main Theorem on p. 417 of [H2] and from [Z2, p. 476, Proposition 4.2] that the set

$$V_i = \{\, x \in V \mid \mathrm{dt}_{\mathrm{Z}}(V,x) \ge i \,\} \qquad (i \ge 0)$$

is a closed subvariety of V. Thus we have a partition $\mathfrak{P}_Z$ of V, by the connected components of the fibers of the function dt_{Z}, and correspondingly a stratifying condition $\mathfrak{Z}(W_1, W_2, x)$, defined to mean that dt_{Z} is constant on a neighborhood of x in W_2. This is a good stratifying condition, see Example 2.5(d). We call the stratification $\mathfrak{S}_{\mathfrak{Z}}$ the *Zariski stratification.* By Proposition 2.7, the Zariski stratification is the coarsest stratification with dt_{Z} constant along each stratum.

One would expect to have a similar stratification for *complex-analytic* locally-hypersurface varieties; but this is not explicitly contained in the papers of Zariski and Hironaka. To be sure, I asked Hironaka during the workshop (September, 1997) whether his key semi-continuity proof applies in the analytic case, and he said not necessarily, there are obstructions to overcome. Thus:

Problem 3.1. Investigate the upper-semicontinuity of dt_{Z} on analytic hypersurfaces, and the possibility of extending the Zariski stratification to this context.

Zariski proved in [Z2, §6] that with V_i as above, $V_i - V_{i+1}$ is smooth, of pure codimension i in V [Z2, p. 508, Theorem 6.4]; and that the closure of a part of $\mathfrak{P}_Z$ is a union of parts [Z2, pp. 510–511]. One naturally asks then whether $\mathfrak{P}_Z = \mathfrak{S}_Z$—what is still missing is condition (ii) of Definition 2.2, that the singular locus of the closure of a part is a union of parts. But that is indeed so, as follows from the equimultiplicity assertion in the next Theorem (which assertion implies that any part meeting the set $\mathrm{Sing}(\overline{P_1})$ of points where $\overline{P_1}$ has multiplicity > 1 is entirely contained in $\mathrm{Sing}(\overline{P_1})$).

Theorem 3.2. *Let V be a purely d-dimensional algebraic $\mathbb{C}$-variety, everywhere of embedding dimension $\leq d+1$. Then for any two parts P_1, P_2 of the partition $\mathfrak{P}_Z$ with $\overline{P_1} \supset P_2$, the Whitney conditions $\mathfrak{W}(\overline{P_1}, P_2, x)$ hold at all $x \in P_2$. So $\overline{P_1}$ is equimultiple along P_2, and $\mathfrak{P}_Z = \mathfrak{S}_Z$ is a Whitney stratification of V.*

Remarks. (a) Let us say that V is *Zariski-equisingular* along a subvariety W at a smooth point x of W if dt_Z is constant on a neighborhood of x in W (see [Z2, p. 472]). The equality $\mathfrak{P}_Z = \mathfrak{S}_Z$ entails that Zariski-equisingularity is the same as $\mathfrak{Z}$-equisingularity (see §2.8).

(b) From Theorem 3.2 we see via Thom-Mather that Zariski-equisingularity implies topological triviality of V along W at x, a result originally due to Varchenko (who actually needed only *one* sufficiently general projection with discriminant topologically trivial along the image of W), see [Va].

(c) In connection with the idea that *analytic* equisingularity should involve something more than Whitney-equisingularity, note the example of Briançon and Speder [Z3, p. 3, (2)] showing that dt_Z need not be constant along the strata of a Whitney stratification. In fact, in [BH] Briançon and Henry show that for families of isolated singularities of surfaces in $\mathbb{C}^3$, constancy of dt_Z is equivalent to constancy of the generic polar multiplicities (i.e., Whitney-equisingularity) and of two *additional* numerical invariants of such singularities.

