Annals of Mathematics, **107** (1978), 151-207

# Desingularization of two-dimensional schemes

By JOSEPH LIPMAN[1]

Section



## Introduction

We present a new proof of the existence of a desingularization for any excellent surface (where "surface" means "two-dimensional reduced noetherian scheme"). The problem of resolution of singularities of surfaces has a long history (cf. the expository article [**25**]). Separate proofs of resolution for arbitrary excellent surfaces were announced by Abhyankar and Hironaka in 1967; to date (1977) full details have not yet been published (but cf. [**2**], [**12**], [**13**], [**14**] and [**15**]). Actually Hironaka's results on "embedded" resolution are stronger than what we shall prove, viz. the following theorem (which nevertheless suffices for many applications).

Unless otherwise indicated, all rings in this paper will be *commutative* and *noetherian*, and all schemes will be *noetherian* and *reduced*. We say that a point $z$ of a scheme $Z$ is *regular* if the stalk $\mathcal{O}_{Z,z}$ of the structure sheaf at $z$ is a regular local ring, and *singular* otherwise; $Z$ is *non-singular* if all its points are regular.

THEOREM. *For a surface $Y$, with normalization $\bar{Y}$, there exists a* desingularization (i.e., *a proper birational map $f: X \to Y$ with $X$ non-singular*) *if and only if the following conditions hold*:

(a) *$\bar{Y}$ is finite over $Y$.*

(b) *$\bar{Y}$ has at most finitely many singular points.*

(c) *For every $y \in \bar{Y}$, the completion of the local ring $\mathcal{O}_{\bar{Y},y}$ is normal.*

(These conditions (a), (b), (c) are of course satisfied if $Y$ is *excellent* [**EGA IV**, § **7.8**].)

0003-486X/78/0107-0001 $ 02.85
© 1978 by Princeton University Press
For copying information, see inside back cover.
[1] Supported by the National Science Foundation under Grant No. MCS75-08333.

The *raison d'être* of this paper must lie in its methods, which rely on homology, duality, and differentials, and so differ markedly from those of Abhyankar and Hironaka. We now describe briefly the basic ideas, assuming for simplicity that $Y$ is irreducible, normal, and proper over a field $k$.

The main point is to show that:

(*) *Among all the normal surfaces proper over $k$ and birationally equivalent to $Y$ there is one—call it $Z$—whose arithmetic genus $\chi(Z) = h^0(\mathcal{O}_Z) - h^1(\mathcal{O}_Z) + h^2(\mathcal{O}_Z)$ is minimal.*[1]

It was suggested by Zariski, in [35], that desingularizing such a $Z$ should not be too difficult. In fact the minimality of $\chi(Z)$ is equivalent with all the singularities of $Z$ being *pseudo-rational* (cf. §(1a)) (for if $g: W \to Z$ is a proper birational map, the Leray spectral sequence gives

$$\chi(Z) - \chi(W) = h^0(R^1 g_* \mathcal{O}_W) \cdots);$$

and in Section 1 we show that pseudo-rational singularities can indeed be resolved, even by successively blowing up points. The proof uses standard techniques; things work out pretty smoothly because of the following two properties of pseudo-rational singularities:

(i) (cf. (1.5)) If $X$ is a normal surface having only pseudo-rational singularities, and $X' \to X$ is obtained by blowing up a point of $X$, then $X'$ is normal and $X'$ has only pseudo-rational singularities.

(ii) (cf. (1.6)) The tangent cone of a pseudo-rational singularity is defined (ideal-theoretically) in some projective space by the vanishing of certain quadratic forms.

* * *

For surfaces over fields, H. Matsumura has had for a long time a proof of (*) based on the theory of Picard varieties (private communication, December 1967). I don't know how to generalize this proof to the case of arbitrary surfaces.

The approach to (*) taken in this paper is inspired by results of Laufer [22, Theorem 3.4]. Let $f: X \to Y$ be a birational map of normal irreducible surfaces, both proper over a perfect field $k$. Let $K$ be the field of rational functions on $X$ and $Y$. Let $\omega_X$ be a dualizing sheaf on $X$; $\omega_X$ can be realized concretely as the sheaf of 2-forms (differentials) of $K/k$ without poles on $X$. *Duality theory* gives isomorphisms of $k$-vector spaces

$$(H^i(X, \omega_X))' \xrightarrow{\sim} H^{2-i}(X, \mathcal{O}_X) \qquad (i = 0, 1, 2)$$

[1] (*) can be reformulated in numerous tantalizing ways. It is equivalent, for example, to the *finite-dimensionality* of $H^1(\mathcal{O}_{\mathscr{R}})$ where $\mathscr{R}$ is the Zariski-Riemann space associated with $Y$ (cf. [17]). It can also be posed as a statement about certain Hilbert-Samuel polynomials in a two-dimensional normal local ring (cf. Remark (B), end of Section (1a)).

(for a $k$-vector space $V$, $V'$ is the dual space $\mathrm{Hom}_k(V, k)$). Furthermore the *vanishing theorem* ((2.4), and cf. (2.3)) gives $R^1f_*(\omega_X) = 0$, so that we have an isomorphism

$$H^1(Y, f_*\omega_X) \xrightarrow{\sim} H^1(X, \omega_X) .$$

There is an obvious inclusion $f_*\omega_X \hookrightarrow \omega_Y$ whose cokernel has *zero-dimensional support*: for $y \in Y$, the dimension of the stalk $(\omega_Y/f_*\omega_X)_y$ is the number of $k$-linearly independent 2-forms with no pole at $y$, but with some pole along a component of $f^{-1}(y)$. Dualizing the exact sequence

$$0 \longrightarrow H^0(f_*\omega_X) = H^0(\omega_X) \longrightarrow H^0(\omega_Y) \longrightarrow H^0(\omega_Y/f_*\omega_X) \longrightarrow$$
$$H^1(f_*\omega_X) = H^1(\omega_X) \longrightarrow H^1(\omega_Y) \longrightarrow H^1(\omega_Y/f_*\omega_X) = 0 ,$$

we obtain an exact sequence

$$0 \longleftarrow H^2(\mathcal{O}_X) \longleftarrow H^2(\mathcal{O}_Y) \longleftarrow H^0(\omega_Y/f_*\omega_X)' \longleftarrow H^1(\mathcal{O}_X) \longleftarrow H^1(\mathcal{O}_Y) \longleftarrow 0 ,$$

which yields the *key expression for "change of arithmetic genus" in terms of differentials*:

$$(**) \qquad \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_X) = h^0(\omega_Y/f_*\omega_X) .$$

From (**) we deduce:

(i) *$Y$ has only pseudo-rational singularities $\Leftrightarrow$ for all $X \to Y$ as above, we have $\omega_Y = f_*\omega_X$.*

(ii) (*) *is true if and only if ($Y$ being fixed) the integers $h^0(\omega_Y/f_*\omega_X)$ are bounded above independently of $X$.*

Now $\omega_Y = (\Omega_Y^2)^{\vee\vee}$, where $\Omega_Y^2 = \Lambda^2(\Omega^1_{Y/k})$ is the sheaf of Kähler two-forms (over $k$) on $Y$, and for any $\mathcal{O}_Y$-module $\mathcal{L}$, $\mathcal{L}^\vee$ is the sheaf $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{O}_Y)$. So there is a natural map $\phi\colon \Omega_Y^2 \to \omega_Y$ whose cokernel has zero-dimensional support; and one sees easily that for any $f\colon X \to Y$ as above,

$$\phi(\Omega_Y^2) \subseteq f_*(\omega_X) .$$

Thus an upper bound as called for in (ii) above is $h^0(\mathrm{coker}\ \phi)$, Q.E.D. for (*)!

If the ground field $k$, of characteristic $p$, is not perfect then we must use differentials over a suitable "admissible" field $k_0$ with $k^p \subseteq k_0 \subseteq k$, $[k : k_0] < \infty$. When we consider the mixed characteristic case, we cannot use differentials at all, but there are other effective ways to represent dualizing sheaves (cf. Section 2, whose main purpose is to work out a suitably general form of (**), viz. Theorem (2.2)).

* * *

The actual proof of reduction to pseudo-rational singularities is given in Section 3. For reasons explained below, this will not be the proof of (*) just indicated, but a variant which, if less striking, is ultimately shorter and has the additional advantage of applying simultaneously to the equicharacteristic and mixed characteristic cases. This proof is based on the existence of "good trace" maps for certain finite algebraic field extensions. A "good trace" is a linear map closely related—in the equicharacteristic case—to the "trace of a differential" studied by Kunz ([**21**, § 1]). In any case, when we deal with *separable* field extensions the ordinary trace map is a good trace, and consequently Section 4, on the existence of good traces, is really necessary only for treating surfaces over a power series ring $k[[U, V]]$ with $k$ an *imperfect* field.

Now what is the trouble with the above-indicated proof based on the relation $\omega_Y = (\Omega_Y^2)^{\vee\vee}$? Since we are interested in arbitrary surfaces, we must work with schemes of finite type over a formal power series ring $k[[U, V]]$ ($k$ a field). In this case the appropriate differential modules to consider are those of [**3**, § 2.3]. To proceed as above we would need a theorem for normal surfaces on the representation of $\omega$ in terms of such differentials. Such a theorem, though true, seems to be available in the literature only for varieties over fields ([**19**], [**21**]). I intend to give a proof for the general case (i.e., schemes over power series rings) elsewhere. To do so here would have made this paper longer than necessary, and that is why the above line of reasoning, though perfectly justifiable, is not entirely followed.

For now, let me just state that two basic ingredients for tying together differentials and dualizing sheaves are: (A) the correct notion of "trace of a differential"; and (B) the existence of "admissible" fields $k_0$ (cf. above; when $[k: k^p] < \infty$, then $k^p$ is admissible, and there is no problem). For our purposes we cannot avoid (B), which is treated in Section 4. (For varieties over fields, cf. [**19**], [**20**]; over power series rings there are additional difficulties because there are infinitely many coefficients floating around; cf. [**16**].) Thus Section 4 is a sort of poor man's substitute for a complete proof of $\omega_Y = (\Omega_Y^2)^{\vee\vee}$.

* * *

The study of surfaces $Y$ such that $\omega_Y = f_*\omega_X$ for all $X$ (cf. above), as a possible step toward resolution of singularities, was proposed to me by Zariski for a thesis problem in 1964. At that time (**) was apparently not known, nor was there available any theory of pseudo-rational singularities. After a few weeks of fruitless effort I turned to other totally unrelated questions, which ultimately led (via the theory of complete ideals in two-dimensional local rings) to an interest in rational singularities$\cdots$. The proof of resolution outlined above (for surfaces over fields) was announced in 1973 [**24**]; the extension to the mixed characteristic case dates from 1974.

* * *

This introduction is concluded with some further remarks on the theorem.

**A.** If a surface $Y$ can be desingularized at all, then there exists a (unique) *minimal desingularization* $f_0: X_0 \to Y$; in other words, for any desingularization $f: X \to Y$ there is a unique map $g: X \to X_0$ such that $f = f_0 \circ g$ [**RS**, page 277, Corollary (27.3)].

**B.** Assume that the surface $Y$ (with normalization $\bar{Y}$) can be desingularized. Consider the sequence

$$\bar{Y} = Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots \longleftarrow Y_n \longleftarrow \cdots$$

where, for each $i \geqq 0$, $Y_{i+1}$ is obtained from $Y_i$ by blowing up all the singular points of $Y_i$ (which can be seen to be finite in number) and then normalizing the resulting surface. *Then for some $n$, $Y_n$ is non-singular* (cf. [**RS**; § 2]; alternatively use [*ibid.*; (26.1), (1.2), and (4.1)]).

Some further light on this result is shed by the remark at the end of Section (1b).

(The desingularization constructed by this canonical process need not be minimal, even for "absolutely isolated" singularities [**32a**, p. 346, (5.11)].

**C.** *If $f: X \to Y$ is a desingularization of a* normal *surface $Y$, then $f$ can be obtained by blowing up an $\mathcal{O}_Y$-ideal $\mathcal{I}$ such that $\mathcal{O}_Y/\mathcal{I}$ has at most zero-dimensional support.*

*Proof.* First of all, a pasting argument reduces us to the case $Y = \mathrm{Spec}(R)$, $R$ a two-dimensional normal local ring. (Consider the finitely many points where the rational map $f^{-1}$ is not defined.) Now let $E_1, \cdots, E_r$ be the irreducible components of $f^{-1}(y)$ ($y$ = closed point of $Y$). The intersection matrix $(E_i \cdot E_j)$ is negative-definite (cf. [**RS**, § 14], where the more efficient proof of Mumford [27, page 6] should have been adapted; cf. also part V in the proof of Theorem (2.4) below). Hence we can find a divisor $E = \sum a_i E_i$, such that $(E \cdot E_i) > 0$ for $i = 1, 2, \cdots, r$, and all the integers $a_i$ are $\leqq 0$ (cf. [**RS**, middle of page 238, remark (ii)]). Setting $\mathcal{L} = \mathcal{O}_X(E) \subseteqq \mathcal{O}_X$, and

$$I = H^0(\mathcal{L}) \subseteqq H^0(\mathcal{O}_X) = R ,$$

we have that $I$ is an ideal of $R$ with $R/I$ of dimension $\leqq 0$. The restriction of $\mathcal{L}$ to $f^{-1}(y)$ is ample (cf. e.g. [**18**, pages 318–319]), so $\mathcal{L}$ itself is ample [**EGA III**, (4.7.1)], and after replacing $E$ by $nE$ ($n \gg 0$), we may assume that $\mathcal{L}$ is *very ample*. [**EGA III**, (2.3.4.1)] shows then that $X = \mathrm{Proj}(\bigoplus_{n \geq 0} I^n)$, the blow-up of $I$.

**D.** Assume that a desingularization $f: X \to Y$ as in the theorem exists. $f$ being proper, $f_*\mathcal{O}_X$ is a *coherent* $\mathcal{O}_Y$-module, and so $\bar{Y} = \mathrm{Spec}(f_*\mathcal{O}_X)$ is finite over $Y$. The singularities of $\bar{Y}$ are to be found among the finitely many points where the rational map $f^{-1}$: $\bar{Y} \longrightarrow X$ is not defined. Thus we have the *necessity* of (a) and (b), and that of (c) is given by [**RS**, page 232, Remark (16.2)].

As for *sufficiency*, we can replace $Y$ by $\bar{Y}$, and then a simple pasting argument shows that it is enough to prove: *if $R$ is a two-dimensional local ring whose completion $\hat{R}$ is normal, then $R$ can be desingularized by blowing up an $\mathfrak{m}$-primary ideal* ($\mathfrak{m}$ = maximal ideal of $R$). Now if there exists a desingularization $\hat{f}$: $\hat{X} \longrightarrow \operatorname{Spec}(\hat{R})$, then this desingularization can be obtained by blowing up an ideal $\hat{I}$ which is primary for $\mathfrak{m}\hat{R}$ (cf. Remark C above); if $f$: $X \longrightarrow \operatorname{Spec}(R)$ is obtained by blowing up the $\mathfrak{m}$-primary ideal $I = \hat{I} \cap R$, then, since $\hat{R}$ is flat over $R$ and $\hat{I} = I\hat{R}$, we have

$$\hat{X} = X \otimes_R \hat{R} \; ,$$

and it follows easily that $f$ is a desingularization of $\operatorname{Spec}(R)$ (cf. beginning of proof of the second Proposition in Section (1b) below; or just use [**EGA** $\mathbf{0}_{\text{IV}}$, (17.3.3(i)]).

So to establish the theorem, it remains to prove:

THEOREM′. *Let $R$ be a complete two-dimensional normal local ring. Then there exists a desingularization $f$: $X \longrightarrow \operatorname{Spec}(R)$.*

## 1. Resolution of analytically normal pseudo-rational singularities



### (1a) Pseudo-rational singularities.

PROPOSITION-DEFINITION (1.1) (cf. [**RS**, page 212, § 9]). *Let $R$ be a two-dimensional local ring. $R$ is said to be* pseudo-rational *if $R$ is normal, and satisfies the following equivalent conditions*:

(i) *For any projective birational map $W \longrightarrow \operatorname{Spec}(R)$ there exists a proper birational map $Z \longrightarrow W$ such that $Z$ is normal and $H^1(Z, \mathcal{O}_Z) = 0$.*

(ii) *For any proper birational map $W \longrightarrow \operatorname{Spec}(R)$, the normalization $\bar{W}$ is finite over $W$, and $H^1(W, \mathcal{O}_{\bar{W}}) = 0$.*

(iii) *The completion $\hat{R}$ is reduced* (i.e. *has no non-zero nilpotents*) *and for every proper birational map $W \longrightarrow \operatorname{Spec}(R)$ with $W$* normal, *we have $H^1(W, \mathcal{O}_W) = 0$.*

*Proof.* Clearly (ii) $\Rightarrow$ (i). Conversely, if $W \to \operatorname{Spec}(R)$ is proper and birational, then (i) together with Chow's lemma [**EGA II**, (5.6.2)] gives us a proper birational $h: Z \to W$ with $Z$ normal and $H^1(Z, \mathcal{O}_Z) = 0$; but then $\bar{W} = \operatorname{Spec}(h_*\mathcal{O}_Z)$ is finite over $W$, and

$$H^1(\bar{W}, \mathcal{O}_{\bar{W}}) = H^1(W, h_*\mathcal{O}_Z) \subseteq H^1(Z, \mathcal{O}_Z) ;$$

thus (i) $\Rightarrow$ (ii). The equivalence of (ii) and (iii) follows from Rees' characterization of "analytically unramified" local rings [**28**].

*Remark.* A two-dimensional normal local ring $R$ is *rational* if there exists a *desingularization* $X \to \operatorname{Spec}(R)$ with $H^1(X, \mathcal{O}_X) = 0$. If $R$ is rational then $R$ is pseudo-rational (cf. [**RS**, page 200, A) and B)]). Conversely, if $R$ is pseudo-rational, and *if* $R$ admits a desingularization $W \to \operatorname{Spec}(R)$, then $H^1(W, \mathcal{O}_W) = 0$, so $R$ is rational. (As pointed out in the introduction (Remark D), if $R$ admits a desingularization then the completion $\hat{R}$ remains normal; and we are about to prove the converse.)

* * *

The present Section 1 is devoted to proving that one can resolve "analytically normal" pseudo-rational singularities by successively blowing up isolated singular points. More precisely:

We say that a surface $Y$ has **only pseudo-rational singularities** if for each singular point $y$ of $Y$, the local ring $\mathcal{O}_{Y,y}$ is two-dimensional and pseudo-rational. An **iterated blow-up** is a composed map $Z \to Y$ of the form

$$Z = Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y ,$$

where each map $Y_i \to Y_{i-1}$ $(0 < i \leqq n)$ is obtained by blowing up a finite set of *closed* points on $Y_{i-1}$.

THEOREM (1.2). *Let $Y$ be a surface having only pseudo-rational singularities. Assume that $Y$ has at most finitely many singular points, and that for each such singular point $y$, the completion of the local ring $\mathcal{O}_{Y,y}$ is normal. Then there exists an iterated blow-up $Z \to Y$ with $Z$ non-singular.*

Theorem (1.2) reduces Theorem′ at the end of the introduction to the following:

THEOREM*. *Let $R$ be a complete two-dimensional normal local ring. Then there exists a proper birational map $W \to \operatorname{Spec}(R)$ such that $W$ has only pseudo-rational singularities.*

(Any $W$ as in Theorem* is a normal surface (use the "dimension formula" [**EGA IV**, (5.5.8)]), and $W$ has only finitely many singularities [**EGA IV**, (6.12.2)]; furthermore, since $R$ is complete, therefore $W$ is excellent, so all the local rings $\mathcal{O}_{W,w}$ $(w \in W)$ have normal completion.)

Theorem* will be proved in Sections 2–4. The following observations will help to bring the problem into focus.

Let $R$ be a two-dimensional normal local ring whose completion $\hat{R}$ is *reduced*. For any proper birational map $Z \to \mathrm{Spec}(R)$ with $Z$ *normal*, let $\lambda_Z$ be the (finite) length of the $R$-module $H^1(Z, \mathcal{O}_Z)$. We set

$$H(R) = \sup_Z(\lambda_Z) \qquad (\leqq \infty),$$

the "sup" being taken over all $Z$ as above. (1.1) (iii) states that $R$ *is pseudo-rational* $\Leftrightarrow H(R) = 0$.

LEMMA (1.3). *For a proper birational map* $g: W \to \mathrm{Spec}(R)$ ($R$ *as above*) *with* $W$ *normal, the following conditions are equivalent*:

(i) $W$ *has only pseudo-rational singularities* (*necessarily finite in number, see above*).

(ii) $\lambda_W = H(R)$.

(iii) *For any proper birational map* $h: Z \to W$ *with* $Z$ *normal, the canonical map*

$$H^1(W, h_*\mathcal{O}_Z) = H^1(W, \mathcal{O}_W) \longrightarrow H^1(Z, \mathcal{O}_Z)$$

*is* bijective; *equivalently*, $R^1h_*\mathcal{O}_Z = 0$.

*Remark.* Clearly a $W$ satisfying these conditions exists if and only if $H(R) < \infty$. So Theorem* states that $H(R) < \infty$ *if $R$ is complete.* (Actually I don't know *any* example of an $R$ with $H(R) = \infty$.)

*Proof of* (1.3). $W$ being a normal surface (see above), $R^1h_*\mathcal{O}_Z$ has support of dimension $\leqq 0$, and so "equivalently" in (iii) follows from the canonical exact sequence

$$\text{(1.3a)}\quad 0 \longrightarrow H^1(W, \mathcal{O}_W) \longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^0(R^1h_*\mathcal{O}_Z) \longrightarrow H^2(W, \mathcal{O}_W) = 0$$

(where $H^2$ vanishes because the fibres of $g$ have dimension $\leqq 1$: cf. **[EGA III, (4.2.2)]**; or else cover $W$ by two affine open subsets and use Čech cohomology).