Here is a sketch of a *proof of Theorem 3.2.* We first need a formulation of $\mathfrak{W}(W_1, W_2, x)$ (x a smooth point of $W_1 \subset W_2$) in terms of the local $\mathbb{C}$-algebra $R := \mathcal{O}_{W_1,x}$ and the prime ideal $\mathfrak{p}$ in R corresponding to W_2 (so that $R/\mathfrak{p}$ is a regular local ring). One fairly close to the geometric definition is, in outline, as follows. We assume for simplicity that W_1 is equidimensional, of dimension d. Let m be the maximal ideal of R, and in the Zariski tangent space $T := \mathrm{Hom}_{R/m}(m/m^2, R/m)$ let T_2 be the tangent space of W_2, i.e., the subspace of maps in T vanishing on $(\mathfrak{p} + m^2)/m^2$. Set $W_1' := \mathrm{Spec}(R)$, $W_2' := \mathrm{Spec}(R/\mathfrak{p})$, let $B \to W_1'$ be the blowup of W_1' along W_2', let Ω be the universal finite differential module of $W_1'/\mathbb{C}$, and let $G \to W_1'$ be the Grassmannian which functorially parametrizes rank d locally free quotients of Ω. Then the canonical map $p\colon B \times_{W_1'} G \to W_1'$ is an isomorphism over $V_0 := W_1' - W_2' - \mathrm{Sing}(W_1')$; and we let Y be the closure of $p^{-1}V_0$ in $B \times_{W_1'} G$. (This Y is the "birational join" of B and the so-called Nash modification of W_1'.)

Any $\mathbb{C}$-rational point y of the closed fiber of $Y \to W_1'$ gives rise naturally, via standard universal properties of B and G, to a pair (E, F) of vector subspaces of T, where E contains T_2 as a codimension-1 subspace, and $\dim F = d$: E consists of all maps in T vanishing on the kernel of the surjection $(\mathfrak{p} + m^2)/m^2 \cong \mathfrak{p}/m\mathfrak{p} \twoheadrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is the 1-dimensional quotient corresponding to the W_1'-homomorphism $\mathrm{Spec}(R/m) \to B \subset \mathbb{P}(\mathfrak{p})$ whose image is the projection of y, $\mathbb{P}(\mathfrak{p})$ being the projective bundle parametrizing 1-dimensional quotients of $\mathfrak{p}$; and F is dual to a similarly-obtained d-dimensional quotient of the R/m-vector space $\Omega/m\Omega \cong m/m^2$.

Then $\mathfrak{W}(W_1, W_2, x)$ holds $\iff$

$\mathfrak{W}(R, \mathfrak{p})$: *$E \subset F$ for all (E, F) arising from points in the closed fiber of $Y \to W_1'$.*

(The idea, which can be made precise with some local analytic geometry, is that for any embedding of W_1 into $\mathbb{C}^t$ ($t = \dim T$) with W_2 linear, the set of pairs (E, F) can naturally be identified with the set of limits of sequences (E_n, F_n) defined as follows: let $x_n \to x$ be a sequence in $W_1 - W_2 - \mathrm{Sing}(W_1)$, let E_n be the join of W_2 and x_n—a linear space containing W_2 as a 1-codimensional subspace—and let F_n be the tangent space to W_1 at x_n.)

It is not hard to see that $\mathfrak{W}(R, \mathfrak{p}) \iff \mathfrak{W}(\hat{R}, \mathfrak{p}\hat{R})$.

This leads to a reduction of the proof of Theorem 3.2 to that of the similar statement for the *formal* Zariski partition of $\mathrm{Spec}(\widehat{\mathcal{O}_{V,x}})$. (That partition is by constancy of formal dimensionality type, defined inductively through *generic* finite map germs—those whose coefficients (see above) are independent indeterminates.) Indeed, we have to prove $\mathfrak{W}(R, \mathfrak{p})$ whenever R is the local ring of the closure of a Zariski-stratum W_1 of V at a $\mathbb{C}$-rational point x, and $\mathfrak{p}$ is the prime ideal corresponding to the Zariski-stratum W_2 through x. (We refer—prematurely—to the parts of the Zariski partition as Zariski-strata.) The dimensionality type of the generic point of $\mathrm{Spec}(R)$ is equal to that of all the closed points of W_1, i.e., to the codimension of W_1 in V (see [H2, p. 419, Theorem] and [Z2, p. 508, Thm. 6.4]); and the analogous statement follows for the generic point of each component of $\mathrm{Spec}(\hat{R})$ (see [Z2, p. 507, bottom])—so that each of those components is the closure of a formal Zariski stratum of $\mathrm{Spec}(\widehat{\mathcal{O}_{V,x}})$ ([Z2, p. 480, Thm. 5.1]).