Similarly, for any proper birational maps $Z \to Z' \to \mathrm{Spec}(R)$ with $Z$ and $Z'$ normal, $H^1(Z', \mathcal{O}_{Z'}) \subseteqq H^1(Z, \mathcal{O}_Z)$, and hence (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). For any proper birational $Z' \to \mathrm{Spec}(R)$, there exists a commutative diagram of proper birational maps

$$\begin{array}{ccc} & Z'' & \\ \swarrow & & \searrow \\ W & & Z' \\ \searrow & & \swarrow \\ & \mathrm{Spec}(R) & \end{array}$$

where $Z''$ is the reduced closed image in $W \times_R Z'$ of the generic point of

Spec($R$) ($Z''$ is the "birational join" of $W$ and $Z'$). The normalization $Z$ of $Z''$ is finite over $Z''$ (since $\hat{R}$ is reduced, cf. [28]); so by (iii) $\lambda_W = \lambda_Z \geqq \lambda_{Z'}$, and (ii) follows.

(iii) follows easily from (i), because two-dimensional regular local rings are *rational*, hence pseudo-rational (preceding remark).

(iii) $\Rightarrow$ (i). Let $S$ be the local ring of a singular point on $W$; $S$ is two-dimensional and normal. Let $g'\colon W' \to \text{Spec}(S)$ be a projective birational map. There exists a projective birational map $g^*\colon W^* \to W$ such that $g' = g^* \times_W \text{Spec}(S)$ (cf. e.g. [EGA III, (2.3.5)]). $\hat{R}$ being reduced, the normalization $Z$ of $W^*$ is finite over $W^*$, so we have a proper birational map $h\colon Z \to W$. By (iii), $R^1h_*\mathcal{O}_Z = 0$, and so $H^1(Z', \mathcal{O}_{Z'}) = 0$, where $Z' = Z \times_W \text{Spec}(S)$ is the normalization of $W'$. Thus by (1.1)(i), $S$ is pseudo-rational. Q.E.D.

COROLLARY (1.4). *Let $R$ be a two-dimensional pseudo-rational local ring with maximal ideal $\mathfrak{m}$ and fraction field $K$. Let $S$ be a normal local ring with maximal ideal $\mathfrak{n}$, such that $R \subseteqq S \subseteqq K$, $\mathfrak{m} \subseteqq \mathfrak{n}$, $S/\mathfrak{n}$ is an algebraic field extension of $R/\mathfrak{m}$, and such that $S$ is essentially of finite type over $R$* (i.e., *$S$ is a localization of a finitely generated $R$-algebra*). *Then $S$ is two-dimensional and pseudo-rational.*

*Proof.* The dimension formula [EGA IV, (5.5.8)] gives dim.($S$) $\leqq 2$, while Zariski's "main theorem" gives $\dim(S/\mathfrak{m}S) \geqq 1$ (unless $S = R$); thus $\dim(S) = 2$.

Clearly there exists a proper birational map $W \to \text{Spec}(R)$ such that $S = \mathcal{O}_{W,w}$ for some $w \in W$; and (1.1)(ii) allows us to assume that $W$ is *normal*. Then $\lambda_W = H(R) = 0$, and the conclusion follows from (1.3).

* * *

The following two remarks, due essentially to Rees [29, page 21], will not really be needed elsewhere in this paper.

(A) (1.3) can be sharpened:

Let $R$ be as in (1.3), and let $W \to \text{Spec}(R)$ be a proper birational map with $W$ normal. For each *closed* point $w \in W$, let $e_w$ be the degree of the residue field of $\mathcal{O}_{W,w}$ over that of $R$. Then

$$H(R) = \lambda_W + \sum_{\substack{w \in W \\ w \text{ closed}}} e_w H(\mathcal{O}_{W,w}) \,. \tag{$\sharp$}$$

(*Proof.* If $H(R) < \infty$, choose $h\colon Z \to W$ with $\lambda_Z = H(R)$, cf. proof of (iii) $\Rightarrow$ (ii) in (1.3); then $Z$ has only pseudo-rational singularities, and the result is given by the exact sequence (1.3a). If $H(R) = \infty$, a similar argument shows that $H(\mathcal{O}_{W,w}) = \infty$ for some $w \in W$.)

($\sharp$) shows that singularities have a tendency to become pseudo-rational, in the following sense: if $0 < H(R) < \infty$, then *either* $H(\mathcal{O}_{W,w}) < H(R)$ for *all*

$w \in W$; *or* there is just one $w$ for which $H(\mathcal{O}_{W,w}) = H(R)$, and then $e_w = 1$, all other singularities on $W$ are pseudo-rational, and $H^1(W, \mathcal{O}_W) = 0$.

(B) Here is another interpretation of $H(R)$.

We consider ideals $I$ in $R$ such that $R/I$ is zero-dimensional, and such that *for all* $n>0$ the ideal $I^n$ is *integrally closed*. For all large $n$, the length of the $R$-module $R/I^n$ is given by a polynomial

$$\mu_0(I)\binom{n}{2} + \mu_1(I)n + \mu_2(I)$$

with integers $\mu_i(I)$. In fact, $\mu_2(I) = \lambda_W$, where $W$ is the (normal) surface obtained by blowing up $I$ [**RS**, (5.2) and (23.2)]. From this it can be shown that

$$H(R) = \sup_I(\mu_2(I)) .$$

Thus, Theorem* says: *if $R$ is complete, then*

$$\sup_I(\mu_2(I)) < \infty .$$

**(1b) Birational stability of (analytically normal) pseudo-rationality.** This part (1b) and the next part (1c) bring out properties of pseudo-rationality which make a relatively simple proof of Theorem (1.2) possible.

Let $(R, \mathfrak{m})$ be a local ring. (The notation signifies that $\mathfrak{m}$ is the unique maximal ideal of $R$.) A *quadratic transform* of $R$ is an $R$-algebra which is $R$-isomorphic to the local ring $\mathcal{O}_{W,w}$ for some *closed* point $w \in W = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathfrak{m}^n)$ (the $R$-scheme obtained by blowing up $\mathfrak{m}$).

For later application, the principal result of (1b) is:

PROPOSITION (1.5). *Let $(R, \mathfrak{m})$ be a pseudo-rational two-dimensional local ring. Then any quadratic transform $S$ of $R$ is again a pseudo-rational two-dimensional local ring. Furthermore, if the completion $\hat{R}$ is normal, then so is $\hat{S}$.*

*Proof.* Let $\bar{W}$ be the normalization of $W = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathfrak{m}^n)$. Then $\bar{W}$ is finite over $W$, and $H^1(\mathcal{O}_{\bar{W}}) = 0$, and so by [**RS**, (7.2)] we have $\mathfrak{m}^n = H^0(\mathfrak{m}^n\mathcal{O}_{\bar{W}})$ for all $n > 0$. This implies that $\mathfrak{m}^n$ is integrally closed (cf. e.g. [*ibid.*, proof of (6.2)]), and hence [*ibid.*, (5.2)] $W$ is already normal (i.e., $W = \bar{W}$). So (1.5) is a consequence of Corollary (1.4) and the following:

PROPOSITION. *Let $(R, \mathfrak{m})$ be a two-dimensional normal local ring with field of fractions $K$. Let $(S, \mathfrak{n})$ be a normal local ring with $R \subseteq S \subseteq K$, $\mathfrak{m} \subseteq \mathfrak{n}$, and such that $S$ is essentially of finite type over $R$* (i.e., *$S$ is a ring of fractions of a finitely generated $R$-algebra*). *If $\hat{R}$ is normal, then so is $\hat{S}$.*

*Proof.* Let $S^* = S \otimes_R \hat{R}$. For any $t > 0$,

$$S^*/\mathfrak{n}^t S^* = (S/\mathfrak{n}^t) \otimes_R \hat{R} = (S/\mathfrak{n}^t) \otimes_{R/\mathfrak{m}^t} (\hat{R}/\mathfrak{m}^t \hat{R}) = S/\mathfrak{n}^t ,$$

and so $\mathfrak{n}S^*$ is a maximal ideal such that the canonical map of $S$ into the localization $T = (S^*)_{\mathfrak{n}S^*}$ induces an *isomorphism of completions* $\hat{S} \xrightarrow{\sim} \hat{T}$. Since $\hat{R}$ is excellent and $T$ is essentially of finite type over $\hat{R}$, for $\hat{T} = \hat{S}$ to be normal it is enough that $T$ be normal [**EGA IV**, (7.8.3) (ii), (iii), (v)].

Since $S$ is normal, and $\dim(S) \leqq 2$ (by the dimension formula [**EGA IV**, (5.5.8)]), therefore $S$ is Cohen-Macaulay, and hence so is $T$ (since $\hat{S} = \hat{T}$, for example). It will suffice therefore to show that $T_P$ is regular for each height-one prime $P$ of $T$.

Note that if $\hat{K}$ is the fraction field of $\hat{R}$, then $S^* \subseteq K \otimes_R \hat{R} \subseteq \hat{K}$, so that $T \subseteq \hat{K}$. If $P \cap \hat{R} \neq \mathfrak{m}\hat{R}$, then

$$\hat{K} \supsetneqq T_P \supseteqq \hat{R}_{P \cap \hat{R}} ;$$

since $\hat{R}_{P\cap\hat{R}}$ is a discrete valuation ring, therefore $T_P = \hat{R}_{P\cap\hat{R}}$ and $T_P$ is regular.

If $P \cap \hat{R} = \mathfrak{m}\hat{R}$, then $Q = P \cap S$ is a prime ideal of $S$, with $Q \cap R = \mathfrak{m}$. So the domain $S/Q$ is essentially of finite type over the field $R/\mathfrak{m}$, and therefore the completion $\hat{S}/Q\hat{S}$ is *reduced*. We have

$$P \supseteqq QT = Q\hat{T} \cap T = Q\hat{S} \cap T$$

so $T/QT$ $(\subseteqq \hat{S}/Q\hat{S})$ is reduced, $P/QT$ is a minimal prime of $T/QT$, and hence $QT_P$ is the maximal ideal of $T_P$. Since the maximal ideal $\mathfrak{n}T \not\subseteqq P$, therefore $Q \neq \mathfrak{n}$, so $S_Q$ is a discrete valuation ring, and the maximal ideal $(QS_Q)T_P$ of $T_P$ is principal; thus $T_P$ is regular. Q.E.D.

*Remark* (not used elsewhere). In view of Remark (A) at the end of Section (1a), the first part of the proof of (1.5) implies that *in the process of desingularizing* Spec($R$) ($R$ *as* (1.3)) *by successively blowing up and normalizing, at most* $H(R)$ *normalizations are actually necessary.*

**(1c) The tangent cone is an intersection of quadrics.**

PROPOSITION (1.6). *Let* $(R, \mathfrak{m})$ *be a two-dimensional normal local ring. Assume that there exists a proper birational map* $f\colon W \to \mathrm{Spec}(R)$ *such that* $\mathfrak{m}\mathcal{O}_W$ *is invertible and* $H^1(W, \mathcal{O}_W) = 0$ *(an assumption which certainly holds if* $R$ *is pseudo-rational). Let* $(z_0, z_1, \cdots, z_\nu)$ *be a minimal basis of* $\mathfrak{m}$, *and let*

$$\theta\colon S = k[Z_0, Z_1, \cdots, Z_\nu] \longrightarrow \bigoplus_{n\geqq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} = G_\mathfrak{m}$$

$$(k = R/\mathfrak{m};\ Z_0, \cdots, Z_\nu \text{ indeterminates})$$

*be the homomorphism of graded* $k$*-algebras for which*

$$\theta(Z_i) = \text{canonical image of } z_i \text{ in } \mathfrak{m}/\mathfrak{m}^2 \qquad (0 \leqq i \leqq \nu) .$$

*Then the kernel of $\theta$ is generated by $\nu(\nu-1)/2$ k-linearly independent quadratic forms $Q_\alpha$ $(1 \leqq \alpha \leqq \nu(\nu-1)/2)$; moreover, if $\nu \geqq 3$ then the k-vector space generated by the linear forms*

$$\partial Q_\alpha/\partial Z_j \qquad (1 \leqq \alpha \leqq \nu(\nu-1)/2;\ 0 \leqq j \leqq \nu)$$

*has dimension* $\geqq \nu - 1$.

*Remark.* More information about the kernel of $\theta$ and its syzygies can be found in Wahl's paper [**33**].

*Proof of* (1.6). Simple considerations show that it is permissible to replace $R$ by the localization of the polynomial ring $R[U]$ at the prime ideal $\mathfrak{m}R[U]$; so we may assume that $R/\mathfrak{m}$ is infinite. Then there exist elements $x, y \in \mathfrak{m}$ and an integer $a > 0$ such that $\mathfrak{m}^{a+1} = (x, y)\mathfrak{m}^a$ (cf. [**30**, pages 153–154]).

*Remarks.* 1 (cf. [**31**]). If $R$ has multiplicity $e$, then the $\mathfrak{m}$-primary ideal $(x, y)$ also has multiplicity $e$ [**30**, page 154, Theorem 2]. $R$ being a Cohen-Macaulay local ring, we have

$$e = \lambda_R(R/(x, y)) \qquad (\lambda_R = \text{length of an } R\text{-module}),$$

and

$$2 = \lambda_R((x, y)/\mathfrak{m}(x, y)) \text{ (Nakayama's lemma)},$$

whence

(1.7) $$e + 1 = \lambda_R(\mathfrak{m}/(x, y)\mathfrak{m}) \geqq \lambda_R(\mathfrak{m}/\mathfrak{m}^2) = \nu + 1 .$$

2. $(x, y)$ is part of a minimal basis of $\mathfrak{m}$ (otherwise, say, $x \in \mathfrak{m}^2 + yR$, whence

$$\mathfrak{m}^{a+1} = (x, y)\mathfrak{m}^a \subseteqq \mathfrak{m}^{a+2} + y\mathfrak{m}^a$$

so that (Nakayama's lemma) $\mathfrak{m}^{a+1} = y\mathfrak{m}^a \subseteqq yR$, which is absurd). The conclusions of (1.6) clearly do not depend on the choice of the minimal basis $(z_0, \cdots, z_\nu)$, so we may assume $x = z_{\nu-1}$, $y = z_\nu$, and correspondingly set $X = Z_{\nu-1}$, $Y = Z_\nu$.

Now the basic point (to be proved below) is that in fact

(1.8) $$\mathfrak{m}^2 = (x, y)\mathfrak{m} .$$

This implies that the kernel of $\theta$ contains $\nu(\nu-1)/2$ elements $Q_{ij}$ $(0 \leqq i \leqq j \leqq \nu - 2)$ of the form

$$Q_{ij} = Z_i Z_j + \sum_{l=0}^{\nu-2} (a_{ijl} Z_l X + b_{ijl} Z_l Y) + e_{ij} X^2 + f_{ij} XY + g_{ij} Y^2$$

(with $a_{ijl}, b_{ijl}, e_{ij}, f_{ij}, g_{ij} \in k$). The $Q_{ij}$ are linearly independent, and if $\nu \geqq 3$, then so are the $\nu - 1$ partial derivatives $\partial Q_{0j}/\partial Z_0$ $(1 \leqq j \leqq \nu - 2)$ and $\partial Q_{01}/\partial Z_1$. Let us show then that the ideal $I$ generated by the $Q_{ij}$ is the entire kernel of $\theta$, i.e., the map $\bar{\theta}\colon S/I \to G_\mathfrak{m}$ induced by $\theta$ is *injective*.

Let $S' = \bigoplus_{n \geqq 0} S'_n$ be the graded $k[X, Y]$-submodule of $S$ generated by the $\nu$ homogeneous elements $1, Z_0, Z_1, \cdots, Z_{\nu-2}$. Clearly the canonical map $S \to S/I$ restricts to a *surjection* $\psi': S' \to S/I$. So we need only check that $\bar\theta \circ \psi': S' \to G_m$ is injective. But if $\bar\theta \circ \psi'$ had a non-zero element of degree (say) $q$ in its kernel, then an easy calculation would give, for $n \geqq q$:

$$\begin{aligned}\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) &\leqq \dim_k(S'_n) - (n-q+1)\\ &= (\nu n + 1) - (n - q + 1)\\ &= (\nu - 1)n + q ,\end{aligned}$$

so that $e \leqq \nu - 1$, contradicting (1.7).

It remains to prove (1.8).

Let $\mathcal{I}$ be the invertible $\mathcal{O}_W$-ideal $\mathfrak{m}\mathcal{O}_W$. Since $\mathfrak{m}^{a+1} = (x, y)\mathfrak{m}^a$, therefore $\mathcal{I}^{a+1} = (x, y)\mathcal{I}^a$, and multiplying by $\mathcal{I}^{-a}$ gives

$$\mathcal{I} = (x, y)\mathcal{O}_W .$$

Note also that

$$\mathfrak{m} \subseteqq H^0(\mathcal{I}) \subseteqq H^0(\mathcal{O}_W) = R$$

and $1 \notin H^0(\mathcal{I})$, so that $H^0(\mathcal{I}) = \mathfrak{m}$.

Now we have an *exact sequence* of $\mathcal{O}_W$-modules

$$0 \longrightarrow \mathcal{O}_W \xrightarrow{\ \alpha\ } \mathcal{I} \oplus \mathcal{I} \xrightarrow{\ \beta\ } \mathcal{I}^2 \longrightarrow 0$$

where $\alpha$ and $\beta$ are given locally by

$$\alpha(t) = tx \oplus (-ty) ,$$
$$\beta(t_1 \oplus t_2) = t_1 y + t_2 x .$$

Since the fibres of $f$ have dimension $\leqq 1$, $H^2$ vanishes on coherent $\mathcal{O}_W$-modules ([EGA III, (4.2.2)]; or note that $W$ can be covered by two affine open subsets $\cdots$). $\mathcal{I} = (x, y)\mathcal{O}_W$ is a homomorphic image of $\mathcal{O}_W^2$, therefore $H^1(\mathcal{I})$ is a homomorphic image of $H^1(\mathcal{O}_W^2) = 0$, whence $H^0(\beta)$ is surjective, i.e.,

$$H^0(\mathcal{I}^2) = (x, y)H^0(\mathcal{I}) = (x, y)\mathfrak{m} .$$

Since $\mathfrak{m}^2 \subseteqq H^0(\mathcal{I}^2)$, we conclude that $\mathfrak{m}^2 = (x, y)\mathfrak{m}$. Q.E.D.

*Remark.* The bijectivity of $\bar\theta \circ \psi'$ (see above) implies that for all $n > 0$

(1.9) $$\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \nu n + 1$$

(whence, in particular, $H^1(W, \mathcal{O}_W) = 0$ for $W = \mathrm{Proj}(\bigoplus_{n \geqq 0} \mathfrak{m}^n)$ [RS, page 253, (23.2)]). (1.9) could also be proved along the lines of [*ibid.*, page 254, (23.3)]; since (1.9) $\Rightarrow (e = \nu) \Rightarrow$ (1.8) (cf. (1.7)) and (1.9) $\Rightarrow (\bar\theta \circ \psi'$ bijective), this would give another proof of Proposition (1.6).

**(1d) An important subspace of $\mathfrak{m}/\mathfrak{m}^2$.** We now review, in a decidedly ad hoc way, some techniques of Hironaka insofar as they are required for proving (1.2). In this context, Corollary (1.21) gives another consequence of pseudo-rationality which simplifies the resolution process.

We begin with some preliminary remarks on *quadratic transforms* and *embedding dimension.* Let $(R, \mathfrak{m})$ be a local ring with residue field $k = R/\mathfrak{m}$, and let $R'$ be a quadratic transform of $R$ (cf. (1b)). Then $\mathfrak{m}R'$ is invertible, say $\mathfrak{m}R' = tR'$ ($t \in \mathfrak{m}$). $R'/tR'$ is the local ring of a closed point on the closed fibre $C_R = \mathrm{Proj}(\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1})$. Let $\nu + 1$ be the embedding dimension of $R$ (i.e., the dimension of the $k$-vector space $\mathfrak{m}/\mathfrak{m}^2$), in symbol:

$$\mathrm{emdim}(R) = \nu + 1 .$$

A minimal basis $(z_0, z_1, \cdots, z_\nu)$ of $\mathfrak{m}$ defines a homomorphism of graded $k$-algebras

$$\theta\colon k[Z] = k[Z_0, Z_1, \cdots, Z_\nu] \longrightarrow \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

where the $Z_i$ are indeterminates, and $\theta(Z_i)$ is the canonical image of $z_i$ in $\mathfrak{m}/\mathfrak{m}^2$ $(0 \leqq i \leqq \nu)$. Corresponding to $\theta$, we get a closed immersion of $C_R$ into the projective space $\mathbf{P}^\nu = \mathrm{Proj}(k[Z])$. So $R'/tR'$ is a homomorphic image of the local ring of a closed point of $\mathbf{P}^\nu$, i.e., of a regular local ring of dimension $\nu$; thus $\mathrm{emdim}(R'/tR') \leqq \nu$, and we conclude that

$$\mathrm{emdim}(R') \leqq \nu + 1 = \mathrm{emdim}(R) . \tag{1.10}$$

The following simple observation gives a useful sufficient condition for strict inequality in (1.10).

LEMMA (1.11). *With preceding notation, let $h \in k[Z]$ be a homogeneous polynomial with $\theta(h) = 0$, and let $H \subseteqq \mathbf{P}^\nu$ be the subscheme $h = 0$ of $\mathbf{P}^\nu$ (so that $C_R \subseteqq H = \mathrm{Proj}(k[Z]/h)$). Let $\xi \in C_R$ be the point whose local ring is $R'/\mathfrak{m}R'$. If $\xi$ is a regular point of $H$ then*

$$\mathrm{emdim}(R') < \mathrm{emdim}(R) .$$

*Proof.* $R'/\mathfrak{m}R' = R'/tR'$ is a homomorphic image of the regular local ring $\mathcal{O}_{H,\xi}$, whose dimension is $<\nu$; thus $\mathrm{emdim}(R'/tR') < \nu$, and so $\mathrm{emdim}(R') < \nu + 1$. Q.E.D.

In attempting to resolve pseudo-rational singularities by blowing up, we will naturally be most interested in the quadratic transforms $R'$ for which $\mathrm{emdim}(R') = \mathrm{emdim}(R)$. We can gain some control over these $R'$ by means of a certain $k$-vector subspace of $\mathfrak{m}/\mathfrak{m}^2$ which corresponds to some bad singularities of the tangent cone.

Let $(R, \mathfrak{m})$, $k[Z]$, $\theta$, be as above. For any homogeneous $h \in k[Z]$ let $\Sigma_h \subseteqq \mathbf{P}^\nu$ be the set of singular (i.e., non-regular) points of the scheme $\mathrm{Proj}(k[Z]/h)$; and set

$$\Sigma_\theta = \bigcap_h \Sigma_h \qquad (\theta(h) = 0)\,.$$

Note that $\theta$ restricts to an *isomorphism*

$$\theta_1\colon (\text{linear forms in } k[Z]) \xrightarrow{\ \sim\ } \mathfrak{m}/\mathfrak{m}^2\,.$$

*We define a k-vector subspace $V_R$ of* $\mathfrak{m}/\mathfrak{m}^2$:

$$V_R = \{v \in \mathfrak{m}/\mathfrak{m}^2 \mid \text{the linear form } \theta_1^{-1}(v) \text{ vanishes everwhere on } \Sigma_\theta\}\,.$$

It is easily checked that $V_R$ depends only on $R$, and not on any choice of basis for $\mathfrak{m}$.

Say that an element $v \in \mathfrak{m}/\mathfrak{m}^2$ *vanishes at the quadratic transform $R'$ of $R$* if the linear form $\theta_1^{-1}(v)$ vanishes at the point $\xi \in C_R \subseteq \mathbf{P}^\nu$ whose local ring is $R'/\mathfrak{m}R'$ (cf. preceding remarks). An equivalent and more intrinsic condition is the following (proof left to the reader): if $v$ is the natural image of, say, $\tilde{v} \in \mathfrak{m}$, then *$v$ vanishes at $R'$ if and only if $\mathfrak{m}R' \neq \tilde{v}R'$.*

Now, as an immediate corollary of (1.11) we have:

COROLLARY (1.12). *If $R'$ is a quadratic transform of $R$ such that* $\text{emdim}(R') = \text{emdim}(R)$ *then all the elements of $V_R$ vanish at $R'$. In particular, if $V_R = \mathfrak{m}/\mathfrak{m}^2$* (i.e., *$\Sigma_\theta$ is empty*) *then for all quadratic transforms $R'$ of $R$ we have* $\text{emdim}(R') < \text{emdim}(R)$.

* * *

Let $\tau_R$ be the codimension of $V_R$ in $\mathfrak{m}/\mathfrak{m}^2$. From (1.12) it appears that as far as resolution is concerned, the smaller $\tau_R$ is, the better. We will see below ((1.19) and (1.20)) that for pseudo-rational two-dimensional $R$, $\tau_R$ is always $\leqq 2$, and almost always $\leqq 1$. For this purpose, and for other valuable information about $V_R$, we need the following very special case of Hironaka's results in [15].

LEMMA (1.13). *Let $k[Z] = k[Z_0, \cdots, Z_\nu]$ be a polynomial ring over a field $k$, $k[Z]_1$ = linear forms in $k[Z]$, $\mathbf{P}^\nu$ =* $\text{Proj}(k[Z])$. *Let $Q \in k[Z]$ be a non-zero quadratic form, and let $\Sigma_Q \subseteq \mathbf{P}^\nu$ be the set of singular points of the scheme* $\text{Proj}(k[Z]/Q)$. *Let $\Sigma$ be a subset of $\Sigma_Q$, and let $V_Q$* (resp. $V_\Sigma$) *be the $k$-vector subspace of $k[Z]_1$ consisting of those linear forms which vanish everywhere on $\Sigma_Q$* (resp. $\Sigma$). *Then*:

(i) *If $V$ is a $k$-vector subspace of $k[Z]_1$ such that $Q \in k[V]$ (the $k$-subalgebra of $k[Z]$ generated by $V$), then $V \supseteq V_Q$.*

(ii) *Conversely, if the characteristic of $k$* (char $k$) *is $\neq 2$, then $Q \in k[V_Q]$, and*

(iii) *if* char $k = 2$ *and the dimension* $\dim_k(V_\Sigma) \geqq \nu - 2$, *then $Q \in k[V_\Sigma]$.*

*Proof.* (i) If $Q \in k[V]$, then clearly any point $\xi \neq (0, 0, \cdots, 0)$ of $k^{\nu+1}$ where all the forms in $V$ vanish represents a point of $\Sigma_Q$, so that $L(\xi) = 0$

for all $L \in V_Q$. Consequently $V \supseteqq V_Q$.

(ii), (iii). Let $(X_0, \cdots, X_\mu)$ be a (vector space) basis of $V_\Sigma$, and expand it to a basis $(X_0, \cdots, X_\mu, Y_0, \cdots, Y_\rho)$ of $k[Z]_1$ $(\rho = \nu - \mu - 1)$. Expressing $Q$ as a form of degree 2 in the $X$'s and $Y$'s and using the fact that

$$\partial Q/\partial X_i \in V_\Sigma \quad (0 \leqq i \leqq \mu)\ , \quad \partial Q/\partial Y_j \in V_\Sigma \quad (0 \leqq j \leqq \rho)\ ,$$

we find that

$$Q = Q'(X) + Q''(Y) \tag{1.14}$$

where

$$Q'(X) \in k[X_0, \cdots, X_\mu] = k[V_\Sigma]\ ,$$
$$Q''(Y) \in k[Y_0, \cdots, Y_\rho]\ ,$$

and where furthermore

$$\partial Q''/\partial Y_j = 0 \qquad (0 \leqq j \leqq \rho)\ . \tag{1.15}$$

If char $k \neq 2$, then (1.15) forces $Q'' = 0$, and (ii) follows (take $\Sigma = \Sigma_Q$).

If char $k = 2$, then (by(1.15)) $Q''$ must be of the form

$$Q'' = a_0 Y_0^2 + \cdots + a_\rho Y_\rho^2 \qquad (a_i \in k)\ . \tag{1.16}$$

Identify the linear space $\Lambda \subseteqq \mathbf{P}^\nu$ defined by $X_0 = X_1 = \cdots = X_\mu = 0$ with $\mathbf{P}^\rho = \operatorname{Proj}(k[Y_0, Y_1, \cdots, Y_\rho]) = \operatorname{Proj}(k[Y])$. *Suppose* $Q'' \neq 0$. I claim that:

(**1.17**) *If a linear form* $L \in k[Y]$ *vanishes everywhere on the singular locus of* $\operatorname{Proj}(k[Y]/Q'')$, *then* $L = 0$.

For, if $L \neq 0$, then $L$ cannot lie in $V_\Sigma$ (which is generated by the $X$'s), so $L(\xi) \neq 0$ for some $\xi \in \Sigma$. But all the $X$'s do vanish at $\xi$, and $Q(\xi) = 0$, so $Q''(\xi) = 0$ (cf. (1.14)); and since $L(\xi) \neq 0$, $\xi$ must be *regular* on $\operatorname{Proj}(k[Y]/Q'') = \Lambda \cap (Q = 0)$; hence $\xi \notin \Sigma_Q$, contradiction.

Now one checks (keeping in mind (1.16)) that if

$$\nu - 2 \leqq \dim_k(V_\Sigma) = \mu + 1 \qquad (\text{i.e., } \rho \leqq 2)\ ,$$

then (1.17) is false. (In fact, using Zariski's mixed Jacobian criterion [34a, page 39, Theorem 11]—or otherwise—one finds that the singular locus of $\operatorname{Proj}(k[Y]/Q'')$ is a proper linear subvariety of $\mathbf{P}^\rho$.) Hence $Q'' = 0$ in this case too, and (iii) is proved. Q.E.D.

COROLLARY (1.18). *Let* $(R, \mathfrak{m})$, $\theta$ *be as in* (1.6), *let* $K_2$ *be the set of quadratic forms in the kernel of* $\theta$, *and let*

$$\Sigma = \bigcap\nolimits_{Q \in K_2} \Sigma_Q \qquad (\textit{notation as in } (1.13))\ .$$

*Then*

$$K_2 \subseteqq k[V_\Sigma]$$

*and consequently* $\dim_k(V_\Sigma) \geqq \nu - 1$. *Furthermore, if* $V$ *is any* $k$*-vector subspace of* $k[Z]_1$ *such that* $K_2 \subseteq k[V]$, *then* $V \supseteq V_\Sigma$.

*Proof.* For $K_2 \subseteq k[V_\Sigma]$ it suffices, by (1.13), that $\dim_k(V_\Sigma) \geqq \nu - 2$; but this inequality is trivial if $\nu \leqq 2$, and if $\nu \geqq 3$ the last assertion of (1.6) even gives $\dim_k(V_\Sigma) \geqq \nu - 1$.

Since $K_2 \subseteq k[V_\Sigma]$ and $\dim_k(K_2) = \nu(\nu - 1)/2$ (cf. (1.6)), we must in fact have $\dim_k(V_\Sigma) \geqq \nu - 1$ for *any* $\nu$.

The proof of the last assertion is similar to the proof of (i) in (1.13).

* * *

With the notation and assumptions of (1.18), since $K_2$ *generates* the kernel of $\theta$ (1.6), it is clear from the definition of $V_R$ that

(1.18a) $$V_\Sigma \cong \theta(V_\Sigma) = V_R .$$

As before, we set

$$\tau_R = \text{codimension of } V_R \text{ in } \mathfrak{m}/\mathfrak{m}^2 .$$

(1.18) and (1.12) now give:

COROLLARY (1.19). *Let* $R$ *be as in* (1.6). *Then* $\tau_R \leqq 2$. *If* $\tau_R = 0$, *then for every quadratic transform* $R'$ *of* $R$, *we have* $\operatorname{emdim}(R') < \operatorname{emdim}(R)$.

Under suitable conditions the inequality $\tau_R \leqq 2$ of (1.19) can be improved: the following generalizes somewhat a remark of Hironaka communicated to me by Wahl.

PROPOSITION (1.20). *Let* $(R, \mathfrak{m})$ *be a two-dimensional normal local ring of embedding dimension* $\nu + 1 \geqq 4$, *let* $\bar{W}$ *be the normalization of* $W = \operatorname{Proj}(\bigoplus_{n\geq 0} \mathfrak{m}^n)$, *and assume that* $\bar{W}$ *is finite over* $W$ *and that* $H^1(\mathcal{O}_{\bar{W}}) = 0$ (*assumptions which certainly hold if* $R$ *is pseudo-rational*[1]). *Then* $\tau_R \leqq 1$.

Before proving (1.20) we note:

COROLLARY (1.21). *With assumptions as in* (1.20), *there is at most one quadratic transform* $R'$ *of* $R$ *such that* $\operatorname{emdim}(R') = \operatorname{emdim}(R)$. *If such an* $R'$ *exists, then it has the same residue field as* $R$.

*Proof of* (1.21). If $\tau_R = 0$ use (1.19). If $\tau_R = 1$ (i.e., $\dim_k(V_R) = \nu$) let $(x, z_1, \cdots, z_\nu)$ be a generating set of $\mathfrak{m}$ such that $V_R$ is generated (as a $k$-vector space) by the images of $z_1, \cdots, z_\nu$ in $\mathfrak{m}/\mathfrak{m}^2$. By (1.12), if $\operatorname{emdim}(R') = \operatorname{emdim}(R)$, then $\mathfrak{m}R' = xR'$ and $z_i/x$ is a non-unit in $R'$ $(1 \leqq i \leqq \nu)$; hence $R'$ must be the localization of $R[z_1/x, \cdots, z_\nu/x]$ at the maximal ideal generated by $x, z_1/x, \cdots, z_\nu/x$, and our assertion follows.

*Proof of* (1.20). To begin with, our assumptions imply that $W$ is

[1] or even if $\hat{R}$ is reduced and $H(O_{\bar{W},w}) = H(R) < \infty$ for some $w \in W$ (§(1a), Remark (A)).

*normal* (cf. proof of (1.5)).

Next, if $\tau_R > 1$, then by (1.19) $\tau_R = 2$, i.e., $\dim_k(V_R) = \nu - 1$. Choose a generating set $(z_0, z_1, \cdots, z_\nu)$ of $\mathfrak{m}$ such that the $k$-vector space $V_R$ is generated by the images of $z_0, z_1, \cdots, z_{\nu-2}$ in $\mathfrak{m}/\mathfrak{m}^2$. Since the kernel of $\theta$ is generated by $\nu(\nu-1)/2$ independent quadratic forms $Q_\alpha \in k[V_\Sigma]$ (cf. (1.6) and (1.18)), and $\dim_k(V_\Sigma) = \dim_k(V_R) = \nu - 1$ (remark preceding (1.19)), we must in fact have

$$\text{(1.22)} \qquad \text{kernel of } \theta = (Z_i Z_j)k[Z_0, \cdots, Z_\nu] \qquad (0 \leqq i \leqq j \leqq \nu - 2)$$
$$= (Z_0, \cdots, Z_{\nu-2})^2 k[Z_0, \cdots, Z_\nu] .$$

Thus in $R$ we have $z_i z_j \in \mathfrak{m}^3$, i.e., with $I = (z_0, \cdots, z_{\nu-2})R$, and $z_{\nu-1} = x$, $z_\nu = y$, we have relations

$$\text{(1.23)} \qquad z_i z_j - G_{ij}(x, y) \in I\mathfrak{m}^2 \qquad (0 \leqq i \leqq j \leqq \nu - 2)$$

where $G_{ij}(U_1, U_2) \in R[U_1, U_2]$ ($U_1$, $U_2$ indeterminates) is a cubic form.

Now, fixing $i$ and $j$, since $\nu \geqq 3$ we can choose $l \neq i$ with $0 \leqq l \leqq \nu - 2$, and then, by (1.23),

$$z_l G_{ij}(x, y) - z_i G_{lj}(x, y) \in I^2\mathfrak{m}^2 \subseteqq \mathfrak{m}^5 ;$$

so if $\bar{G}$ is the natural image of $G$ in $k[U_1, U_2]$, we have

$$Z_l \bar{G}_{ij}(Z_{\nu-1}, Z_\nu) - Z_i \bar{G}_{lj}(Z_{\nu-1}, Z_\nu) \in \text{kernel of } \theta ,$$

and since $l \neq i$ it follows at once from (1.22) that $\bar{G}_{ij} = 0$, i.e., *all the coefficients of* $G_{ij}$ *lie in* $\mathfrak{m}$. This implies that $W$ is *not normal* (contradiction), as follows:

Let $R'$ be a quadratic transform of $R$. Then $R'/\mathfrak{m}R'$ is the local ring of a point of the subscheme of $\mathbf{P}^\nu$ defined by the ideal $(Z_0, \cdots, Z_{\nu-2})^2$. Hence the images of the $z_i$ $(0 \leqq i \leqq \nu - 2)$ in $\mathfrak{m}/\mathfrak{m}^2$ vanish at $R'$, so the principal ideal $\mathfrak{m}R'$ is generated either by $x$ or by $y$, say $\mathfrak{m}R' = xR'$; and if $I' = (z_0/x, \cdots, z_{\nu-2}/x)R'$, then

$$\text{(1.23a)} \qquad \sqrt{xR'} = (I', xR') \neq xR' .$$

From (1.23) we get

$$(z_i/x)(z_j/x) - xG_{ij}(1, y/x) \in xI' .$$

Since all the coefficients of $G_{ij}$ lie in $\mathfrak{m} \subseteqq xR'$, we deduce that

$$(I')^2 \subseteqq xI' + x^2R' .$$

It follows that for any valuation ring $R_v \supseteqq R'$, we have $I'R_v \subseteqq xR_v$, and so

$$(I'/x) \subseteqq \text{ integral closure of } R' .$$

Since $I' \nsubseteqq xR'$ (1.23a), $R'$ cannot be normal.

(1e) **Proof of Theorem (1.2).** We are now ready to prove Theorem (1.2).

With $Y$ as in (1.2), let $y_1, y_2, \cdots, y_n$ be all the singular points of $Y$, and set

$$\nu_i + 1 = \text{emdim}(\mathcal{O}_{Y,y_i}) \qquad (1 \leqq i \leqq n) ,$$

$$\nu(Y) = \max_{1\leqq i\leqq n}(\nu_i) \qquad (=1 \text{ if } Y \text{ is non-singular}) .$$

Assume that $\nu(Y) > 1$. Choose $i$ such that $\nu_i = \nu(Y)$, and let $Y' \to Y$ be obtained by blowing up $y_i$. Then (1.5) implies that for each singular point $y$ of $Y'$, the local ring $\mathcal{O}_{Y',y}$ is two-dimensional and pseudo-rational. $Y'$ has at most finitely many singular points (use [**EGA IV**, (6.12.2)] and the fact that $Y'$ is a normal surface). Furthermore,

$$\nu(Y') \leqq \nu(Y) \qquad (\text{cf. } (1.10)) .$$

Now repeat the procedure with $Y'$ in place of $Y$, to get $Y'' \to Y'$, etc. Theorem (1.25) below obviously implies that after a finite number of steps we obtain a surface $Y^*$ with $\nu(Y^*) < \nu(Y)$, and from this Theorem (1.2) follows at once.

A *quadratic sequence* is a sequence (finite or infinite) of homomorphisms of local rings

(1.24) $$R_0 \longrightarrow R_1 \longrightarrow R_2 \longrightarrow \cdots$$

such that for each $i > 0$, $R_i$ is a quadratic transform of $R_{i-1}$, $R_{i-1} \to R_i$ being the canonical map. (Often, but not always, the maps will just be inclusions.)

THEOREM (1.25). *Let $(R, \mathfrak{m})$ be a non-regular two-dimensional pseudo-rational local ring with normal completion $\hat{R}$. Then there exist only finitely many quadratic sequences* (1.24) *for which $R = R_0$, $R_i$ is an $R$-sub-algebra of the fraction field of $R$ for all $i \geqq 0$, and*

$$\text{emdim}(R_0) = \text{emdim}(R_1) = \text{emdim}(R_2) = \cdots .$$

(*In particular, the number of members of such a sequence is bounded above by an integer depending only on $R$.*)

*Proof.* Keep in mind that for any quadratic sequence as in (1.25), *all* the $R_i$ ($i \geqq 0$) are two-dimensional pseudo-rational local rings with normal completions (cf. (1.5)).

We have $\tau_R \leqq 2$ (1.19). If $\tau_R = 0$ then the assertion is trivial (1.19).

Suppose next that $\tau_R = 1$. Then if $R_1$ exists at all, it is *uniquely determined* by $R$ (proof of (1.21)). Set $R_1 = R'$ (assuming that $R_1$ exists). Let $k = R/\mathfrak{m}$, and let $(x, z_1, \cdots, z_\nu)$ be a minimal generating set of $\mathfrak{m}$ such that the $k$-vector space $V_R$ is generated by the images of $(z_1, \cdots, z_\nu)$ in $\mathfrak{m}/\mathfrak{m}^2$; then we have (cf. proof of (1.21)) $\mathfrak{m}R' = xR'$, the maximal ideal $\mathfrak{m}'$ of $R'$ is generated by $(x, z'_1, \cdots, z'_\nu)$ where $z'_i = z_i/x$ $(1 \leqq i \leqq \nu)$, and the residue field $R'/\mathfrak{m}' = k$.

Let

$$\theta\colon k[X, Z_1, \cdots, Z_\nu] \longrightarrow \bigoplus_{n\geqq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

be as usual, and let $Q$ be a quadratic form in the kernel of $\theta$. By (1.18a) we have

$$V_\Sigma = kZ_1 + kZ_2 + \cdots + kZ_\nu\ , \tag{1.26}$$

and by (1.18),

$$Q = Q(Z_1, \cdots, Z_\nu) \in k[Z_1, \cdots, Z_\nu]\ .$$

Since $\theta(Q) = 0$, there is a relation in $R$ of the form

$$Q^*(z_1, \cdots, z_\nu) \in \mathfrak{m}^3$$

where $Q^* \in R[Z_1, \cdots, Z_\nu]$ is a quadratic form whose natural image in $k[Z_1, \cdots, Z_\nu]$ is $Q$. Dividing by $x^2$, we see that in $R'$,

$$Q^*(z'_1, \cdots, z'_\nu) \in xR'\ .$$

Since $\operatorname{emdim}(R') = \operatorname{emdim}(R) = \nu + 1$, therefore $x \notin (z'_1, \cdots, z'_\nu)R'$, so we must in fact have

$$Q^*(z'_1, \cdots, z'_\nu) \in x\mathfrak{m}'\ .$$

Hence if

$$\theta'\colon k[X, Z_1, \cdots, Z_\nu] \longrightarrow \bigoplus_{n\geqq 0} (\mathfrak{m}')^n/(\mathfrak{m}')^{n+1}$$

is as usual, then the kernel of $\theta'$ contains a quadratic form $Q'(X, Z_1, \cdots, Z_\nu)$ such that

$$Q'(0, Z_1, \cdots, Z_\nu) = Q(Z_1, \cdots, Z_\nu)\ . \tag{1.27}$$

I claim that $\tau_{R'} \leqq 1$. For, by (1.18)

$$Q' \in k[V_{\Sigma'}] \qquad \text{(self-evident notation)}$$

whence, by (1.27),

$$Q \in k[V']$$

where $V'$ is the image of $V_{\Sigma'}$ under the $k$-homomorphism $\psi\colon k[X, Z_1, \cdots, Z_\nu] \to k[Z_1, \cdots, Z_\nu]$ for which $\psi(X) = 0$ and $\psi(Z_i) = Z_i$ $(1 \leqq i \leqq \nu)$. Since $Q$ is an arbitrary member of $K_2$ (notation of (1.18)) we have $K_2 \subseteqq k[V']$, so by (1.18) $V' \supseteqq V_\Sigma$. Hence (cf. (1.26))

$$\nu = \dim_k(V_\Sigma) \leqq \dim_k(V') \leqq \dim_k(V_{\Sigma'}) = \dim_k(V_{R'})$$

(the last equality by (1.18a)), so that

$$\tau_{R'} = \operatorname{emdim}(R') - \dim_k(V_{R'}) = \nu + 1 - \dim_k(V_{R'}) \leqq 1\ .$$

Furthermore *if the image of $x$ in $(\mathfrak{m}')/(\mathfrak{m}')^2$ is in $V_{R'}$, then $\tau_{R'} = 0$.* For then $\Sigma'$ lies in the hyperplane $X = 0$, and hence is contained in $\Sigma_0$, the intersection of the singular sets of the schemes

$$(Q' = 0) \cap (X = 0) = \operatorname{Proj}(k[Z_1, \cdots, Z_\nu]/Q) \qquad (0 \neq Q \in K_2) .$$

If $V_0$ is the space of linear forms in $k[Z_1, \cdots, Z_\nu]$ vanishing on $\Sigma_0$, then $K_2 \subseteq k[V_0]$ (cf. proof of (1.18)), so $V_0 \supseteq V_\Sigma$ (1.18), and therefore all the $Z_i$ ($1 \leqq i \leqq \nu$) are in $V_0$, i.e., $\Sigma_0$ is *empty*. Thus $\Sigma'$ is empty, and

$$\dim_k(V_{R'}) = \dim_k(V_{\Sigma'}) = \nu + 1 \qquad \text{(cf. (1.18a))}$$

i.e., $\tau_{R'} = 0$.