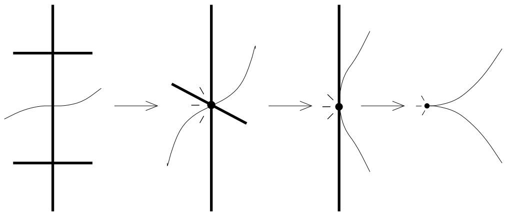
To complete the reduction, observe that if $\widehat{R_1}, \dots, \widehat{R_s}$ are the quotients of $\hat{R}$ by its minimal primes then $\mathfrak{W}(\hat{R}, \mathfrak{p}\hat{R}) \iff \mathfrak{W}(\widehat{R_i}, \mathfrak{p}\widehat{R_i})$ for all $i = 1, 2, \dots, s$. (The corresponding geometric statement is clear.)

Once we are in the formal situation, where $V = \mathrm{Spec}(\mathbb{C}[[X_1, \dots, X_{d+1}]]/(f))$ with f an irreducible formal power series, [Z2, p. 490, Prop. 5.3] gives an embedding of V into $\mathbb{C}^{d+1}$ such that for almost all *linear* projections $\pi\colon \mathbb{C}^{d+1} \to \mathbb{C}^d$, if w is either the generic point of W_1 or the generic point of W_2 or the closed point of V then the dimensionality type of the point πw on the discriminant hypersurface Δ_π of $\pi|V$ is one less than that of w, while the dimension of the closure $\overline{\pi w}$ is the same as the dimension of $\overline{w}$. It follows then from [Z2, p. 491, Prop. 5.4] that $\pi(\overline{W_i})$ $(i = 1, 2)$ is the closure of a Zariski stratum of Δ_π, except when $\overline{W_i} = V$.

At this point we have reached a situation which is much like the one treated by Speder in [S]. In the context of local analytic geometry, Speder shows that a certain version of equisingularity does imply the Whitney conditions. That version is not the same as the one we have called Zariski-equisingularity: while it is also based on an induction involving the discriminant, the induction is with respect to *one* sufficiently general linear projection, whereas Zariski's induction is with respect to *almost all* projections. Nevertheless there are enough similarities in the two approaches that Speder's arguments can now be adapted to give us what we want. Note that Speder asserts only that equisingularity gives the Whitney conditions when $\overline{W_1} = V$; but in view of the remark about $\pi(\overline{W_i})$ in the preceding paragraph, the remaining cases can be treated *in the present situation* by reasoning like that in section IV of [S]. Details (copious) are left to the reader.

4 Equiresolvable stratifications

A standard way to analyze singularities is to *resolve* them, and on the resulting manifold to study the inverse image of the original singularity. For example, after three point-blowups the cusp at the origin of the plane curve $Y^2 = X^3$ is replaced by a configuration of three lines crossing normally, a configuration which represents the complexity of the original singularity.



Quite generally, the embedded topological type of any plane plane curve singularity is determined by the intersection matrix of its inverse image—a collection of normally crossing projective lines—on a minimal embedded desingularization.

This suggests one sense in which a family of singularities could be regarded as being equisingular: if its members can be resolved simultaneously, in such a way that their inverse images all "look the same."

Several ways to make such an idea precise come to mind. Here is one. With notation as in section 2.4, define the *local equiresolvability* condition $ER(W_1, W_2, x)$ to mean there exists an embedding of the germ (W_1, x) into a smooth germ $(M, 0)$, a proper birational (or bimeromorphic) map of manifolds $f\colon M' \to M$ such that $f^{-1}W_1$ is a divisor with smooth components having only normal crossing intersections—so that $f^{-1}W_1$ (considered as a *reduced* variety) has an obvious stratification, by multiplicity—and such that $f^{-1}W_2$ is a union of strata, each of which f makes into a C^∞-fiber bundle, with smooth fibers, over W_2.

Proposition 4.1. (Cf. [T2, p. 401].) *The condition ER is stratifying (see §2.4). Hence, by Proposition 2.7, every variety has a coarsest equiresolvable stratification.*

Proof. Let $W_2 \subset W_1 \subset V$, with W_2 smooth and connected. The question being local, we may embed W_1 in some open $M \subset \mathbb{C}^n$. Let $f_2\colon M_2 \to M$ be the blowup of $W_2 \subset M$, so that $f_2^{-1}W_2$ is a divisor in M_2. Let $f_1\colon M' \to M_2$ be an embedded resolution of $f_2^{-1}W_1$, and set $f := f_2 f_1$, so that $f^{-1}W_1 \subset M'$ is a finite union of normally crossing smooth divisors, as is its subvariety $f^{-1}W_2$ (also a divisor in M'). After removing some proper closed subvarieties from W_2 we may assume that each intersection W of components of $f^{-1}W_2$ is mapped onto W_2 and, by the theorem of Bertini-Sard, that the restriction of f maps each such W *submersively* onto W_2. Now Thom's first isotopy lemma ([GMc, p. 41], or [Ve, p. 311, Thm. 4.14]) applied to the multiplicity stratification—clearly a Whitney stratification—of $f^{-1}W_2$, shows that each stratum is, via f, a smooth C^∞-fiber bundle over W_2. ○