If $\tau_{R'} = 0$, then we are done (1.19). Otherwise, we can repeat the whole preceding argument with $R_1$ in place of $R$ and $R_2$ in place of $R'$, but with the *same* $x$. If (1.25) is false, then continuing in this manner, we find a quadratic sequence whose existence contradicts the following proposition:

PROPOSITION (1.28). *Let $R_0$ be any (noetherian) local ring of dimension $>0$ whose completion $\hat{R}_0$ has an* isolated singularity (i.e., *for any non-maximal prime ideal $\mathfrak{p}$ in $\hat{R}_0$, the localization $(\hat{R}_0)_\mathfrak{p}$ is regular*). *Let $R_0 \to R_1 \to R_2 \to \cdots$ be an infinite quadratic sequence satisfying*:

(i) *there is an element $x$ in $R_0$ such that for all $n \geqq 0$, $\mathfrak{m}_n R_{n+1} = x R_{n+1}$ ($\mathfrak{m}_n$ is the maximal ideal of $R_n$), and*

(ii) *for all $n \geqq 0$, $R_{n+1}$ has the same residue field as $R_n$* (i.e., *the natural inclusion $R_n/\mathfrak{m}_n \hookrightarrow R_{n+1}/\mathfrak{m}_{n+1}$ is an* isomorphism).

*Then for all sufficiently large $N$ the local ring $R_N$ is regular.*

Before proving (1.28), let us consider the case $\tau_R = 2$. In this case, $\nu = 2$ (1.20). A discussion of this case, with many more details than we need here, is given in [**RS**, pages 264-268]. (N.B. The integer $\tau$ of *loc. cit.* is $3 - \tau_R$.) A perusal of this discussion shows that we need only consider the case where, for a suitable basis $(x, y, z)$ of $\mathfrak{m}$, we have a relation of the form

$$z^2 - xy^2 \in z\mathfrak{m}^2 + (x, y)^3\mathfrak{m} . \tag{1.29}$$

Setting $R_1 = (R', \mathfrak{m}')$, we have two possibilities (cf. [*ibid.*, page 266]): **either** $R'$ is the unique quadratic transform of $R$ such that $\mathfrak{m}'$ is generated by $(y, x/y, z/y)$, in which case $\tau_{R'} = 0$, and $R_2$ does not exist (1.19); **or** $R'$ is the localization of $k[x, y/x, z/x]$ at the maximal ideal generated by $x$, $y' = y/x$, $z' = z/x$ (so that $R'/\mathfrak{m}' = k$). In this latter case, we have (after dividing (1.29) by $x^2$) a relation

$$(z')^2 - x(y')^2 \in xz'R' + x^2R' .$$

If $\tau_{R'} < 2$, then we have a previously considered case. If $\tau_{R'} = 2$, then **either** we have case (III b) or (III c) of [*loc. cit.*, page 265], in which case there are at most three possibilities for $R_2$ and none for $R_3$; or we have, after replacing $z'$ by $z' + \alpha x$ and $y'$ by $y' + \beta x$ for suitable $\alpha, \beta \in R'$, a relation

$$(z')^2 - x(y')^2 \in z'(\mathfrak{m}')^2 + (x, y')^3\mathfrak{m}' .$$

Now we can repeat the argument with $R_1$ in place of $R$ and $R_2$ in place of $R'$. Continuing in this way, we obtain, as before, a contradiction with (1.28).

It remains to prove (1.28).

**(1f) Proof of Proposition (1.28).**

**(I)** To begin with, we can reduce to the case where $R_0$ is *complete*, as follows: Let $\phi: (R, \mathfrak{m}) \to (R_*, \mathfrak{m}_*)$ be a homomorphism of local rings with $\phi(\mathfrak{m}) \subset (\mathfrak{m}_*)$, such that the induced map of completions $\hat{\phi}: \hat{R} \to \hat{R}_*$ is an isomorphism; in particular, $R_*$ is *flat* over $R$, via $\phi$. Let $(R', \mathfrak{m}')$ be a quadratic transform of $R$. Then $\mathfrak{n} = \mathfrak{m}'(R' \otimes_R R_*)$ is a maximal ideal of $R' \otimes_R R_*$; the localization $R'_* = (R' \otimes_R R_*)_\mathfrak{n}$ is a quadratic transform of $R_*$; and the natural map $\phi': R' \to R'_*$ induces an isomorphism of completions. (The proofs of these statements, and the consequent reduction to the case where $R_0$ is complete, are left to the reader.) Assume then that $R_0$ *is complete, and has an isolated singularity.*

**(II)** Note that $x$ *is not a zero-divisor in* $R_n$ for $n > 0$, since $xR_n = \mathfrak{m}_{n-1}R_n$ is *invertible*. In particular, $x \notin x^2R_1 = \mathfrak{m}_0^2R_1$, so $x \notin \mathfrak{m}_0^2$, and we can extend $x$ to a minimal generating set $(x, y_1, \cdots, y_\nu)$ of $\mathfrak{m}_0$.

I claim that *there exist elements* $\rho_{ij}$ *in* $R_0$ $(i = 1, 2, \cdots, \nu; 0 \leq j < \infty)$ *such that, for each* $j \geq 0$, $\mathfrak{m}_j$ *is generated in* $R_j$ *by the elements* $x, y_{ij}$ $(i = 1, 2, \cdots, \nu)$ *where*

$$y_{ij} = \phi_j(x)^{-j}\phi_j(y_i - \rho_{i0} - \rho_{i1}x - \rho_{i2}x^2 - \cdots - \rho_{ij}x^j) ,$$

$\phi_j$ being the composed map $R_0 \to R_1 \to \cdots \to R_j$. (From now on, to avoid cluttering up formulas, we treat only the case where all the maps $\phi_j$ are *inclusions*. So for example, we can write

$$y_{ij} = x^{-j}(y_i - \rho_{i0} - \rho_{i1}x - \cdots - \rho_{ij}x^j) .$$

To obtain the proof for the general case, the concerned reader can insert $\phi_j$'s in the following arguments as required.) The assertion is clear for $j = 0$, with $\rho_{i0} = 0$ for all $i$. We proceed by induction from $j$ to $j + 1$. By assumption, $R_{j+1}$ is the localization $R^*_\mathfrak{q}$, where

$$R^* = R_j\left[\frac{y_{1j}}{x}, \cdots, \frac{y_{\nu j}}{x}\right] \subseteq R_{j+1}$$

and

$$\mathfrak{q} = \mathfrak{m}_{j+1} \cap R^* .$$

Since $R_{j+1}$ has the same residue field as $R_0$, we can find $\rho_{i,j+1} \in R_0$ such that

$$\frac{y_{ij}}{x} - \rho_{i,j+1} \in \mathfrak{m}_{j+1} .$$

Then the canonical map

$$R_j/\mathfrak{m}_j \longrightarrow R^* \Big/ \left(x, \frac{y_{1j}}{x} - \rho_{1,j+1}, \cdots, \frac{y_{\nu j}}{x} - \rho_{\nu,j+1}\right)$$

is *bijective*, so that $(x, \cdots, (y_{ij}/x) - \rho_{i,j+1}, \cdots)$ is a maximal ideal, necessarily equal to $\mathfrak{q}$. The assertion for $j + 1$ follows.

**(III)** Next, we define elements $z_i \in R_0$:

$$z_i = y_i - \sum_{j=0}^{\infty} \rho_{ij} x^j \qquad (i = 1, 2, \cdots, \nu) .$$

Then

$$x^{-j} z_i \equiv y_{ij} \pmod{x R_j} \qquad (0 \leqq j < \infty)$$

and so $\mathfrak{m}_j$ is generated by $(x, x^{-j}z_1, \cdots, x^{-j}z_\nu)$. It follows that for any $j \geqq 0$,

$$x^j \notin (z_1, \cdots, z_\nu)R_0 = \mathfrak{p}_0 \ \text{(say)} .$$

So $R_0/\mathfrak{p}_0$ has dimension $\geqq 1$, and since its maximal ideal is generated by the image of $x$, $R_0/\mathfrak{p}_0$ is a discrete valuation ring; thus $\mathfrak{p}_0$ is a prime ideal. Similarly for any $j \geqq 0$, the ideal

$$\mathfrak{p}_j = (x^{-j}z_1, \cdots, x^{-j}z_\nu)R_j$$

is prime, $R_j/\mathfrak{p}_j$ being a discrete valuation ring with maximal ideal generated by the image of $x$.

For $j < j'$, we have $\mathfrak{p}_j \subset \mathfrak{p}_{j'}$, $x \notin \mathfrak{p}_{j'}$; so $\mathfrak{p}_j$ is the contraction (inverse image) of $\mathfrak{p}_{j'}$ in $R_j$. Hence, since blowing up a point induces an isomorphism outside the point in question, we have *isomorphisms*

$$(R_0)_{\mathfrak{p}_0} \xrightarrow{\sim} (R_1)_{\mathfrak{p}_1} \xrightarrow{\sim} (R_2)_{\mathfrak{p}_2} \xrightarrow{\sim} \cdots .$$

Note that by assumption $(R_0)_{\mathfrak{p}_0}$ is a regular local ring, of dimension, say, $d$. So the same is true of $(R_j)_{\mathfrak{p}_j}$ ($j \geqq 0$), and therefore $R_j$ has Krull dimension $\geqq d + 1$. We will show: *for all sufficiently large $N$, $\mathfrak{p}_N$ is generated by $d$ elements (whence $\mathfrak{m}_N$ is generated by $d + 1$ elements).* For such $N$, therefore, $R_N$ is *regular*, of dimension $d + 1$, as desired.

**(IV)** Fix an integer $j$, set $R = R_j$, $\mathfrak{p} = p_j$, $z'_i = x^{-j}z_i$. After rearranging, we may assume that $(z'_1, \cdots, z'_\mu)$ ($\mu \leqq \nu$) is a minimal basis of $\mathfrak{p}$. Clearly for $N \geqq 0$, $\mathfrak{p}_{j+N}$ is generated by $x^{-N}z'_1, \cdots, x^{-N}z'_\mu$. It will therefore suffice to show that *if $\mu > d$, then for all sufficiently large $N$, $\mathfrak{p}_{j+N}$ is generated by a* *proper subset of* $\{x^{-N}z'_1, \cdots, x^{-N}z'_\mu\}$.

Recall that $R_\mathfrak{p}$ is regular, of dimension $d$, and that $R/\mathfrak{p}$ is a discrete valuation ring, with maximal ideal $x(R/\mathfrak{p})$. We may therefore assume that $z'_1, \cdots, z'_d$ generate $\mathfrak{p}R_\mathfrak{p}$, whence ($\mu$ being $> d$) there is a relation

$$w_\mu z'_\mu = \sum_{i=1}^d w_i z'_i \qquad (w_\mu \notin \mathfrak{p})\,; \tag{1.30}$$

furthermore, for suitable units $\varepsilon_i$ in $R$, and suitable $q_i$ with $0 \leqq q_i \leqq \infty$, we have

$$w_i \equiv \varepsilon_i x^{q_i} \pmod{\mathfrak{p}} \qquad (i = 1, 2, \cdots, d, \mu)$$

(by convention $x^\infty = 0$), so that (1.30) gives

$$\varepsilon_\mu x^{q_\mu} z'_\mu \equiv \sum_{i=1}^d \varepsilon_i x^{q_i} z'_i \pmod{\mathfrak{p}^2} \qquad (q_\mu < \infty)\,. \tag{1.31}$$

Let

$$q = \min(q_1, \cdots, q_d, q_\mu)$$

so that $q \leqq q_\mu < \infty$. Dividing (1.31) by $x^{2q}$, we obtain, in $R_{j+q}$,

$$\varepsilon_\mu x^{q_\mu - q}(x^{-q}z'_\mu) \equiv \sum_{i=1}^d \varepsilon_i x^{q_i - q}(x^{-q}z'_i) \pmod{\mathfrak{p}_{j+q}^2}\,.$$

Since one of the exponents $q_i - q$ ($i = 1, 2, \cdots, d, \mu$) is 0, we see (Nakayama's lemma) that in the basis $(x^{-q}z'_1, \cdots, x^{-q}z'_\mu)$ of $\mathfrak{p}_{j+q}$, at least one of the members $x^{-q}z'_i$ ($i = 1, 2, \cdots, d, \mu$) is redundant.

This completes the proof of Proposition (1.28) and of Theorem (1.2).

(1g) **Uniformization of rank two valuations by blowing up and normalizing.** From (1.28) we can deduce a special case (needed in Section 3) of "local uniformization."

Let $R$ be a noetherian local domain, and let $v$ be a valuation of the fraction field $K$ of $R$, such that $v$ dominates $R$ (i.e., $v(r) \geqq 0$ for all $r \in R$, and $v(r) > 0$ if $r$ is a non-unit in $R$). The "normal transform" of $R$ along $v$ is the unique local ring $R^*$ of a point on the normalization of the blow-up of (the closed point of) Spec($R$), such that $R^*$ is dominated by $v$. The "normal sequence along $v$ determined by $R$" is the sequence

$$R = R_0 < R_1 < R_2 < \cdots \tag{1.32}$$

where for each $i \geqq 0$, $R_{i+1}$ is the normal transform of $R_i$ along $v$.

PROPOSITION (1.33). *Let $R$ be a two-dimensional local ring whose completion $\hat{R}$ is normal. Let $v$ be a rank two valuation of the fraction field of $R$ such that $v$ dominates $R$, and let* (1.32) *be the normal sequence along $v$ determined by $R$. Then for some $N$, the local ring $R_N$ is regular.*

*Proof.* We first remark that *all the local rings $R_n$ are essentially of* *finite type over R, and have normal completions.* The proof is by induction on $n$: if $\hat{R}_i$ is normal, then by Rees [**28**], $R_{i+1}$ is essentially of finite type over $R_i$, and $\hat{R}_{i+1}$ is normal (cf. Proposition in §(1b)).

Next, we recall that *the valuation ring* $R_v$ *of* $v$ *is* $\bigcup_{n\geq 0} R_n$ (cf. [**34**, Theorem 10, page 681] for surfaces over fields; a similar argument works in the general case—use the following results in [**38**]: Corollary, page 21; Proposition 1, page 330; Corollary 2, page 339; and the argument in the middle of page 392).

Now $v$ is *discrete* of rank two ([**1**, page 330, Theorem 1], or [**38**, page 338, Corollary 1]), and hence the maximal ideal of $R_v$ is principal, generated, say, by $x$. Furthermore, for each $n \geqq 0$, the residue field $k_v$ of $R_v$ is finite algebraic over the residue field $k_n$ of $R_n$ (*loc. cit.*). Hence for some $m$ we have $x \in R_m$ and $k_v = k_m$. Without loss of generality assume that $m = 0$; then from (1.28) it follows that $v$ dominates a *regular* local ring $R'$ essentially of finite type over $R_0$ (namely some member of the quadratic sequence "along $v$" starting with $R_0$).

Since $R_v = \bigcup_{n\geq 0} R_n$, therefore $R_n$ contains $R'$ for some $n$, and $R_n$ is essentially of finite type over $R'$ (cf. beginning of this proof). From (1.4) we get that $R_n$ is *pseudo-rational*. Then from (1.28) and (1.5), we conclude that $R_N$ is regular for some $N \geqq n$.

## 2. Duality and vanishing



**(2a) The dual of $H^1$.** The purpose of Section 2 is to establish an appropriate generalization of the relation $(**)$ of the introduction. To state things precisely, we need some preliminary definitions.

Let $R$ be a two-dimensional regular local ring, with fraction field $K$, and let $S$ be a two-dimensional regular local ring with $R \subseteqq S \subseteqq K$. Then [**1**, page 343, Theorem 3] there is a unique *quadratic sequence*

$$R = R_0 \subset R_1 \subset \cdots \subset R_n = S$$

(i.e., each $R_i$ $(0 < i \leqq n)$ is a quadratic transform (§(1b)) of $R_{i-1}$). *We define a fractionary $S$-ideal* $\omega_S \supseteqq S$ by induction as follows:

$$\omega_R = R\ ,$$
$$\omega_{R_{i+1}} = \omega_{R_i}(\mathfrak{m}_i R_{i+1})^{-1} \qquad (i \geqq 0)$$

where $\mathfrak{m}_i$ is the maximal ideal of $R_i$ (so that $\mathfrak{m}_i R_{i+1}$ is an invertible $R_{i+1}$-ideal).

Using the fact that any proper birational map of non-singular surfaces can be factored into a sequence of blow-ups ([**36**, page 46]; cf. also [**RS**, page 204]) we see easily that *for any proper birational map* $f\colon Y \to \operatorname{Spec}(R)$ *with* $Y$ *non-singular there is a unique invertible* $\mathcal{O}_Y$*-submodule* $\boldsymbol{\omega}_Y$ *of the constant sheaf* $K\mathcal{O}_Y$ *(=sheaf of rational functions on* $Y$*) such that for any closed point* $y \in Y$*, with local ring* $S = \mathcal{O}_{Y,y}$*, the stalk* $\boldsymbol{\omega}_{Y,y}$ *coincides with* $\boldsymbol{\omega}_S$*.*

Now let $L$ be a finite algebraic field extension of $K$, and fix a *non-zero* $K$-linear map $T\colon L \to K$. (For example $T$ could be "Trace" if $L$ is separable over $K$.) Let $f\colon Y \to \operatorname{Spec}(R)$, $\boldsymbol{\omega}_Y$, be as above, and for any $y \in Y$ let $\bar{\mathcal{O}}_y$ be the integral closure in $L$ of the local ring $\mathcal{O}_{Y,y}$. Set

$$C_Y = \{\mu \in L \mid T(\mu\bar{\mathcal{O}}_y) \subseteq \boldsymbol{\omega}_{Y,y} \text{ for all } y \in Y\}\ .$$

In particular, set

$$C = C_{\operatorname{Spec}(R)}\ .$$

I claim that $C_Y \subseteq C$. For this, it clearly suffices to show that

$$R = \textstyle\bigcap_{y\in Y} \boldsymbol{\omega}_{Y,y}\ \left(=H^0(Y, \boldsymbol{\omega}_Y)\right)\ . \tag{2.1}$$

But if $f(y) = \mathfrak{p} \in \operatorname{Spec}(R)$, and $\mathfrak{p}$ is not the maximal ideal in $R$, then $\boldsymbol{\omega}_{Y,y} = R_{\mathfrak{p}}$, and from this (2.1) is immediate.

Here is the main result.

THEOREM (2.2). *Let* $(R, \mathfrak{m})$ *be a two-dimensional regular local ring, with fraction field* $K$*, and let* $L$ *be a finite algebraic field extension of* $K$ *such that for every proper birational* $f\colon Y \to \operatorname{Spec}(R)$ *with* $Y$ *non-singular, the normalization* $\bar{Y}$ *of* $Y$ *in* $L$ *is finite over* $Y$.[1] *Let* $I$ *be the injective hull of the* $R$*-module* $R/\mathfrak{m}$*. Then, with* $C_Y \subseteq C$ *as above, there is an isomorphism of* $R$*-modules*

$$C/C_Y \xrightarrow{\ \sim\ } \operatorname{Hom}_R\left(H^1(\bar{Y}, \mathcal{O}_{\bar{Y}}), I\right)\ .$$

The precise role played by Theorem (2.2) in the proof of Theorem* §(1a) is explained at the beginning of Section 3.

Theorem (2.2) is a consequence of the following two results. First there is a "duality theorem" (2.3), which is actually a special case of a corollary of Grothendieck's local and global duality theorems (cf. Appendix (2d) below). In order to make this paper independent of the massive machinery of duality theory, I will give an ad hoc proof; it should however be said that the first proof I had was the one presented in the appendix. The result itself was

[1] It suffices for this that $L \otimes_K \hat{K}$ be *reduced*, where $\hat{K}$ is the fraction field of the completion $\hat{R}$ (cf. [28] and [**EGA** $0_{\text{IV}}$, (23.1.7)]).

suggested by Laufer's use of Serre duality in [**22**].

Secondly there is a "vanishing theorem" (2.4) which generalizes a result of Laufer ([**22**, Theorem 3.2]).[1]

THEOREM (2.3). *Let* $(R, \mathfrak{m})$ *be a two-dimensional regular local ring and let* $f\colon Y \to \mathrm{Spec}(R)$ *be a proper birational map with* $Y$ *non-singular. Set* $\omega = \omega_Y$, *and let* $E = f^{-1}(\{\mathfrak{m}\})$ *be the closed fibre on* $Y$. *Then* $H_E^2(\omega) = I$, *the injective hull of* $R/\mathfrak{m}$; *and for all coherent* $\mathcal{O}_Y$*-modules* $\mathcal{F}$, *the Yoneda map*

$$H_E^1(\mathcal{F}) \longrightarrow \mathrm{Hom}_R\big(\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \omega), H_E^2(\omega)\big)$$

*is an isomorphism.*

*Explanation.* To any element

$$0 \longrightarrow \omega \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \omega)$ associate the coboundary map $H_E^1(\mathcal{F}) \to H_E^2(\omega)$. This gives a pairing

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \omega) \times H_E^1(\mathcal{F}) \longrightarrow H_E^2(\omega)$$

to which the map in (2.3) corresponds.

THEOREM (2.4). *Let* $A$ *be a two-dimensional normal semilocal ring with Jacobson radical* $\mathfrak{n}$, *let* $g\colon X \to \mathrm{Spec}(A)$ *be a proper birational map with* $X$ *normal, and let* $E_1, \cdots, E_n$ *be the irreducible components of* $E = X \otimes_A (A/\mathfrak{n})$. *If* $\mathcal{L}$ *is an invertible* $\mathcal{O}_X$*-module such that* $(\mathcal{L}\cdot E_i) \leqq 0$ *for all* $i$, *then* $H_E^1(\mathcal{L}) = 0$.

*Explanation.* $E_i$ is a curve which is proper over a field $k = A/\mathfrak{m}$, $\mathfrak{m}$ some maximal ideal of $A$; $(\mathcal{L}\cdot E_i)$ is by definition the degree (over $k$) of the invertible $\mathcal{O}_{E_i}$-module $j^*\mathcal{L}$, where $j\colon E_i \to X$ is the inclusion map.[2]

Before proving (2.3) and (2.4), let us deduce (2.2). Let $\pi\colon \bar{Y} \to Y$ be the canonical (finite) map. It is easily seen that

$$C_Y \cong H^0(Y, \mathcal{C}_Y)$$

where $\mathcal{C}_Y$ is the $\mathcal{O}_Y$-module

$$\mathcal{C}_Y = \mathcal{H}om_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\bar{Y}}, \omega) \qquad (\omega = \omega_Y)\,.$$

Furthermore, since $Y$ is regular, of dimension 2, both $\pi_*\mathcal{O}_{\bar{Y}}$ and $\mathcal{C}_Y$ are *locally*

[1] For Laufer's result in any characteristic $p \geqq 0$, cf. [32, page 21, Prop. (2.6)]. For higher-dimensional generalizations (in char 0), cf. [5].

[2] In the proofs of (2.3) and (2.4) we will use freely the properties of intersection numbers given in §§ 10 and 13 and page 221 of [RS]. *Correction*: In Lemma (10.1) of *loc. cit.*, replace $\mathcal{F} \oplus \mathcal{N}$ by $\mathcal{F} \otimes \mathcal{N}$; also, in the proof replace both $i_*i^*\mathcal{F}$ and $i_*i^*\mathcal{G}$ by $\beta\alpha(\mathcal{F}) + \gamma(\mathcal{G})$, where $\alpha, \beta, \gamma$ are given by

$$\mathcal{F} \xrightarrow[\alpha]{} i_*i^*\mathcal{F} \xrightarrow[\beta]{\sim} i_*i^*\mathcal{G} \xleftarrow[\gamma]{} \mathcal{G}\,.$$

*free* $\mathcal{O}_Y$-modules, of rank $[L\colon K]$. Hence

$$\begin{aligned} H^1(\bar{Y}, \mathcal{O}_{\bar{Y}}) &= H^1(Y, \pi_*\mathcal{O}_{\bar{Y}}) \\ &= H^1\big(Y, \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{C}_Y, \omega)\big) \\ &= \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{C}_Y, \omega) \end{aligned}$$

and (2.3) gives an isomorphism

$$H^1_E(\mathcal{C}_Y) \xrightarrow{\sim} \operatorname{Hom}_R\big(H^1(\bar{Y}, \mathcal{O}_{\bar{Y}}), I\big)\,.$$

So to prove (2.2), we need an isomorphism

$$C/C_Y \xrightarrow{\sim} H^1_E(\mathcal{C}_Y)\,.$$

Let

$$U = Y - E = \operatorname{Spec}(R) - \{\mathfrak{m}\}\,.$$

Then we have an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0_E(\mathcal{C}_Y) & \longrightarrow & H^0(Y, \mathcal{C}_Y) & \longrightarrow & H^0(U, \mathcal{C}_Y) & \longrightarrow & H^1_E(\mathcal{C}_Y) \longrightarrow & H^1(Y, \mathcal{C}_Y) \\ & \| & & \| & & \| & & & \\ & 0 & & C_Y & & C & & & \end{array}$$

and we need only show that $H^1(Y, \mathcal{C}_Y) = 0$. But

$$\begin{aligned} H^1(Y, \mathcal{C}_Y) &= H^1\big(Y, \mathcal{H}om(\pi_*\mathcal{O}_{\bar{Y}}, \omega)\big) \\ &= \operatorname{Ext}^1_{\mathcal{O}_Y}(\pi_*\mathcal{O}_{\bar{Y}}, \omega) \end{aligned}$$

is a finite-length $R$-module which is Matlis-dual $\big($by (2.3)$\big)$ to

$$H^1_E(\pi_*\mathcal{O}_{\bar{Y}}) = H^1_{\bar{E}}(\mathcal{O}_{\bar{Y}}) \qquad \big(\bar{E} = \pi^{-1}(E)\big)\,;$$

and by (2.4), applied to the map $\bar{Y} \longrightarrow \operatorname{Spec}(\bar{R})$ ($\bar{R}$ = integral closure of $R$ in $L$) we have $H^1_{\bar{E}}(\mathcal{O}_{\bar{Y}}) = 0$. Q.E.D.

**(2b) Proof of Theorem (2.3).**

LEMMA (2.5). *$\omega = \mathcal{O}(\mathcal{K})$, where $\mathcal{K}$ is the unique effective divisor supported on $E$ such that for every effective divisor $D$ supported on $E$ we have*

$$(\mathcal{K}\cdot D) = -(D\cdot D) - 2\chi(D) \tag{2.6}$$

$\big($*where $\chi(D) = h^0(\mathcal{O}_D) - h^1(\mathcal{O}_D)$, $h^i(\cdot)$ being the length of the $R$-module $H^i(Y, \cdot)$*$\big)$.

*Proof.* It is clear that $\omega = \mathcal{O}(\mathcal{K})$ for some effective divisor $\mathcal{K}$ supported on $E$. If (2.6) holds for $\mathcal{K}$, then the "uniqueness" statement follows from the negative definiteness of the intersection matrix $\big((E_i\cdot E_j)\big)$.

$f\colon Y \longrightarrow \operatorname{Spec}(R)$ can be realized as a succession of $n$ blow-ups, where $n$ is the number of irreducible components of $E$. We proceed by induction on

$n$. There being nothing to prove if $n = 0$, it suffices, then, to show that if (2.5) holds for $Y$, and if $g\colon Y' \to Y$ is obtained by blowing up a closed point $y \in Y$, then (2.5) also holds for $Y'$.

Let $E_1, \cdots, E_n$ be the (reduced) irreducible components of $E$, let $E_0' = g^{-1}(y)$, and let $E_i'$ be the *proper transform* of $E_i$ on $Y'$, i.e., $E_i' = g^{-1}(E_i) - \varepsilon_i E_0'$, where $\varepsilon_i = 1$ if $y$ lies on $E_i$ and $\varepsilon_i = 0$ otherwise $(1 \leq i \leq n)$. Then $E_0', E_1', \cdots, E_n'$ are the irreducible components of $E' = (f \circ g)^{-1}(\{\mathfrak{m}\})$.

Setting $\boldsymbol{\omega}_Y = \mathcal{O}(\mathcal{K})$, $\boldsymbol{\omega}_{Y'} = \mathcal{O}(\mathcal{K}')$, we have (by the definition of $\boldsymbol{\omega}$)

$$\mathcal{K}' = g^{-1}(\mathcal{K}) + E_0' .$$

Now $g^{-1}(\mathcal{K})\cdot E_0' = 0$ [**RS**, bottom of page 227], and $(E_0'\cdot E_0') = -\chi(E_0')$, so that

$$(\mathcal{K}'\cdot E_0') = (E_0'\cdot E_0') = -(E_0'\cdot E_0') - 2\chi(E_0') .$$

Furthermore, for $1 \leq i \leq n$ we have

$$\begin{aligned}(\mathcal{K}'\cdot E_i') &= \big(g^{-1}(\mathcal{K})\cdot E_i'\big) + (E_0'\cdot E_i') \\ &= (\mathcal{K}\cdot E_i) + \varepsilon_i(E_0'\cdot E_i') \qquad \big([ibid.]\big) .\end{aligned}$$

The inductive hypothesis gives

$$(\mathcal{K}\cdot E_i) = -(E_i\cdot E_i) - 2\chi(E_i) .$$

Since $g$ maps $E_i'$ isomorphically onto $E_i$, therefore

$$\chi(E_i) = \chi(E_i') .$$

Finally,

$$\begin{aligned}(E_i'\cdot E_i') &= \big(g^{-1}(E_i)\cdot E_i'\big) - \varepsilon_i(E_0'\cdot E_i') \\ &= (E_i\cdot E_i) - \varepsilon_i(E_0'\cdot E_i') \qquad \big([ibid.]\big) .\end{aligned}$$

Combining all these equalities, we get

$$(\mathcal{K}'\cdot E_i') = -(E_i'\cdot E_i') - 2\chi(E_i') .$$

Thus

$$(\mathcal{K}'\cdot D') = -(D'\cdot D') - 2\chi(D') \tag{2.7}$$

holds (on $Y'$) if $D'$ is any one of the irreducible components of $E'$. Since both sides of (2.7) are additive in $D'$ (cf. [**RS**, middle of page 249]), (2.7) must hold for *any* effective $D'$ supported on $E'$. Q.E.D.

LEMMA (2.8). *Let $\mathcal{K}$ be as in* (2.5). *Then for any effective divisor $D$ on $Y$ supported in $E$, the $R$-module $H^1(Y, \mathcal{O}(D))$ has length*

$$\lambda_D = \frac{1}{2}(\mathcal{K} - D\cdot D) .$$

*Proof.* We have an exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0 .$$

Since $H^1(\mathcal{O}_Y) = 0$ (because $R$ is *rational*, §(1a)), and $H^2(\mathcal{O}_Y) = 0$ (because the fibres of $f$ have dimension $\leqq 1$), therefore we have an *isomorphism*

$$H^1(\mathcal{O}(D)) \xrightarrow{\sim} H^1(\mathcal{O}_D(D)) .$$

Furthermore, the restrictions of $\mathcal{O}_Y$ and $\mathcal{O}(D)$ to $U = Y - E$ are both equal to $\mathcal{O}_U$, and we see from the natural commutative diagram

$$\begin{array}{ccccc} 0 = H^0_E(\mathcal{O}(D)) & \longrightarrow & H^0(Y, \mathcal{O}(D)) & \longrightarrow & H^0(U, \mathcal{O}(D)) \\ & & \uparrow \gamma & & \| \\ & & H^0(Y, \mathcal{O}_Y) & \longrightarrow & H^0(U, \mathcal{O}_Y) \\ & & \| & & \| \\ & & R & = & H^0(U, \mathcal{O}_U) \end{array}$$

that $\gamma$ is *bijective*, whence

(2.9) $$H^0(\mathcal{O}_D(D)) \subseteqq H^1(\mathcal{O}_Y) = 0 .$$

It follows that

$$\lambda_D = -h^0(\mathcal{O}_D(D)) + h^1(\mathcal{O}_D(D)) = -\chi(\mathcal{O}_D(D)) .$$

But

$$(D \cdot D) = \chi(\mathcal{O}_D(D)) - \chi(D)$$

([**RS**, top of page 223]), and by (2.5)

$$(\mathcal{K} \cdot D) = -(D \cdot D) - 2\chi(D)$$

so that

$$\begin{aligned} (\mathcal{K} - D \cdot D) &= -2[(D \cdot D) + \chi(D)] \\ &= -2\chi(\mathcal{O}_D(D)) \\ &= 2\lambda_D . \end{aligned}$$

Q.E.D.

COROLLARY (2.10). *$H^1(Y, \omega) = 0$, and hence $H^2_E(\omega) = I$.*

*Proof.* Since $\omega = \mathcal{O}(\mathcal{K})$, (2.8) shows that $H^1(Y, \omega) = 0$. Setting $U = Y - E = \operatorname{Spec}(R) - \{\mathfrak{m}\}$, we have $H^1(U, \mathcal{O}_U) = I$ [**10**, Propositions 4.10 and 4.13]. Finally, we have an exact sequence

$$\begin{array}{ccccccc} 0 = H^1(Y, \omega) & \longrightarrow & H^1(U, \omega) & \longrightarrow & H^2_E(\omega) & \longrightarrow & H^2(Y, \omega) = 0 . \\ & & \| & & & & \\ & & H^1(U, \mathcal{O}_U) & & & & \end{array}$$

Q.E.D.

We proceed now with the proof of (2.3). Since $Y$ is obtained from $\operatorname{Spec}(R)$ by a sequence of blow-ups, there is an effective divisor $D$ on $Y$ supported in $E$ such that the invertible $\mathcal{O}_Y$-ideal $\mathcal{O}(-D)$ is *very ample.* So, with $\mathcal{O}(p) = \mathcal{O}(-pD)$ $(p = 0, \pm 1, \pm 2, \cdots)$, there is for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ an exact sequence

$$\mathcal{O}(-m)^s \xrightarrow{\alpha} \mathcal{O}(-n)^t \xrightarrow{\beta} \mathcal{F} \longrightarrow 0$$

with suitable positive integers $s, t, m, n$ $(m \geqq n)$. Here $n$ can be chosen as large as we please. For any $R$-module $M$, set

$$M^* = \operatorname{Hom}_R(M, H^2_E(\omega)) = \operatorname{Hom}_R(M, I) \qquad (\text{cf. } (2.10)) .$$

We have then a commutative diagram with exact rows

$$\begin{array}{ccccccc} H^2_E(\mathcal{O}(-m)^s) & \longrightarrow & H^2_E(\mathcal{O}(-n)^t) & \longrightarrow & H^2_E(\mathcal{F}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \operatorname{Hom}(\mathcal{O}(-m)^s, \omega)^* & \longrightarrow & \operatorname{Hom}(\mathcal{O}(-n)^t, \omega)^* & \longrightarrow & \operatorname{Hom}(\mathcal{F}, \omega)^* & \longrightarrow & 0 . \end{array} \tag{2.11}$$

We want to show that

(**2.12**) *For any sufficiently large n the natural map*

$$\delta\colon H^2_E(\mathcal{O}(-n)) \longrightarrow \operatorname{Hom}(\mathcal{O}(-n), \omega)^*$$

*is bijective.*

In view of (2.11), (2.12) will imply that for *any* coherent $\mathcal{F}$ the natural map

$$H^2_E(\mathcal{F}) \longrightarrow \operatorname{Hom}(\mathcal{F}, \omega)^*$$

is bijective. We can then prove (2.3) as follows:

If $\mathcal{G}$ is the kernel of $\beta$, we have a commutative diagram

$$\begin{array}{ccccccc} H^1_E(\mathcal{O}(-n)^t) & \longrightarrow & H^1_E(\mathcal{F}) & \longrightarrow & H^2_E(\mathcal{G}) & \longrightarrow & H^2_E(\mathcal{O}(-n)^t) \\ \downarrow & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ \operatorname{Ext}^1(\mathcal{O}(-n)^t, \omega)^* & \longrightarrow & \operatorname{Ext}^1(\mathcal{F}, \omega)^* & \longrightarrow & \operatorname{Hom}(\mathcal{G}, \omega)^* & \longrightarrow & \operatorname{Hom}(\mathcal{O}(-n)^t, \omega)^* \end{array} \tag{2.13}$$

with exact rows, the last two vertical arrows being isomorphisms, and the first two arising from Yoneda pairings. If $n$ is sufficiently large, then

$$\operatorname{Ext}^1(\mathcal{O}(-n)^t, \omega) = H^1(\omega \otimes \mathcal{O}(n))^t = 0 .$$

*Furthermore,* for any $n \geqq 0$,

$$H^1_E(\mathcal{O}(-n)^t) = 0 . \tag{2.14}$$

(This special case of Theorem (2.4) can be proved directly as follows: $\mathcal{O}(1)$, being very ample, is generated by its global sections; since $\mathcal{O}(1) \subseteqq \mathcal{O}_Y$, we conclude that

$$\mathcal{O}(1) = \mathfrak{n}\mathcal{O}_Y \tag{2.15}$$

for some $\mathfrak{m}$-primary ideal $\mathfrak{n}$ in $R$. Then, as in the first part of the proof of (2.4) (§(2c) below), we see that

$$H^1_E(\mathcal{O}(-n)) = \varinjlim_t H^0(\mathcal{O}(-n-t)/\mathcal{O}_Y)\ ;$$

but, with $r = n + t$,

$$H^0(\mathcal{O}(-n-t)/\mathcal{O}_Y) = H^0(\mathcal{O}_{rD}(rD)) = 0 \qquad \text{(cf. (2.9)) .)}$$

Now from (2.13) we deduce that

$$H^1_E(\mathcal{F}) \longrightarrow \operatorname{Ext}^1(\mathcal{F}, \boldsymbol{\omega})^*$$

is bijective, proving (2.3).

It remains to prove (2.12). Set $\boldsymbol{\omega}(n) = \boldsymbol{\omega} \otimes \mathcal{O}(n)$, so that

$$\operatorname{Hom}(\mathcal{O}(-n), \boldsymbol{\omega}) = H^0(Y, \boldsymbol{\omega}(n))\ .$$

The restrictions of $\boldsymbol{\omega}(n)$ and $\mathcal{O}(-n)$ to $U = Y - E$ are both equal to $\mathcal{O}_U$. We have exact sequences (cf.(2.14))

$$\begin{array}{c} 0 = H^1_E(\mathcal{O}(-n)) \longrightarrow H^1(Y, \mathcal{O}(-n)) \longrightarrow H^1(U, \mathcal{O}(-n)) \longrightarrow H^2_E(\mathcal{O}(-n)) \longrightarrow H^2(Y, \mathcal{O}(-n)) = 0 \\ \| \\ H^1(U, \mathcal{O}_U) = I \end{array} \qquad \text{(cf. proof of (2.10))}$$

and

$$0 = H^0_E(\boldsymbol{\omega}(n)) \longrightarrow H^0(\boldsymbol{\omega}(n)) \xrightarrow{\ \gamma\ } H^0(U, \boldsymbol{\omega}(n)) = H^0(U, \mathcal{O}_U) = R$$

from which we derive a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(\mathcal{O}(-n)) & \longrightarrow & I & \longrightarrow & H^2_E(\mathcal{O}(-n)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta'} & & \downarrow{\wr} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & (R/H^0(\boldsymbol{\omega}(n)))^* & \longrightarrow & R^* & \xrightarrow[\gamma^*]{} & H^0(\boldsymbol{\omega}(n))^* & \longrightarrow & 0\ , \end{array}$$

where $I \to R^*$ is the natural isomorphism, and $\delta$ is the map of (2.12). To show that $\delta$ is bijective, we must show that $\delta'$ is bijective, i.e. (since $\delta'$ is clearly injective) that the $R$-module $(R/H^0(\omega(n)))^*$ has the same length as $H^1(\mathcal{O}(-n))$, viz. (by (2.8))

$$\frac{1}{2}(\mathcal{K} - nD \cdot nD)\ .$$

Note that any finite-length $R$-module $M$ has the same length as $M^*$ [26, page 526, Theorem 4.2]. Without loss of generality, assume that $n$ is large enough so that

$$\boldsymbol{\omega} = \mathcal{O}(\mathcal{K}) \subseteq \mathcal{O}(-n) \qquad \text{(cf. (2.15))}\ ;$$

and that furthermore, $H^1(Y, \omega(n)) = 0$. Replacing $D$ by $nD$, we may assume that $n = 1$, so that $\mathcal{O}(-n) = \mathcal{O}(D)$, $\omega(n) = \mathcal{O}(\mathcal{K} - D)$ and

$$H^1(\mathcal{O}(\mathcal{K} - D)) = 0 .$$

It suffices then to show that the $R$-module $R/H^0(\mathcal{O}(\mathcal{K} - D))$ has length $1/2(\mathcal{K} - D\cdot D)$.

The exact sequences

$$0 \longrightarrow H^0(\mathcal{O}(\mathcal{K} - D)) \longrightarrow \underset{\displaystyle \overset{\|}{R}}{H^0(\mathcal{O}_Y)} \longrightarrow H^0(\mathcal{O}_{D-\mathcal{K}}) \longrightarrow H^1(\mathcal{O}(\mathcal{K} - D)) = 0$$

and

$$0 = H^1(\mathcal{O}_Y) \longrightarrow H^1(\mathcal{O}_{D-\mathcal{K}}) \longrightarrow H^2(\mathcal{O}(\mathcal{K} - D)) = 0$$

show that the length of $R/H^0(\mathcal{O}(\mathcal{K} - D))$ is $\chi(D - \mathcal{K})$. Now, by (2.6),

$$(\mathcal{K}\cdot D - \mathcal{K}) = -(D - \mathcal{K}\cdot D - \mathcal{K}) - 2\chi(D - \mathcal{K}) ,$$

i.e.,

$$\begin{aligned} 2\chi(D - \mathcal{K}) &= -(D - \mathcal{K}\cdot D - \mathcal{K}) - (\mathcal{K}\cdot D - \mathcal{K}) = -(D\cdot D - \mathcal{K}) \\ &= (\mathcal{K} - D\cdot D) \end{aligned}$$

and the conclusion follows. Q.E.D.

(2c) **Proof of Theorem** (2.4). When $X$ is non-singular, the proof of (2.4) is quite short (cf. [**32**, page 21, Prop. (2.6)]). (And the general case is easily reduced to the non-singular case whenever there exists a desingularization $X' \to X$!) Our proof is basically the same as that of *loc. cit*; its length is due to the unavoidable technical complications resulting from the presence of singular points on $X$.

(I) We first recall some terminology. (For more details cf. [4].) A fractionary $\mathcal{O}_X$-ideal $\mathcal{I}$ is a non-zero coherent $\mathcal{O}_X$-submodule of the sheaf of rational functions on $X$. For such an $\mathcal{I}$, we set $\mathcal{I}' = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}; \mathcal{O}_X)$; $\mathcal{I}'$ is also a fractionary $\mathcal{O}_X$-ideal. We say that $\mathcal{I}$ is *divisorial* if $\mathcal{I} = \mathcal{J}'$ for some fractionary ideal $\mathcal{J}$, or equivalently, $\mathcal{I} = \mathcal{I}''$. Note that when $\mathcal{I} \subseteq \mathcal{O}_X$, $\mathcal{I}''$ is locally the intersection of those primary components of $\mathcal{I}$ which belong to height one primes.