Following Teissier, [T1, p. 107, Définition 3.1.5], we can define a stronger equiresolvability condition ER^+ by adding to ER the requirement that the *not-necessarily-reduced* space $f^{-1}W_2$ be locally analytically trivial over each $w \in W_2$, i.e., germwise the product of W_2 with the fiber $f^{-1}w$. *Condition* ER^+ *is still stratifying:* with notation as in the preceding proof, set $E_2 := f_2^{-1}W_2$, and realize the desingularization f_1 as a composition of blowups $f_{i+1}\colon M_{i+1} \to M_i$ $(i \geq 2)$ of smooth $C_i \subset M_i$ such that if $E_{j+1} := f_j^{-1}(C_j \cup E_j)$ $(j \geq 2)$, then C_i has normal crossings with E_i; then note for any $z \in M_i$ at which both C_i and $(f_2 f_3 \dots f_i)^{-1}W_2$ are locally analytically trivial, that $(f_2 f_3 \dots f_i f_{i+1})^{-1}W_2$ is locally analytically trivial everywhere along $f_{i+1}^{-1}z$; and finally argue as before, using Bertini-Sard etc.

The result that ER^+ is stratifying is similar to [T1, p. 109, Prop. 2]; but Teissier's formulation of strong simultaneous resolution refers to birational rather than embedded resolution, and so seems a priori weaker than ER^+. It would be good to know whether in fact Teissier's condition is equivalent to ER^+.

It would also be good to know whether either of the stratifying conditions ER and ER^+ is *good* (see §2.4), or in case they are not, whether there is a convincing formulation of equiresolvability which does lead to a good stratifying condition.

Teissier showed that strong simultaneous resolution along any smooth subvariety W of the hypersurface V implies the Whitney conditions for the pair $(V - \mathrm{Sing}(V), W)$, see [T1, p. 111, Prop. 4].

A converse, Whitney $\Rightarrow$ strong simultaneous resolution, was proved by Laufer (see [L]) in case V is the total space of a family of isolated two-dimensional surface singularities, for example if V is three-dimensional, $\mathrm{Sing}(V) =: W$ is a nonsingular curve, and there is a retraction $V \to W$ whose fibers are the members of the family.

The basic motivation behind this paper is our interest in possible relations between Zariski-equisingularity and simultaneous resolution. Such relations played a prominent role in Zariski's thinking—see e.g., [Z1, p. 490, F, G. H] and [Z3, p. 7].

To begin with, Zariski showed that if $\mathrm{dt}_{\mathrm{Z}}(V, x) = 1$ then $\mathrm{Sing}(V)$ is smooth, of codimension 1 at x, and that for some neighborhood V' of x in V, the Zariski stratification of V' *is* equiresolvable (in a strong sense, via successive blowups of the singular locus, which stays smooth of codimension 1 until it disappears altogether)—see [Z3, p. 93, Thm. 8.1]). Zariski's result involves birational resolution, but can readily be extended to cover embedded resolution.

The *converse,* that any simultaneous resolution of a family of plane curve singularities entails equisingularity, was shown by Abhyankar [A]. This converse fails in higher dimension, for instance for the Briançon-Speder family of surface singularities referred to in Remark (c) following Theorem 3.2, a family which is Whitney-equisingular and hence—by the above-mentioned result of Laufer—strongly simultaneously resolvable, but not Zariski-equisingular.

We mention in passing a different flavor of work, by Artin, Wahl, and others, on simultaneous resolution and infinitesimal equisingular deformations of normal surface singularities, see e.g., [Wa].

At any rate, a positive answer to the next question would surely enhance the appeal of defining analytic equisingularity via dt_{Z}.

Problem 4.2. Is the Zariski stratification of a hypersurface V equiresolvable in some reasonable sense? For example, is it an ER- or ER^+-stratification?