(II) We have

$$\begin{aligned} H^1_E(\mathcal{L}) &= \varinjlim_t \operatorname{Ext}^1(\mathcal{O}_X/\mathfrak{n}^t\mathcal{O}_X, \mathcal{L}) \\ &= \varinjlim_t H^0(\mathcal{E}xt^1(\mathcal{O}_X/\mathfrak{n}^t\mathcal{O}_X, \mathcal{L})) . \end{aligned}$$

(The last equality comes from the canonical exact sequence

$$H^1(\mathcal{H}om) \longrightarrow \mathrm{Ext}^1 \longrightarrow H^0(\mathcal{E}xt^1) \longrightarrow H^2(\mathcal{H}om).)$$

Since

$$\mathcal{H}om(\mathcal{O}_X/\mathfrak{n}^t\mathcal{O}_X, \mathfrak{L}) = 0 = \mathcal{E}xt^1(\mathcal{O}_X, \mathfrak{L}) ,$$

therefore

$$\begin{aligned}\mathcal{E}xt^1(\mathcal{O}_X/\mathfrak{n}^t\mathcal{O}_X, \mathfrak{L}) &= \mathcal{H}om(\mathfrak{n}^t\mathcal{O}_X, \mathfrak{L})/\mathcal{H}om(\mathcal{O}_X, \mathfrak{L}) \\ &= \mathfrak{L} \otimes (\mathcal{I}_t/\mathcal{O}_X) \qquad \big(\mathcal{I}_t = \mathcal{H}om(\mathfrak{n}^t\mathcal{O}_X, \mathcal{O}_X)\big) .\end{aligned}$$

So it will suffice to show:

(**2.16**) *For every divisorial fractionary ideal $\mathcal{I} \supseteqq \mathcal{O}_X$ such that $\mathcal{I}/\mathcal{O}_X$ is supported in $E$, we have*

$$H^0(\mathfrak{L} \otimes \mathcal{I}/\mathcal{O}_X) = 0 .$$

Let $\mathcal{P}_1, \cdots, \mathcal{P}_n$ be the defining ideals of the irreducible components $E_1, \cdots, E_n$. (The $\mathcal{P}_i$ are locally prime, of height one.) We will show that:

(**2.17**) *If $\mathcal{I} \neq \mathcal{O}_X$ $\big(\mathcal{I}$ as in (2.16)$\big)$ then, for some $i$,*

$$E_i \subseteqq \mathrm{Supp}(\mathcal{I}/\mathcal{O}_X)$$

*and*

$$H^0\big(\mathfrak{L} \otimes \mathcal{I}/(\mathcal{P}_i\mathcal{I})''\big) = 0 .$$

Since $\mathcal{I} \supsetneqq (\mathcal{P}_i\mathcal{I})'' \supseteqq \mathcal{O}_X$, (2.16) follows from (2.17) by an obvious induction.

(**III**) Now $\mathcal{I}/(\mathcal{P}_i\mathcal{I})''$ is the extension by zero of a *torsion-free rank one* $\mathcal{O}_E$-module $\mathcal{I}_i$. Let $\mathfrak{L}_i$ be the restriction of $\mathfrak{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{E_i}$ to $E_i$ (so that $\mathfrak{L}_i$ is an invertible sheaf on $E_i$). In order to prove (2.17) let us assume that whenever $E_i \subseteqq \mathrm{Supp}(\mathcal{I}/\mathcal{O}_X)$ we have

$$H^0(E_i, \mathfrak{L}_i \otimes \mathcal{I}_i) \neq 0 ,$$

and then derive a contradiction.

Let $\pi\colon Z \to X$ be obtained by blowing up $\mathcal{I}$, and let $\mathcal{J} = \mathcal{I}\mathcal{O}_Z$, so that $\mathcal{J}$ is an invertible $\mathcal{O}_Z$-module containing $\mathcal{O}_Z$. Let $E_1', \cdots, E_m'$ be the irreducible components of $\pi^{-1}(E)$. Set $\mathcal{J}_j = \varepsilon_j^*\mathcal{J}$ where $\varepsilon_j\colon E_j' \hookrightarrow Z$ is the inclusion, and set

$$\mathfrak{L}_j' = (\pi \circ \varepsilon_j)^*\mathfrak{L} .$$

Then, I claim,

$$H^0(E_j', \mathfrak{L}_j' \otimes \mathcal{J}_j) \neq 0$$

for all $j$ such that $E_j' \subseteqq \mathrm{Supp}(\mathcal{J}/\mathcal{O}_Z)$. Indeed, if $\pi(E_j')$ is a single point, then the (non-zero) sheaf $\mathfrak{L}_j' \otimes \mathcal{J}_j$ is generated by its sections over $E_j'$. If, on the other hand, $\pi(E_j') = E_i$, then $\pi$ induces a birational map $E_j' \to E_i$, and $\mathcal{I}_i\mathcal{O}_{E_j'}$ is a torsion-free rank one $\mathcal{O}_{E'_j}$-module which is a homomorphic image of the invertible sheaf $\mathcal{J}_j$, so that in fact $\mathcal{J}_j = \mathcal{J}_i\mathcal{O}_{E'_j}$; furthermore, $E_i \subseteq \mathrm{Supp}(\mathcal{J}/\mathcal{O}_X)$, so $H^0(\mathcal{L}_i \otimes \mathcal{J}_i) \neq 0$, and consequently $H^0(\mathcal{L}'_j \otimes \mathcal{J}_j) \neq 0$.

Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{E'_j} \longrightarrow \mathcal{L}'_j \otimes \mathcal{J}_j \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}$ has support of dimension $\leqq 0$, and so

$$\mathrm{degree}(\mathcal{L}'_j \otimes \mathcal{J}_j) = \chi(\mathcal{L}'_j \otimes \mathcal{J}_j) - \chi(\mathcal{O}_{E'_j}) = h^0(\mathcal{G}) \geqq 0 .$$

By assumption $\mathcal{L}_i$ has degree $\leqq 0$ on $E_i$; hence [**RS**, page 214, (10.2)] $\mathcal{L}'_j$ has degree $\leqq 0$ on $E'_j$, and [*ibid.*]

$$\mathrm{degree}(\mathcal{J}_j) = \mathrm{degree}(\mathcal{L}'_j \otimes \mathcal{J}_j) - \mathrm{degree}(\mathcal{L}'_j) \geqq 0 ,$$

i.e.,

$$\chi(\mathcal{J}_j) - \chi(\mathcal{O}_{E'_j}) \geqq 0 . \tag{2.18}$$

Now $\mathcal{J} = \mathcal{O}(C)$ for some effective Cartier divisor $C$ on $Z$ ($C \neq 0$). From (2.18) we will deduce that

$$(C \cdot C) = \chi(\mathcal{J}/\mathcal{O}_Z) - \chi(\mathcal{O}_Z/\mathcal{J}^{-1}) \geqq 0 \tag{2.19}$$

(cf. [**RS**, top of page 223] for the first equality); and finally (2.19) will be shown to be impossible.

(**IV**) To deduce (2.19) from (2.18) we can use dévissage (cf. [**18**, page 298, Corollaries 1 and 2]); or argue explicitly as follows:

For any closed subscheme $D$ of $C$, set $\mathcal{J}_D = \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_D$, and

$$\lambda(D) = \sum_x \lambda(\mathcal{O}_{D,x})$$

where $x$ runs through all non-closed points of $D$, and $\lambda(\mathcal{O}_{D,x})$ is the length of the Artin ring $\mathcal{O}_{D,x}$. We show by induction on $\lambda(D)$ that

$$\chi(\mathcal{J}_D) - \chi(\mathcal{O}_D) \geqq 0 ,$$

this being obvious when $\lambda(D) = 0$. (For $D = C$ we get (2.19).)

If $\lambda(D) > 0$, let $\mathcal{P}$ be the defining ideal in $\mathcal{O}_D$ of one of the one-dimensional reduced irreducible subschemes $E$ of $D$; and let $\mathcal{A} = (0) : \mathcal{P}$. We have an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{\bar{D}} \longrightarrow 0 \tag{2.20}$$

where

$$\lambda(\bar{D}) = \lambda(D) - r$$

with $r > 0$ ($r$ is the generic rank of the $\mathcal{O}_E$-module $\mathcal{A}$). Since $\mathcal{J}$ is invertible, we can tensor (2.20) with $\mathcal{J}$ and take Euler characteristics to conclude that

$$[\chi(\mathcal{J}_D) - \chi(\mathcal{O}_D)] - [\chi(\mathcal{J}_{\bar{D}}) - \chi(\mathcal{O}_{\bar{D}})] = [\chi(\mathcal{J} \otimes \mathcal{A}) - \chi(\mathcal{A})] .$$

By the inductive assumption, $\chi(\mathcal{J}_{\bar{D}})-\chi(\mathcal{O}_{\bar{D}})\geqq 0$, and by (2.18), $\chi(\mathcal{J}_E)-\chi(\mathcal{O}_E)\geqq 0$, so it will suffice to show that

$$\chi(\mathcal{J} \otimes \mathcal{A}) - \chi(\mathcal{A}) = r(\chi(\mathcal{J}_E) - \chi(\mathcal{O}_E)) . \tag{2.21}$$

But since $\mathcal{A}$ is generically (on $E$) isomorphic to $\mathcal{O}_E^r$, [**RS**, page 214, Lemma (10.1)] (cf. also footnote following Theorem (2.4) above) gives

$$\chi(\mathcal{J} \otimes \mathcal{A}) - \chi(\mathcal{J} \otimes \mathcal{O}_E^r) = \chi(\mathcal{A}) - \chi(\mathcal{O}_E^r)$$

which is just a rearrangement of (2.21).

(**V**) It remains to be shown that $(C\cdot C) < 0$ *for any effective non-zero Cartier divisor* $C$ *on* $Z$ *with support in* $Z \otimes_A (A/\mathfrak{n})$. (Note that if $Z$ is non-singular this is just the well-known "negative-definiteness of the intersection matrix.")

From [**RS**, page 223, Prop. (13.1), d)] we obtain

$$\chi(nC) = -(C\cdot C)\binom{n}{2} + \chi(C)\cdot n \qquad (n \geqq 0)$$

so we need to show that

$$\lim_{n\to\infty} \frac{1}{n^2}\chi(nC) > 0 .$$

The exact sequence (for $n > 0$):

$$\begin{array}{c} \qquad\qquad\qquad A \\ \qquad\qquad\qquad \| \\ 0 \longrightarrow H^0(\mathcal{O}(-nC)) \longrightarrow H^0(\mathcal{O}_Z) \longrightarrow H^0(\mathcal{O}_{nC}) \longrightarrow H^1(\mathcal{O}(-nC)) \longrightarrow H^1(\mathcal{O}_Z) \longrightarrow H^1(\mathcal{O}_{nC}) \longrightarrow 0 \end{array}$$

shows that

$$h^1(\mathcal{O}_{nC}) \leqq h^1(\mathcal{O}_Z) ,$$
$$h^0(\mathcal{O}_{nC}) \geqq \lambda(A/H^0(\mathcal{O}(-nC))$$

($\lambda$ = length of an $A$-module, $h^i(\cdot) = \lambda(H^i(\cdot))$); so it suffices to show that

$$\liminf_{n\to\infty} \frac{1}{n^2}\lambda(A/H^0(\mathcal{O}(-nC)) > 0 .$$

Let $E$ be an irreducible component of $C$, let $S = \mathcal{O}_{Z,E}$ (a one-dimensional local domain whose maximal ideal contains $\mathfrak{n}$), let $\bar{S}$ be the integral closure of $S$ (so that $\bar{S}$ is a semi-local Dedekind domain [4, page 31, Cor. 2]), and let $v$ be a discrete valuation whose valuation ring $R_v$ is the localization of $\bar{S}$ at one of its maximal ideals. Let

$$P_n = \{a \in A \mid v(a) \geqq n\} .$$

Then $P_n \supseteqq H^0(\mathcal{O}(-nC))$; so it will suffice to show:

$$\liminf_{n\to\infty} \frac{1}{n^2}\lambda(A/P_n) > 0 . \tag{2.22}$$

Set $P_1 = \mathfrak{m}$ (a maximal ideal in $A$). There exist elements $\alpha$, $\beta$ in $A$ such that $v(\alpha) = v(\beta) = r$ (say), and the image of $\alpha/\beta$ in the residue field of $R_v$ is transcendental over $A/\mathfrak{m}$. For any integer $a > 0$, if $F(U, V) \in A[U, V]$ is a homogeneous polynomial of degree $a$, whose image $\bar{F}(U, V)$ in $(A/\mathfrak{m})[U, V]$ is non-zero, then

$$v(F(\alpha, \beta)) = ar .$$

(Otherwise we would have $v(F(\alpha/\beta, 1)) > 0$, contradicting the just-mentioned transcendence property of $\alpha/\beta$.) Hence

$$\lambda(P_{ar}/P_{ar+1}) \geqq a + 1 .$$

(For, $P_{ar}/P_{ar+1}$ is an $A/\mathfrak{m}$-vector space in which the images of the elements $\alpha^i\beta^j$ $(i + j = a)$ are linearly independent.) Thus

$$\lambda(P_{ar}/P_{(a+1)r}) \geqq a + 1$$

and so for any integer $b > 0$,

$$\lambda(A/P_{br}) \geqq 1 + 2 + \cdots + b = \frac{b(b+1)}{2} .$$

Hence, for any integer $n = br + c$ $(b > 0,\ r > c \geqq 0)$ we have

$$\frac{1}{n^2}\lambda(A/P_n) > \frac{1}{2r^2}\left(\frac{b^2r^2}{n^2}\right) > \frac{1}{2r^2}\left(1 - \frac{r}{n}\right)^2$$

and (2.22) is proved. Q.E.D.

**(2d) Appendix: Duality with supports.** We outline the proof of a duality theorem which combines local and global Grothendieck duality.[1] The theorem contains (2.3), and is given here for completeness; as indicated in the remarks preceding (2.3), this appendix is not needed elsewhere in the paper.

Let $(A, \mathfrak{m})$ be a local ring such that $X = \mathrm{Spec}(A)$ admits a *residual complex* $R^\bullet$. For example $A$ could be essentially of finite type over a Gorenstein local ring [**11**, pages 299, 306]. We may assume that $R^\bullet$ is *normalized* [**11**, page 276]. Let $f\colon Y \to X$ be a proper morphism, let $E = f^{-1}\{\mathfrak{m}\}$ be the closed fibre, and let $R^\bullet_Y = f^\Delta R^\bullet$ (so that $R^\bullet_Y$ is a residual complex on $Y$ [**11**, page 318]). Let $n$ be the Krull dimension of $Y$, and let

$$\omega_{Y/A} = H^{-n}(R^\bullet_Y) .$$

[1] A related result is given in [**11a**, page 48, Prop. (5.2)].

*Remarks.* 1. If *all* the local rings $\mathcal{O}_{Y,y}$ of closed points $y$ are $n$-dimensional and Cohen-Macaulay, then $R_Y^\bullet[-n]$ ($=R_Y^\bullet$ shifted $n$ places) is an *injective resolution* of $\boldsymbol{\omega}_{Y/A}$.

2. With the notation of (2.3), we have

$$\boldsymbol{\omega}_Y = \boldsymbol{\omega}_{Y/R} .$$

This follows from (2.5); or it can be shown directly from the definitions involved.

THEOREM. *With the preceding notation, let $I$ be the injective hull of the $A$-module $A/\mathfrak{m}$; and let $\Gamma_E$ denote "sections supported in $E$." Then for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ (or, more generally, any complex in $D_c^b(Y)$* **[11**, page 85]) *there is an isomorphism (in the derived category of the category of $A$-modules)*

$$\mathbf{R}\Gamma_E(\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, R_Y^\bullet), I) .$$

*Hence, if $Y$ is locally Cohen-Macaulay, of pure dimension $n$, there exist isomorphisms (for all $i$)*

$$H_E^i(\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Ext}\,_{\mathcal{O}_Y}^{n-i}(\mathcal{F}, \boldsymbol{\omega}_{Y/A}), I) .$$

*Proof.* The duality theorem [**11**, page 379] gives an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, R_Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathbf{R}f_*\mathcal{F}, R^\bullet) .$$

The complex $\mathbf{R}f_*\mathcal{F}$ may be taken to be of the form $f_*(\mathcal{I}^\bullet)$, where $\mathcal{I}^\bullet$ is a *quasicoherent* injective resolution of $\mathcal{F}$; since $f_*(\mathcal{I}^\bullet)$ is quasi-coherent, with coherent cohomology ($f$ being proper), we can apply local duality [**11**, page 278] to get an isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Hom}_{\mathcal{O}_X}(\mathbf{R}f_*\mathcal{F}, R^\bullet), I) .$$

Finally, since $\Gamma_E = \Gamma_{\mathfrak{m}} \circ f_*$, we have

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}f_*\mathcal{F}) = \mathbf{R}\Gamma_E(\mathcal{F}) ,$$

and the assertion follows.

### 3. Reduction to pseudo-rational singularities

We want to prove Theorem* of Section (1a). Let us first see how Theorem (2.2) can help us to do this.

Let $\bar{R}$ be a complete two-dimensional normal local ring. There exists a complete two-dimensional regular local ring $R \subseteq \bar{R}$ such that $\bar{R}$ is a finite $R$-module. Let $K$ (resp. $L$) be the fraction field of $R$ (resp. $\bar{R}$). It clearly suffices to show that there exists an iterated blow-up $f: Y \to \mathrm{Spec}(R)$ (i.e., $f$ is obtained by successively blowing up closed points) such that *the normal-*

*ization* $\bar{Y}$ *of* $Y$ *in* $L$ *has only pseudo-rational singularities.* (Note that since $R$ is complete, $\bar{Y}$ is finite over $Y$, hence proper over $\mathrm{Spec}(\bar{R})$.)

What we want, then, is that *for any proper birational map* $h\colon Z \to \bar{Y}$, *with* $Z$ *normal, the canonical map* $H^1(\mathcal{O}_{\bar{Y}}) \to H^1(\mathcal{O}_Z)$ *should be bijective* (cf. (iii) in Lemma (1.3)). The following lemma allows us to restrict our attention to those $Z$ which are of the form $\bar{Y}_1$, $Y_1 \to Y$ being an iterated blow-up.

LEMMA (3.1) ("Elimination of indeterminacies"). *For any* $Z \to \bar{Y}$ *as above, there is an iterated blow-up* $Y_1 \to Y$ *such that the normalization* $\bar{Y}_1$ *of* $Y_1$ *in* $L$ *dominates* $Z$, i.e., *there is a commutative diagram*

$$\begin{array}{ccccc} \bar{Y}_1 & \longrightarrow & Z & \longrightarrow & \bar{Y} \\ \downarrow & & & & \downarrow \\ Y_1 & \longrightarrow \cdot \longrightarrow & \cdots & \longrightarrow \cdot \longrightarrow & Y \end{array}$$

*(iterated blow-up)*.

*Proof.* Note that all the maps involved are of finite type. Let $E_1, \cdots, E_n$ be those irreducible curves on $Z$ whose image on $\bar{Y}$ is a single point, and let $v_1, \cdots, v_n$ be the corresponding discrete valuations of $L$. Because of Zariski's "Main Theorem," we need only arrange that the restriction of each $v_i$ to $K$ be the valuation associated to an irreducible curve on $Y_1$; and this is possible by [**38**, page 392]. Q.E.D.

Thus, we need an iterated blow-up $f\colon Y \to \mathrm{Spec}(R)$ such that for every iterated blow-up $Y_1 \to Y$, we have

$$\lambda(H^1(\mathcal{O}_{\bar{Y}})) \geqq \lambda(H^1(\mathcal{O}_{\bar{Y}_1}))$$

where "$\lambda$" denotes "length" (of an $R$-module). But by Theorem (2.2) (and Matlis duality [**26**, page 526, Theorem 4.2]),

$$\lambda(H^1(\mathcal{O}_{\bar{Y}})) = \lambda(C/C_Y)\,.$$

*So the problem becomes one of "maximizing"* $\lambda(C/C_Y)$.

Specifically, it suffices to show:

(**3.2**) *There exists an iterated blow-up* $f'\colon Y' \to \mathrm{Spec}(R)$ *and an integer* $N$ *such that for any iterated blow-up* $g\colon Y \to Y'$, *and with* $C, C_Y$ *as in Theorem* (2.2), *we have that* $\mathfrak{m}^N C \subseteqq C_Y$ ($\mathfrak{m}$ = *maximal ideal of* $R$).

For then the $R$-module $C/C_Y$ has length bounded by that of $C/\mathfrak{m}^N C$, and the length of $H^1(\bar{Y}, \mathcal{O}_{\bar{Y}})$ is bounded by the same integer (independent of $Y$!). So among all iterated blow-ups $g\colon Y \to Y'$, there exists one for which $H^1(\bar{Y}, \mathcal{O}_{\bar{Y}})$ has maximal length, and then, by the preceding remarks, $\bar{Y}$ has only pseudo-rational singularities.

*Remark* (not used hereafter). (3.2) will be proved as it stands; but observe that if $Y'$, $N$ are as in (3.2), and if $Y_0 \to \mathrm{Spec}(R)$ is *any* iterated blow-up, then, as in (3.1), there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\text{iterated blow-up}} & Y' \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & \mathrm{Spec}(R) , \end{array}$$

and it follows that

$$\mathfrak{m}^N C \subseteqq C_Y \subseteqq C_{Y_0}$$

(i.e., (3.2) holds with $Y' = \mathrm{Spec}(R)$).

* * *

As a first step toward the proof of (3.2), we gather together some elementary observations:

LEMMA (3.3). *The following conditions are equivalent*:

(i) *There exists* $d \neq 0$ *in* $R$ *such that* $dC \subseteqq C_Y$ *for all non-singular* $Y$ *proper and birational over* $\mathrm{Spec}(R)$.

(ii) $\bigcap_Y C_Y \neq 0$, *where the intersection is over all* $Y$ *as in* (i).

(iii) *There exists a non-zero K-linear map* $T'$: $L \to K$ *such that for any two-dimensional regular local ring* $S$ *with* $R \subseteqq S \subseteqq K$, *we have*

$$T'(\bar{S}) \subseteqq \omega_S$$

*where* $\bar{S}$ *is the integral closure of* $S$ *in* $L$, *and* $\omega_S$ *is as at the beginning of Section* 2.

*Proof.* (i) $\Rightarrow$ (ii): Since $\bar{R}$ is a finite $R$-module, it is clear that $C \neq 0$, so $dC \neq 0$.

(ii) $\Rightarrow$ (iii): Recall that $C$, $C_Y$ are defined with respect to some fixed non-zero $K$-linear map $T$: $L \to K$ (cf. Section 2). Set $T' = \mu T$, where

$$0 \neq \mu \in \bigcap_Y C_Y .$$

Every $S$ is of the form $\mathcal{O}_{Y,y}$ for some $Y$ as in (i) and $y \in Y$; by definition of $C_Y$, we have

$$T(\mu\bar{\mathcal{O}}_{Y,y}) \subseteqq \omega_{Y,y} ,$$

i.e.,

$$T'(\bar{S}) \subseteqq \omega_S .$$

(iii) $\Rightarrow$ (i): Let $T$: $L \to K$ be as above. Then $T' = \rho T$ for some $0 \neq \rho \in L$. If $C'_Y$ is defined for $T'$ in the same way that $C_Y$ is defined for $T$, then $C_Y = \rho C'_Y$, and similarly $C = \rho C'$. Hence we may assume that $T' = T$.

Let $(e_1, e_2, \cdots, e_n)$ be a $K$-vector space basis of $L$, with each $e_i \in \bar{R}$. Let

$d$ be the determinant of the matrix $(T(e_ie_j))$. Then $d \neq 0$, and $d \in R$ (since $T(\bar{R}) \subseteqq \omega_R = R$). Given $Y$ as in (i), and a closed point $y \in Y$, set $S = \mathcal{O}_{Y,y}$. For any $s = \sum_{i=1}^n \alpha_i e_i \in \bar{S}$ ($\alpha_i \in K$), we have $T(se_j) \in \omega_S$, i.e.,

$$\textstyle\sum_{i=1}^n \alpha_i T(e_ie_j) \in \omega_S \qquad (j = 1, 2, \cdots, n)\,.$$

Then Cramer's rule gives $d\alpha_i \in \omega_S$, so that $ds \in \sum_i \omega_S e_i$. Since $e_i \in \bar{R}$, therefore $T(Ce_i) \subseteqq \omega_R = R$ (definition of $C$), and so

$$T(dC\bar{S}) \subseteqq T(C(\textstyle\sum_i \omega_S e_i)) \subseteqq \omega_S\,.$$

Thus

$$T(dC\bar{\mathcal{O}}_y) \subseteqq \omega_{Y,y}$$

for all closed points $y \in Y$, and hence also for *all* $y \in Y$; i.e., $dC \subseteqq C_Y$. Q.E.D.

A non-zero $K$-linear map $T'\colon L \to K$ as in (3.3) (iii) willl be called a **good trace.**

If *either* $\bar{R}$ has characteristic zero *or* $\bar{R}$ has a perfect residue field, then by a theorem of Nagata[1], $R$ can be chosen so that $L$ is *separable* over $K$; and then the usual trace map $\mathrm{Tr}\colon L \to K$ is a good trace, since for all $S$ we have

$$0 \neq \mathrm{Tr}(\bar{S}) \subseteqq S \subseteqq \omega_S\,.$$

Thus the conditions in (3.3) hold in this case.

In the remaining case, when $\bar{R}$ contains a field and the residue field of $\bar{R}$ is not perfect, then for any choice of $R$ *there still exists a good trace* $L \to K$; the proof of this fact will occupy most of Section 4.

The rest of Section 3 will be devoted mainly to a proof of:

(**3.4**) *The equivalent conditions in* (3.3) *imply* (3.2) (*and hence Theorem**).

*To review*: after proving (3.4) we will be done, except in the equicharacteristic, non-perfect residue field case, where we need Section 4 to ensure that the conditions in (3.3) do indeed hold. Actually, Section 4 will provide *another proof* (independent of the rest of Section 3) for (3.2) in the *equicharacteristic case.* In fact if $\bar{R}$ contains a field, then for any fixed choice of $R$ we will show the following (cf. Corollary 4.7): *for any height one prime* $\mathfrak{p}$ *in* $R$, *there exist* $e_1, e_2, \cdots, e_n$, *and a good trace* $T\colon L \to K$ *such that*

$$d = \det(T(e_ie_j)) \notin \mathfrak{p}\,.$$

Since $dC \subseteqq \bigcap_Y C_Y$ (cf. proof of (3.3)), it follows that the annihilator of $C/\bigcap C_Y$ is $\mathfrak{m}$-primary, and so (3.2) holds with $Y' = \mathrm{Spec}(R)$.

[1] Cf. Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. **28** (1953), page 276, Theorem 3.

In the *equicharacteristic, perfect residue field* case, we can also replace (3.4) by the following argument:

*Exercise*: Assume that $\bar{R}$ contains a field, and that the residue field $k$ of $\bar{R}$ is *perfect*. Let $\mathfrak{q}$ be a height one prime in $\bar{R}$. Then $R$ can be chosen so that $\bar{R}$ is *étale* over $R$ at $\mathfrak{q}$ (cf. [**3**, page 123, Satz 4.1.12] or [**EGA IV**, (18.11.10)]); in other words, $\bar{R}$ is a localization of

$$R[U]/P(U) = R[u] \subseteq \bar{R}$$

where $P(U)$ is a monic polynomial such that $P'(u) \notin \mathfrak{q}$ ($P'$ is the derivative $dP/dU$). Show that

$$P'(u)C \subseteq \bigcap_Y C_Y$$

(use the fact that $P'(u)\bar{S} \subseteq S[u]$ for all $S$). As in the preceding remark, deduce (3.2).

*Remark* (not used elsewhere). The purpose of Sections 3 and 4 is to prove Theorem*; but conversely the main results in these sections follow from Theorem* (and Theorem (2.2)). That is, let $R$, $K$, $L$ be as in (2.2), let $\bar{R}$ be the integral closure of $R$ in $L$, and suppose that there exists a proper birational map $Z \to \mathrm{Spec}(\bar{R})$ such that $Z$ has only pseudo-rational singularities. Then there exists an iterated blow-up $Y_1 \to \mathrm{Spec}(R)$ such that $\bar{Y}_1$ dominates $Z$ (cf. (3.1)), and $\bar{Y}_1$ has only pseudo-rational singularities (cf. (1.4)).

Furthermore, it is easily seen that there exists a non-zero $K$-linear map $T: L \to K$ such that $T(\mathcal{O}_{\bar{Y}_1}) \subseteq \omega_{Y_1}$; and *this $T$ is a good trace.* (Indeed, for any iterated blow-up $Y' \to Y_1$, Theorem (2.2) implies that $C_{Y'} = C_{Y_1}$, and since $1 \in C_{Y_1}$, therefore $T(\mathcal{O}_{\bar{Y}'}) \subseteq \omega_{Y'} \cdots$.)

We proceed now with the *proof of* (3.4). Let $d \neq 0$ be in $R$. For any iterated blow-up $Y \to \mathrm{Spec}(R)$, let $d_Y$ be the "proper transform on $Y$ of the curve $d = 0$," i.e.,

$$d_Y = \{y \in Y \mid \mathfrak{m}\mathcal{O}_{Y,y} \not\subset \sqrt{d\mathcal{O}_{Y,y}}\}\ .$$

($d_Y$ is the closed subset of $Y$ defined by the coherent $\mathcal{O}_Y$-ideal $\sqrt{d\mathcal{O}_Y}: \mathfrak{m}\mathcal{O}_Y$.) We shall prove below:

LEMMA (3.5). *There exists an iterated blow-up $Y' \to \mathrm{Spec}(R)$ such that, if $\pi: \bar{Y}' \to Y'$ is the normalization in $L$, then each closed point in $\pi^{-1}(d_{Y'})$ is a pseudo-rational singularity of $\bar{Y}'$.*

(3.4) can be deduced from (3.5) as follows: Fix a $d$ as in (3.3) (i), and let $Y'$ be as in (3.5). From the definition of $d_{Y'}$ it follows easily that there is an integer $N$ such that, *for all $y' \in Y' - d_{Y'}$, we have*

(3.6) $$\mathfrak{m}^N \mathcal{O}_{Y',y'} \subseteqq d\mathcal{O}_{Y',y'} .$$

After enlarging $N$ if necessary, we may assume that

(3.7) $$\mathfrak{m}^N(C/C_{Y'}) = 0 .$$

(This can easily be deduced from the definition of $C_{Y'}$, or from Theorem (2.2).)

Now let $g$: $Y \to Y'$ be any iterated blow-up. Let us show that $\mathfrak{m}^N C \subseteqq C_Y$ (thereby proving (3.2)); in other words, *for any* $\mu \in L$, *if* $T(\mu\bar{R}) \subseteqq R$ *then, for all* $y \in Y$

$$T(\mathfrak{m}^N \mu\bar{\mathcal{O}}_y) \subseteqq \omega_y$$

(where $\bar{\mathcal{O}}_y$ is the integral closure of $\mathcal{O}_{Y,y}$ in $L$, and $\omega_y$ is the stalk $\omega_{Y,y}$).

There are two cases to consider. First, if $y' = g(y) \in Y' - d_{Y'}$ then (3.6) implies that $\mathfrak{m}^N \mu\bar{\mathcal{O}}_y \subseteqq d\mu\bar{\mathcal{O}}_y$; since $d\mu \in dC \subseteqq C_Y$, we have

$$T(\mathfrak{m}^N \mu\bar{\mathcal{O}}_y) \subseteqq T(C_Y\bar{\mathcal{O}}_y) \subseteqq \omega_y .$$

Second, suppose that $y' = g(y) \in d_{Y'}$. Let $g_1$: $Z \to Y'$ be the iterated blow-up obtained from the iterated blow-up $g$ by *omitting* those blow-ups for which the point blown up does not lie over $y'$. Then $g$ factors as

$$Y \xrightarrow[g_2]{} Z \xrightarrow[g_1]{} Y'$$

and we have

$$\mathcal{O}_y = \mathcal{O}_{Y,y} = \mathcal{O}_{Z,g_2(y)} ;$$

so we may as well assume that $g_1 = g$. Then, if $\pi$: $\bar{Y}' \to Y'$ is the normalization in $L$, the map of normalizations $\bar{g}$: $\bar{Y} \to \bar{Y}'$ induces an *isomorphism*

$$\bar{Y} - \bar{g}^{-1}\pi^{-1}(y') \xrightarrow{\sim} \bar{Y}' - \pi^{-1}(y') ,$$

so that $R^1\bar{g}_*(\mathcal{O}_{\bar{Y}})$ *is concentrated on* $\pi^{-1}(y')$. Since $y' \in d_{Y'}$, all the points of $\pi^{-1}(y')$ are *pseudo-rational* singularities of $\bar{Y}'$, and so $R^1\bar{g}_*(\mathcal{O}_{\bar{Y}}) = 0$. This implies that the canonical map $H^1(\bar{Y}', \mathcal{O}_{\bar{Y}'}) \to H^1(\bar{Y}, \mathcal{O}_{\bar{Y}})$ is *bijective*, whence (Theorem (2.2)), the $R$-modules $C/C_{Y'}$ and $C/C_Y$ have the same length; since $C_Y \subseteqq C_{Y'}$, we conclude that

$$C_{Y'} = C_Y .$$

From (3.7) we now obtain $\mathfrak{m}^N(C/C_Y) = 0$, and hence, again, $T(\mathfrak{m}^N \mu\bar{\mathcal{O}}_y) \subseteqq \omega_y$.

It remains to prove (3.5). Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the height one primes in $R$ containing $d$, and let $v_i$ $(1 \leqq i \leqq r)$ be the valuation with valuation ring $R_{\mathfrak{p}_i}$. Each $R/\mathfrak{p}_i$ is a one-dimensional local domain whose integral closure in its field of fractions $K_i$ is a semi-local Dedekind domain [4; page 30]; so the valuations $\bar{w}_{ij}$ of $K_i$ whose valuation ring contains $R/\mathfrak{p}_i$ are discrete, of rank one, and finite in number. Each of these valuations gives rise to a rank two discrete valuation $w_{ij}$ composite with $v_i$. (The valuation ring of $w_{ij}$ is the inverse image of that of $\bar{w}_{ij}$ under the natural map $R_{\mathfrak{p}_i} \to K_i$.)

LEMMA (3.8). *Let $Y \to \mathrm{Spec}(R)$ be an iterated blow-up, and let $y$ be a closed point of $d_Y$. Then $\mathcal{O}_{Y,y}$ is dominated by one of the valuations $w_{ij}$* (i.e., *$w_{ij}(\alpha) \geqq 0$ for any $\alpha \in \mathcal{O}_{Y,y}$, and $w_{ij}(\alpha) > 0$ if $\alpha$ is a non-unit*).

*Proof.* One checks that $y \in d_Y$ if and only if $\mathcal{O}_{Y,y} \subseteqq R_{\mathfrak{p}_i}$ for some $\mathfrak{p}_i$ as above; then $R_{\mathfrak{p}_i}$ is actually the localization of $\mathcal{O}_{Y,y}$ at some height one prime $\mathfrak{q}$ with $\mathfrak{q} \cap R = \mathfrak{p}_i$, and the conclusion follows from the fact that $\mathcal{O}_{Y,y}/\mathfrak{q}$ is dominated by some $\bar{w}_{ij}$. Q.E.D.

Let $v_{ijk}$ be the finitely many extensions of $w_{ij}$ to $L$; the $v_{ijk}$ are rank two discrete valuations of $L$. So Proposition (1.33) gives:

COROLLARY (3.9). *Let*

$$\mathrm{Spec}(\bar{R}) = Z_0 \longleftarrow Z_1 \longleftarrow Z_2 \longleftarrow \cdots$$

*be such that for each $n \geqq 0$, $Z_{n+1}$ is obtained from $Z_n$ by blowing up the centers $z_{ijkn}$ on $Z_n$ of all the valuations $v_{ijk}$ and then normalizing* ($z_{ijkn}$ *is the unique* (*closed*) *point on $Z_n$ whose local ring is dominated by $v_{ijk}$*). *Then for some $n$, $z_{ijkn}$ is a regular point of $Z_n$ for all $i$, $j$, $k$.*

Now, by (3.1), we can find an iterated blow-up $Y' \to \mathrm{Spec}(R)$ such that, $\pi$: $\bar{Y}' \to Y'$ being the normalization in $L$, there is a proper birational map $h$: $\bar{Y}' \to Z_n$. By (3.8) and [**38**, page 31, Theorem 13], each closed $x \in \pi^{-1}(d_{Y'})$ is the center on $\bar{Y}'$ of some $v_{ijk}$. Hence $h(x) = z_{ijkn}$ is regular, hence rational, and so (Corollary (1.4)) $x$ is pseudo-rational. Q.E.D.

## 4. Existence of good trace maps



Let $k$ be a field of characteristic $p \geqq 0$. Let $R$ be a formal power series ring in two variables over $k$, $R = k[[U_1, U_2]]$; let $K$ be the fraction field of $R$, and let $L$ be a finite algebraic field extension of $K$. In Section (4a) we prove the existence of a "good trace" map $T$: $L \to K$ (cf. remarks following (3.3)). In fact, we shall show (Corollary (4.7)) that for each height one prime $\mathfrak{p}$ in $R$, there exists such a $T$ which is "unramified at $\mathfrak{p}$" (cf. remarks following (3.4) for the significance of this result).

$T$ will appear as the trace map between certain modules of differentials over an "admissible" field $k_0 \subseteqq k$. (The existence of such $k_0$ is established in

Section (4b); difficulty occurs only when $[k\colon k^p] = \infty$, since otherwise $k^p$ is admissible.) The notion of "trace of a differential" is studied extensively by Kunz in [21];[1] we include in (4a) an exposition of those parts of [21] which are needed for the desired results on good traces.

(4a) **Differentials and traces.** Let $k$ be a field and let $\mathfrak{C}$ be the category of complete local $k$-algebras with residue field finite over $k$. Let $\mathfrak{B}$ be the category of *reduced* $k$-algebras $B$ for which there exists an $R \in \mathfrak{C}$ and a $k$-algebra homomorphism $R \to B$ such that $B$ is essentially of finite type over $R$ (i.e., $B$ is $R$-isomorphic to a ring of fractions of some finitely-generated $R$-algebra).

Let $p \geqq 0$ be the characteristic of $k$. When $p > 0$ let $k^p$ be the field consisting of $p^{\text{th}}$ powers of elements of $k$; and when $p = 0$ let $k^p = k$. We shall say that a field $k_0$ is a **base field** (under $k$) if $k^p \subseteqq k_0 \subseteqq k$ and $[k\colon k_0] < \infty$.

For any such $k_0$, and any $R \in \mathfrak{C}$, there is a $k_0$-derivation $d_R$ of $R$ into a finitely generated $R$-module $\Omega(R/k_0)$ which is *universal* for $k_0$-derivations of $R$ into *finitely generated* $R$-modules [3, page 47, Satz 2.1.5].

Given $R \to B$ as above, there is a "universal extension" of $d_R$ to a derivation $d_B\colon B \to \Omega(B/k_0)$. Explicitly, if "$\Omega_{k_0}$" denotes "universal module of $k_0$-differentials," and $I$ is the kernel of the natural surjection $\Omega_{k_0}R \to \Omega(R/k_0)$ then we have a natural commutative diagram of $B$-module maps, with exact rows and columns:

$$
\begin{array}{ccccccc}
B \otimes_R I & \longrightarrow & B \otimes_R \Omega_{k_0} R & \longrightarrow & B \otimes_R \Omega(R/k_0) & \longrightarrow & 0 \\
\Vert & & \downarrow & & \downarrow & & \\
B \otimes_R I & \longrightarrow & \Omega_{k_0} B & \xrightarrow{\ \gamma\ } & \Omega(B/k_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega_R B & = & \Omega_R B & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
\tag{4.1}
$$

($\Omega_R B$ is the universal module of $R$-differentials of $B$). The middle row defines $\Omega(B/k_0)$, and $d_B$ is the composition of $\gamma$ with the universal derivation $B \to \Omega_{k_0}B$. The last column shows that $\Omega(B/k_0)$ *is a finitely generated $B$-module.*

By [3, page 60, Satz 2.3.10], *the pair* $(\Omega(B/k_0), d_B)$ *depends only on the $k$-algebra $B$* (and *not* on the choice of $R \to B$).

[1] I hope to present elsewhere a treatement—based on Grothendieck's residue symbol—which is more general (it applies to relative complete intersections, rather than just pairs of fields) and more uniform (for example, all finite field extensions—separable or not—are dealt with simultaneously).

*Example* (4.2). Suppose $p > 0$, and let $R \to B$ be as above, with $R$ the formal power series ring $k[[U_1, \cdots, U_n]]$. Then $\Omega(B/k_0)$ is the universal module of $k_0[[U_1^p, \cdots, U_n^p]]$-differentials of $B$ (and $d_B$ is the universal $k_0[[U_1^p, \cdots, U_n^p]]$-derivation).

Now let $R = k[[U_1, U_2]]$, let $K$ be the fraction field of $R$, and let $S$ be a two-dimensional regular local ring with $R \subseteqq S \subseteqq K$. Our next task is to bring out a connection between differentials and the $S$-module $\omega_S$ (cf. Section 2).

Let $k_0$ be a base field, with $[k: k_0] = p^e$ ($e = 0$ if $p = 0$). Then $\Omega(R/k_0)$ is *free*, of rank $e + 2$, with free generators, say, $\xi_1, \xi_2, \cdots, \xi_{e+2}$ (cf. (4.2)). For any $B$ in the category $\mathfrak{B}$, and any integer $n \geqq 0$, we set

$$\Omega^n(B/k_0) = \Lambda_B^n \Omega(B/k_0)$$

(where "$\Lambda$" denotes "exterior power"). Then, since (as is easily seen)

$$\Omega(K/k_0) = K \otimes_R \Omega(R/k_0) ,$$

therefore

$$\Omega^{e+2}(K/k_0) = K\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{e+2} .$$

We have a canonical map of $S$-modules

$$\Omega^{e+2}(S/k_0) \longrightarrow \Omega^{e+2}(K/k_0) = K \otimes_S \Omega^{e+2}(S/k_0)$$

whose image we denote by

$$\omega'_S \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{e+2}$$

so that $\omega'_S \supseteqq S$ is an $S$-submodule of $K$.

LEMMA (4.3). *Let $S$, $\omega'_S$ be as above, and let $\omega_S$ be as in Section 2. Then $\omega'_S \subseteqq \omega_S$.*

*Proof.* There is a quadratic sequence

$$R = R_0 < R_1 < \cdots < R_n = S .$$

For $0 \leqq i \leqq n$, let $\omega'_i = \omega'_{R_i}$ and $\omega_i = \omega_{R_i}$. We proceed by induction. For $i = 0$, we have

$$\omega'_0 = R = \omega_0 .$$

Now assume, for some $i$ with $0 \leqq i < n$, that $\omega'_i \subseteqq \omega_i$. $R_{i+1}$ is a localization of $R_i[x/y]$ for suitable generators $x$, $y$ of the maximal ideal $\mathfrak{m}_i$ of $R_i$; hence (cf. (4.1)) the $R_{i+1}$-module $\Omega(R_{i+1}/k_0)$ is generated by the *natural image* of $\Omega(R_i/k_0)$ and by

$$d(x/y) = y^{-1}(dx - (x/y)dy)$$

($d: R_{i+1} \to \Omega(R_{i+1}/k_0)$ being the canonical derivation). It follows that

$$\omega'_{i+1} \subseteqq y^{-1}R_{i+1}\omega'_i \, .$$

By definition (Section 2)

$$\omega_{i+1} = (\mathfrak{m}_i R_{i+1})^{-1}\omega_i = y^{-1}R_{i+1}\omega_i$$

and hence $\omega'_{i+1} \subseteqq \omega_{i+1}$. Q.E.D.

We can now describe the construction of some good traces. Once again, let $R = k[[U_1, U_2]]$, with fraction field $K$, and let $L$ be a finite field extension of $K$. In Section (4b) (Lemma (4.11)), we will show that *for some base field $k_0$ with, say, $[k: k_0] = p^e$, the $L$-vector space $\Omega(L/k_0)$ has dimension $e + 2$.* (Here $e = 0$ when $p = 0$.) Such a $k_0$ is said to be *admissible* for $L$ (Definition (4.10)). If $L$ is separable over $K$, then any base field is admissible for $L$; and when $p > 0$ and $L$ is not necessarily separable over $K$, then $k_0$ is admissible for $L$ if and only if $L$ is linearly disjoint from $k_0^{p^{-1}}[[U_1, U_2]]$ over $R$ (cf. (4.8) and (4.9)). (In particular, $k^p$ is admissible if $[k: k^p] < \infty$.) In any case we see that *if $k_0$ is admissible for $L$, then $k_0$ is also admissible for any field $L'$ with $K \subseteqq L' \subseteqq L$.*

Now let $K \subseteqq L' \subseteqq L'' \subseteqq L$ be fields such that either

(a) $L''$ is separable over $L'$, or

(b) $L''$ is purely inseparable, of degree $p$, over $L'$.