Initial discouragement is generated by an example of Luengo [Lu], a family of quasi-homogeneous two-dimensional hypersurface singularities which is equisingular in Zariski's sense, but cannot be resolved simultaneously by blowing up smooth centers. There is however another way. That is the subject of the next section.

5 Simultaneous resolution of quasi-ordinary singularities

We pursue the question "Does Zariski-equisingularity imply equiresolvability?" (A positive answer would settle Problem 4.2 if equiresolvability were expressed by a good stratifying condition, see §2.8.)

In fact we ask a little more: if $x \in V$, with V a hypersurface in $\mathbb{C}^{d+1}$, does there exist a neighborhood M of x in $\mathbb{C}^{d+1}$ and a proper birational (or bimeromorphic) map of manifolds $f\colon M' \to M$ such that $f^{-1}V$ is a divisor with smooth components having only normal crossing intersections and such that for *every* Zariski-stratum W of $V \cap M$, $f^{-1}W$ is a union of multiplicity-strata of $f^{-1}(V \cap M)$, each of which is made by f into a C^∞-fiber bundle, with smooth fibers, over W? We could further require that the not-necessarily-reduced space $f^{-1}W$ be locally analytically trivial over W.

As mentioned above, Zariski gave an affirmative answer when $\mathrm{dt}_{\mathrm{Z}}\, x = 1$. We outline now an approach to the question which most likely gives an affirmative answer when $\mathrm{dt}_{\mathrm{Z}}\, x = 2$. (At this writing, I have checked many, but not all, of the details.) Roughly speaking, this approach is the classical desingularization method of Jung, applied to families. The main roadblock to extending it inductively to dimensionality types ≥ 3 is indicated below (Problem 5.1).

For simplicity, assume that $\dim V = 3$ and that $\mathrm{dt}_{\mathrm{Z}}\, x = 2$, so that the Zariski 2-stratum is a non-singular curve W through x. By definition we can choose a finite map germ whose discriminant Δ has the image of W as its Zariski 1-stratum; and after reimbedding V we may assume this map germ to be induced by a linear map $\pi\colon \mathbb{C}^4 \to \mathbb{C}^3$. In what follows, we need only *one* such π.

According to Zariski, there is an embedded resolution $h\colon M^3 \to \mathbb{C}^3$ of Δ such that $h^{-1}\pi(W)$ is a union of strata. (We abuse notation by writing $\mathbb{C}^3$ for a suitably small neighborhood of $\pi(x)$ in $\mathbb{C}^3$.) Let $g\colon M^4 = M^3 \times_{\mathbb{C}^3} \mathbb{C}^4 \to \mathbb{C}^4$ be the projection, a birational map of manifolds. There is a finite map from the codimension-1 subvariety $V' := g^{-1}V \subset M^4$ to M^3 whose discriminant, being contained in $h^{-1}\Delta$, has normal crossings. This means that the singularities of V' are all *quasi-ordinary*, see [L2]. Moreover, as Zariski showed (see [Z2, p. 514]), the Zariski-strata of V are all étale over the corresponding strata of Δ, so their behavior under pullback through $V' \to V$ is essentially the same as that of the strata of Δ under h, giving rise to nice fiber-bundle structures. More precisely, if $\mathfrak{S}'$ is the pullback on V' of the multiplicity stratification on the normal crossings divisor $g^{-1}\Delta$, and S is any Zariski-stratum of V, then $g^{-1}S$ is a union of $\mathfrak{S}'$-strata, each of which g makes into a C^∞-fiber bundle over S. Thus we have achieved an "equisimplification," but not yet an equiresolution—the singularities of V along W have simultaneously been made quasi-ordinary.

So now we need only deal with quasi-ordinary singularities, keeping in mind however that V' is no longer a localized object—it contains projective subvarieties in the fibers over the original singularities of V. Fortunately, the above-defined stratification $\mathfrak{S}'$ can be characterized *intrinsically* (even *topologically*) on V', see [L3, §6.5]. So we can forget about the projection $V' \to M^3$ via which $\mathfrak{S}'$ was determined, and deal directly with this canonical stratification on V' (closely related, presumably, with the Zariski stratification, though $V' \to M^3$ may not be sufficiently general at every point of V'). The problem is thus reduced to showing that *the canonical stratification $\mathfrak{S}'$ on the quasi-ordinary space V' is equiresolvable.*

Any germ (V', y) (assumed, for simplicity, irreducible) is given by the vanishing of a polynomial

$$Z^m + a_1(X,Y,t)Z^{m-1} + \cdots + a_m(X,Y,t) \qquad (a_i \in \mathbb{C}[[X,Y,t]])$$

whose discriminant is of the form

$$\delta = X^a Y^b \epsilon(X,Y,t), \qquad \epsilon(0,0,0) \neq 0.$$

(Here t is a local parameter along W.) The roots of this polynomial are fractional power series $\zeta_i(X^{1/n}, Y^{1/n}, t)$; and since $\delta = \prod(\zeta_i - \zeta_j)$, we have, for some non-negative integers a_{ij}, b_{ij},

$$\zeta_i - \zeta_j = X^{a_{ij}/n} Y^{b_{ij}/n} \epsilon_{ij}(X^{1/n}, Y^{1/n}, t), \qquad \epsilon_{ij}(0,0,0) \neq 0.$$

Modulo some standardization, the monomials $X^{a_{ij}/n}Y^{b_{ij}/n}$ so obtained are called the *characteristic monomials* of the quasi-ordinary germ (V', y). They provide a very effective tool for studying such germs. They are higher dimensional generalizations of the characteristic pairs of plane curve singularities (which are always quasi-ordinary), and they control many basic features of (V', y), for example the number of components of the singular locus, the multiplicities of these components on V' and at the origin, etc. (See [L2], [L3] for more details.) In particular, two quasi-ordinary singularities have the same embedded topology iff they have the same characteristic monomials [G]. It is therefore natural to ask:

Problem 5.1. Do the characteristic monomials of a quasi-ordinary singularity determine a canonical embedded resolution procedure?

This question makes sense in any dimension, and would arise naturally in any attempt to extend the preceding argument inductively to higher dimensionality types. A positive answer would imply that the canonical stratification of any quasi-ordinary hypersurface is equiresolvable.

At this point, if we weren't concerned with embedded resolution (which we need to be, if there is to be any possibility of induction), we could *normalize* V' to get a family of cyclic quotient singularities whose structure is completely determined by the characteristic monomials. Simultaneously resolving such a family is old hat (see [L1, Lecture 2]), leading to the following result (for germs of families of two-dimensional hypersurface singularities): *If there is a projection with an equisingular branch locus, then we have simultaneous resolution; and if that projection is transversal (i.e., its direction does not lie in the Zariski tangent cone) then we have strong simultaneous resolution, and hence (by Teissier) Whitney-equisingularity.*

In particular, the above-mentioned example of Luengo *is* in fact strongly simultaneously resolvable—though not by one of the standard blowup methods.

For *embedded resolution,* canonical algorithms are now available [BM], [EV] (see also [O] for a two-dimensional precursor), so that Problem 5.1 is quite concrete: verify that the invariants which drive a canonical procedure, operating on a quasi-ordinary singularity parametrized as above (by the ζ_i), are completely determined by the characteristic monomials.

The idea is then that the intersections of V' with germs of smooth varieties transversal to the strata form families of quasi-ordinary singularities with the same characteristic monomials, and so any resolution of one member of the family should propagate along the entire stratum—whence the equiresolvability

A key point in dimension 2 is *monoidal stability:* any permissible blowup of a 2-dimensional quasi-ordinary singularity is again quasi-ordinary, and the characteristic monomials of the blown up singularity depend only on those of the original one and the center of blowing up—chosen according to an algorithm depending only on the characteristic monomials; explicit formulas are given in [L2, p. 170]. When it comes to total transform, there are of course some complications; but monoidal stability can still be worked out for a configuration consisting of a quasi-ordinary singularity *together with* a normal crossings divisor such as arises in the course of embedded resolution. In principle, then, one should be able to use an available canonical resolution process and see things through. Some preliminary work along these lines was reported on at the workshop by Ban and McEwan.

Unfortunately such monoidal stability fails in higher dimensions, even for such simple quasi-ordinary singularities as the origin on the threefold $W^4 = XYZ$. So we need to look into:

Problem 5.2. Find a condition on singularities weaker than quasi-ordinariness, but which is monoidally stable, and which can be substituted for quasi-ordinariness in Problem 5.1.

Careful analysis of how the above-mentioned canonical resolution procedures work on quasi-ordinary singularities could suggest an answer.

Added in proof. Villamayor has come up with an affirmative answer to problem 4.2, based on an approach involving quasi-ordinary singularities, but different from the one suggested in §5—see [Vi].

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