Then (cf. [20a, pages 286-288]) there exists a *surjective* $L'$-linear "trace" map

$$\tau: \Omega^{e+2}(L''/k_0) \longrightarrow \Omega^{e+2}(L'/k_0) \tag{4.4}$$

satisfying:

(a′) *if $L''$ is separable over $L'$*, so that (as is easily checked) $\Omega(L''/k_0) = L'' \otimes_{L'} \Omega(L'/k_0)$, then

$$\tau = \mathrm{Tr} \otimes 1: L'' \otimes_{L'} \Omega^{e+2}(L'/k_0) \longrightarrow L' \otimes_{L'} \Omega^{e+2}(L'/k_0) \tag{4.4a}$$

where $\mathrm{Tr}: L'' \to L'$ is the usual trace map, and $1$ is the identity map of $\Omega^{e+2}(L'/k_0)$;

(b′) *if $L'' = L'(\alpha)$, with $\alpha \notin L'$, $\alpha^p \in L'$*, then, for any $a_2, \cdots, a_{e+2}$ in $L'$, and

$$\lambda = c_0 + c_1\alpha + \cdots + c_{p-1}\alpha^{p-1} \qquad (c_i \in L')$$

we have

$$\tau(\lambda d''\alpha d''a_2 \cdots d''a_{e+2}) = c_{p-1}d'(\alpha^p)d'a_2 \cdots d'a_{e+2} \tag{4.4b}$$

(where $d' = d_{L'}: L' \to \Omega(L'/k_0)$ is the canonical derivation, and similarly $d'' = d_{L''}$. Note that $d'(\alpha^p) \neq 0$, since otherwise $\Omega(L''/k_0)$ would have dimension $e + 3$).

Next let

$$K = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_m = L$$

be fields such that $L_1$ is separable over $L_0$, and for $1 \leqq i < m$, $L_{i+1}$ is purely inseparable, of degree $p$, over $L_i$ (when $p = 0$, we take $m = 1$). Let

$$\tau_i \colon \Omega^{e+2}(L_i/k_0) \longrightarrow \Omega^{e+2}(L_{i-1}/k_0) \qquad (0 < i \leqq m)$$

be as in the preceding discussion.

Choose $y_1, y_2, \cdots, y_{e+2}$ in $\bar{R}$ such that $dy_1, dy_2, \cdots, dy_{e+2}$ generate $\Omega(L/k_0)$ ($d\colon L \to \Omega(L/k_0)$ being the canonical derivation). Let $T\colon L \to K$ be the unique $K$-linear map such that for all $\lambda \in L$ we have

$$T(\lambda)\xi_1\xi_2 \cdots \xi_{e+2} = (\tau_1\tau_2 \cdots \tau_m)(\lambda dy_1 dy_2 \cdots dy_{e+2}) \,. \tag{4.5}$$

(Here, as before, $\{\xi_1, \cdots, \xi_{e+2}\}$ is a free basis of $\Omega(R/k_0)$.) $T \neq 0$, since each $\tau_i$ is surjective.

*Claim*: *$T$ is a good trace.*

For justifying the claim, some more *notation* will be convenient. Let $A$ be any subring of $L$ such that $A$ contains $R$ and $A$ is essentially of finite type over $R$. Let $L^{\#}$ be the fraction field of $A$. We denote by $\Omega^*(A)$ the image of the canonical map $\Omega^{e+2}(A/k_0) \to \Omega^{e+2}(L^{\#}/k_0)$.

The following key technical result (and its proof) are essentially contained in **[21]** (Satz 2.15).

LEMMA (4.6). *Let $\tau$ be as in* (4.4). *Let $A$ be a Dedekind domain with fraction field $L'$, such that $R \subseteqq A$ and $A$ is essentially of finite type over $R$; and let $\bar{A}$ be the integral closure of $A$ in $L''$ ($\bar{A}$ is a finite $A$-module, since $R$ is complete). Then*:

(a) $$\tau(\Omega^*(\bar{A})) \subseteqq \Omega^*(A) \,.$$

(b) (i) *If $L''$ is* purely inseparable *over $L'$, then for $\lambda \in L''$ we have*

$$[\tau(\lambda\Omega^*(\bar{A})) \subseteqq \Omega^*(A)] \Longleftrightarrow \lambda \in \bar{A} \,.\text{[1]} \tag{4.6'}$$

(ii) *If $L''$ is* separable *over $L'$, and if $\Omega(\bar{A}/k_0)$ is a* torsion-free *$\bar{A}$-module, then* (4.6′) *holds for all $\lambda \in L''$.*

For convenience, a proof of (4.6) is given below. But first let us see how (4.6) (a) can be used to show that $T$ is a good trace. If $S$ is any two-dimensional regular local ring with $R \subseteqq S \subseteqq K$, $\mathfrak{q}$ is a height one prime ideal in $S$, and $\bar{S}$ is the integral closure of $S$ in $L$, then, with $dy_1, dy_2, \cdots, dy_{e+2}$ as in the definition of $T$, we have

$$\bar{S} dy_1 dy_2 \cdots dy_{e+2} \subseteqq \Omega^*(\bar{S}_{\mathfrak{q}}) \,;$$

from (4.6)(a) and (4.3) we obtain

[1] In other words, $\tau$ generates the $\bar{A}$-module $\mathrm{Hom}_A(\Omega^*(\bar{A}), \Omega^*(A))$.

$$(\tau_1\tau_2 \cdots \tau_m)\Omega^*(\bar{S}_\mathfrak{q}) \subseteqq \Omega^*(S_\mathfrak{q}) = \Omega^*(S)_\mathfrak{q} \subseteqq (\omega_S)_\mathfrak{q}(\xi_1\xi_2 \cdots \xi_{e+2}) ,$$

and conclude that

$$T(\bar{S}) \subseteqq (\omega_S)_\mathfrak{q} .$$

But since $\omega_S$ is invertible, $\omega_S$ is the intersection of all its localizations of the form $(\omega_S)_\mathfrak{q}$; so

$$T(\bar{S}) \subseteqq \omega_S ,$$

and we are done.

Using 4.6(b) we can get even more: as indicated in the remarks after (3.4), the following result can replace (3.4) in the equicharacteristic case.

COROLLARY (4.7). *With previous notation, let $\mathfrak{p}$ be a height one prime ideal in $R$. Then there exist elements $e_1, \cdots, e_n \in \bar{R}$ $(n = [L: K])$, and a good trace $T: L \to K$, such that the determinant* $\det(T(e_ie_j))$ *lies in $R - \mathfrak{p}$.*

*Proof.* Choose a free basis $e'_1, \cdots, e'_n$ of the $R_\mathfrak{p}$-module $\bar{R}_\mathfrak{p}$ (integral closure in $L$); and let $r \in R - \mathfrak{p}$ be such that

$$e_i = re'_i \in \bar{R} \qquad (1 \leqq i \leqq n) .$$

Then for any $K$-linear map $T: L \to K$,

$$\det(T(e_ie_j)) = r^{2n}\det(T(e'_ie'_j)) ;$$

so it will suffice to find a good trace $T$ such that $\det(T(e'_ie'_j))$ is a unit in $R_\mathfrak{p}$. (Recall that for a good trace $T$, we have

$$T(e_ie_j) \in T(\bar{R}) \subseteqq \omega_R = R ,$$

and hence

$$\det(T(e'_ie'_j)) \in R_\mathfrak{p} .)$$

We have first to modify the base field $k_0$. Let

$$K = L_0 \subseteqq L_1 \subseteqq \cdots \subseteqq L_m = L$$

be as before, and let $R'$ be the integral closure of $R$ in $L_1$. In Section 4b (Proposition (4.12)) we will show that, by shrinking $k_0$ if necessary, we can arrange that *for all height one primes $\mathfrak{p}$ in $R$, the $R'_\mathfrak{p}$-module $\Omega(R'_\mathfrak{p}/k_0)$ is free* ($R'_\mathfrak{p} = R' \otimes_R R_\mathfrak{p}$). This extra condition on $k_0$ allows us to assume that (4.6′) holds for $A = R_\mathfrak{p}$, $\bar{A} = R'_\mathfrak{p}$ (cf. (4.6)(b)(ii)).

Now let $\Omega_f = \Omega_f(\bar{R}_\mathfrak{p})$ be the image of the canonical map

$$\Omega(R_\mathfrak{p}/k_0) \longrightarrow \Omega(L/k_0) .$$

Then clearly $\Omega_f$ is locally, hence globally, a *free* $\bar{R}_\mathfrak{p}$-module of rank $e + 2$ ($\bar{R}_\mathfrak{p}$ is a *semi-local* Dedekind domain); and

$$\Omega^*(\bar{R}_\mathfrak{p}) = \Lambda^{e+2}_{\bar{R}_\mathfrak{p}}(\Omega_f) .$$

Choose $y_1, \cdots, y_{e+2} \in \bar{R}$ such that $\{dy_1, \cdots, dy_{e+2}\}$ is a free $\bar{R}_\mathfrak{p}$-basis of $\Omega_f$, and let $T$ be the corresponding good trace (cf. (4.5) and the argument following (4.6)). Then from (4.6′) it follows (by an obvious induction on $m$) that, for $\lambda \in L$,

$$(\tau_1\tau_2 \cdots \tau_m)(\lambda\Omega^*(\bar{R}_\mathfrak{p})) \subseteqq \Omega^*(R_\mathfrak{p}) \Longleftrightarrow \lambda \in \bar{R}_\mathfrak{p} ,$$

i.e. (since $\Omega^*(\bar{R}_\mathfrak{p}) = \bar{R}_\mathfrak{p} dy_1 dy_2 \cdots dy_{e+2}$ and $\Omega^*(R_\mathfrak{p}) = R_\mathfrak{p}\xi_1 \cdots \xi_{e+2}$):

$$T(\lambda\bar{R}_\mathfrak{p}) \subseteqq R_\mathfrak{p} \Longleftrightarrow \lambda \in \bar{R}_\mathfrak{p} .$$

Thus, for $c_1, c_2, \cdots, c_n \in K$, we have

$$\begin{aligned} & T\left(\textstyle\sum_{i=1}^n c_i e_i' e_j'\right) \in R_\mathfrak{p} && \text{for all } j = 1, 2, \cdots, n , \\ \Longleftrightarrow\ & c_i \in R_\mathfrak{p} && \text{for all } i = 1, 2, \cdots, n . \end{aligned}$$

In particular if $(c_{li}')$ is the inverse (over $K$) of the matrix $(T(e_i'e_j'))$, then $c_{li}' \in R_\mathfrak{p}$ for all $l$, $i$. Thus $\det(T(e_i'e_j'))$ is a unit in $R_\mathfrak{p}$. Q.E.D.

Finally, we outline the proof of (4.6). Localizing at the maximal ideals of $A$, we reduce easily to the case where $A$ is *local*. Suppose first that $L''$ is purely inseparable, of degree $p$, over $L'$. As in the proof of (4.7), we let $\Omega_f(A)$ be the image of $\Omega(A/k_0) \to \Omega(L'/k_0)$, so that $\Omega_f(A)$ is a free $A$-module of rank $e + 2$, and

$$\Omega^*(A) = \Lambda^{e+2}_A \Omega_f(A) .$$

$\Omega_f(\bar{A})$ is defined similarly.

Let $d'a_1, d'a_2, \cdots, d'a_{e+2}$ generate $\Omega_f(A)$, where $a_i \in A$ $(1 \leqq i \leqq e + 2)$ and $d'$: $L' \to \Omega(L'/k_0)$ is the canonical derivation. It is easily seen that $\bar{A} = A[\alpha]$, with $\alpha^p \in A$. Hence $\Omega(L''/k_0)$ is generated by $d''\alpha, d''a_1, \cdots, d''a_{e+2}$, subject to the single relation

$$d''(\alpha^p) = \textstyle\sum_{i=1}^{e+2} b_i d'' a_i = 0$$

where the $b_i \in A$ are determined by

$$d'(\alpha^p) = \textstyle\sum_{i=1}^{e+2} b_i d' a_i .$$

Clearly $\Omega_f(\bar{A}) = \Omega_f(A[\alpha])$ is also generated (over $\bar{A}$) by $d''\alpha, d''a_1, \cdots, d''a_{e+2}$ subject to $\sum b_i d'' a_i = 0$. Since $\Omega_f(\bar{A})$ is free of rank $e + 2$, the $b_i$ must generate the unit ideal in $\bar{A}$, whence some $b_i$, say, $b_1$, is a unit in $A$. Thus, we may assume that $a_1 = \alpha^p$; and we have

$$\Omega^*(\bar{A}) = \Lambda^{e+2}_{\bar{A}}\, \Omega_f(\bar{A}) = \bar{A} d''\alpha d'' a_2 \cdots d'' a_{e+2} .$$

From this and from (4.4b) we find (keeping in mind that $\bar{A} = A[\alpha]$) that (4.6′) holds, and this proves (4.6) for the purely inseparable case.

Assume next that $L''$ is *separable* over $L'$. As above, let $\{d'a_1, d'a_2, \cdots, d'a_{e+2}\}$ be a free basis of the $A$-module $\Omega_f(A/k_0)$ and similarly let $\{d''\alpha_1, d''\alpha_2, \cdots, d''\alpha_{e+2}\}$ be an $\bar{A}$-free basis of $\Omega_f(\bar{A}/k_0)$. Then

$$d''a_i = \sum_{j=1}^{e+2} c_{ij} d''\alpha_j \qquad (c_{ij} \in \bar{A}) ;$$

and if $\Delta$ is the determinant $\det(c_{ij})$, then $\Delta \neq 0$, and

$$\Omega^*(\bar{A}) = \bar{A} d''\alpha_1 \cdots d''\alpha_{e+2} = \Delta^{-1}(\bar{A} \otimes_A \Omega^*(A)) ;$$

furthermore, $\Delta\bar{A} \supseteq$ the Kähler different $\mathfrak{d}$ of $\bar{A}$ over $A$, and $\Delta\bar{A} = \mathfrak{d}$ if $\Omega(\bar{A}/k_0)$ is torsion-free. (Indeed, if $\Omega_t(A)$, $\Omega_t(\bar{A})$ are the torsion submodules of $\Omega(A/k_0)$, $\Omega(\bar{A}/k_0)$ respectively, and if $\mathfrak{f}_0$ is the $0^{\text{th}}$ Fitting ideal of the cokernel of the natural map $\bar{A} \otimes_A \Omega_t(A) \to \Omega_t(\bar{A})$, then one shows easily that $\mathfrak{d} = \Delta\mathfrak{f}_0$.)

The desired conclusions follow now from (4.4a) and the well-known *equality of Kähler and Dedekind differents*, which tells us that for $\lambda \in L$,

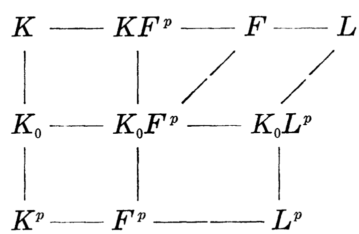
$$\mathrm{Tr}(\lambda\mathfrak{d}^{-1}\bar{A}) \subseteq A \iff \lambda \in \bar{A}$$

("Tr" denoting the usual trace map $L'' \to L'$). Q.E.D.

(4b) **Admissible base fields.**

LEMMA (4.8). *Let $K \subseteq L$ be two fields of characteristic $p > 0$, with $L$ finitely generated (as a field) over $K$, of transcendence degree, say, $t$. Let $K_0$ be a field between $K^p$ and $K$, such that $[K: K_0] = p^f < \infty$. Then the dimension $D$ of the $L$-vector space $\Omega_{K_0}L$ (Kähler differentials of $L$ over $K_0$) satisfies $D \geqq t + f$, with equality if and only if $K_0^{p^{-1}}$ and $L$ are linearly disjoint over $K$.*

*Proof.* Let $K \subseteq F \subseteq L$, with $F$ a purely transcendental field extension of $K$, of transcendence degree $t$ (so that $[L: F] < \infty$ and $[F: KF^p] = p^t$).




Clearly $K$ and $F^p$ are linearly disjoint (l.d.) over $K^p$. So, first of all, $K$ and $K_0F^p$ are l.d. over $K_0$, i.e.,

$$[KF^p: K_0F^p] = [K: K_0] = p^f$$

whence

$$[F: K_0F^p] = [F: KF^p][KF^p: K_0F^p] = p^{t+f} ;$$

and secondly, $K_0$ and $F^p$ are l.d. over $K^p$, whence

$$(K_0 \text{ and } L^p \text{ l.d. over } K^p) \Longleftrightarrow (K_0F^p \text{ and } L^p \text{ l.d. over } F^p)\,.$$

Now

$$[K_0L^p: K_0F^p] \leqq [L^p: F^p] = [L: F]$$

and

$$[L: K_0L^p][K_0L^p: K_0F^p] = [L: F][F: K_0F^p] \qquad (=[L: K_0F^p])$$

so

$$[L: K_0L^p] \geqq [F: K_0F^p] = p^{t+f} \qquad \text{(see above)}$$

$$\begin{aligned} &\text{with equality} \Longleftrightarrow (K_0F^p \text{ and } L^p \text{ l.d. over } F^p)\,;\\ &\text{i.e., (as above)} \Longleftrightarrow (K_0 \text{ and } L^p \text{ l.d. over } K^p)\\ &\text{i.e.,} \qquad\qquad \Longleftrightarrow (K_0^{p^{-1}} \text{ and } L \text{ l.d. over } K)\,. \end{aligned}$$

But $[L: K_0L^p] = p^D$ (cf. [37, page 127]), so (4.8) is proved.

Now let $k$ be a field of characteristic $p \geqq 0$, and let $h: R \to B \in \mathfrak{B}$ be as before (beginning of §(4a)). By [3, page 63, Satz 2.3.13] *the integral closure $\bar{R}$ of $R$ in $B$ is a subring of $B$ which depends only on $B$*; furthermore, *$\bar{R}$ is finite over $R$*. Let $n(B)$ be the Krull dimension of the semi-local ring $\bar{R}$. Note that $B$ has a $k$-subalgebra $R'$ which is $k$-isomorphic to a formal power series ring $k[[U_1, \cdots, U_n]]$ ($n = n(B)$), and over which $B$ is essentially of finite type. (Take $R'$ to be a subring of the complete local ring $h(R)$, with $h(R)$ finite over $R'$.)

If $B$ is a domain, let $t(B)$ be the transcendence degree of $B$ over $\bar{R}$ (or over $R'$) where $\bar{R}$, $R'$ are as above.

COROLLARY (4.9) (cf. [3, page 87, Lemma 3.1.3]). *Let $L \in \mathfrak{B}$ be a field, $n = n(L)$, $t = t(L)$. Let $k_0$ be a base field,* i.e., *$k^p \subseteqq k_0 \subseteqq k$ and* $[k: k_0] =$ *(say) $p^e < \infty$ (when $p=0$ set $e=0$). Then the $L$-vector space $\Omega(L/k_0)$ has dimension $\geqq n + t + e$, with equality if $k_0 = k^p$ ($=k$ when $p = 0$).*

*Proof.* We have just seen that $L$ is essentially of finite type over a $k$-subalgebra $R \cong k[[U_1, \cdots, U_n]]$. When $p>0$, in view of example (4.2), we can apply (4.8), with $K$ the fraction field of $R$ and $K_0$ the fraction field of $k_0[[U_1^p, \cdots, U_n^p]]$ (so that $[K: K_0] = p^{n+e}$). (Note that if $k_0 = k^p$ then $K_0 = K^p$.)

Now suppose that $p = 0$, or, more generally, that $L$ is *separable* over the fraction field $K$ of $R$. $\Omega(R/k)$ is a free $R$-module of rank $n$ (generated by $dU_1, \cdots, dU_n$). It will suffice to show that the sequence

$$0 \longrightarrow L \otimes_R \Omega(R/k) \xrightarrow{\ \psi\ } \Omega(L/k) \longrightarrow \Omega_R L \longrightarrow 0$$

(cf. (4.1)) is *exact,* since $\Omega_R L$ has dimension $t$. We need only check that $\psi$ is injective, i.e., that the dual map

$$\psi^*: \mathrm{Hom}_L(\Omega(L/k), L) \longrightarrow \mathrm{Hom}_L(L \otimes_R \Omega(R/k), L) = \mathrm{Hom}_R(\Omega(R/k), L)$$

is surjective; but this surjectivity follows easily from the fact that every derivation of $R$ into $L$ extends to a derivation of $L$ into $L$ (because $L$ is separable over $K$). Q.E.D.

DEFINITION (4.10). Let $B \in \mathfrak{B}$ be a *regular local ring*, $n = n(B)$, $t = t(B)$, and let $k_0$ be a base field, so that $k^p \subseteqq k_0 \subseteqq k$, and $[k: k_0] < \infty$, say $[k: k_0] = p^e$. (When $p = 0$, set $k_0 = k$ and $e = 0$.) We say that $k_0$ is *admissible for* $B$ if the $B$-module $\Omega(B/k_0)$ is free of rank $n + t + e$.

In Section (4a), the following two results, (4.11) and its generalization (4.12), were needed. (Note that (4.12) was used only in the proof of Corollary (4.7), and hence is not required for proving the existence of a good trace.)

LEMMA (4.11) (cf. also [**EGA** $\mathbf{0}_{\mathrm{IV}}$, §(21.8)] and [**16**, page 96, Folgerung 3.8]). *Let $R$ be the formal power series ring $k[[U_1, \cdots, U_n]]$, let $K = k((U_1, \cdots, U_n))$ be the fraction field of $R$, and let $L \supseteqq K$ be a finitely generated field extension of $K$. Then any base field $k_*$ contains a base field $k_0$ such that $k_0$ is admissible for $L$.*

*Remark.* (4.11) is trivial when $[k: k^p] < \infty$ (since then $k^p$ is admissible, cf. (4.9)).

*Proof of* (4.11). We may assume that $p > 0$. For any base field $k_0 \subseteqq k_*$, set $G_0 = k_0^{p^{-1}}((U_1, \cdots, U_n))$. Recall (cf. (4.8) and proof of (4.9)) that $k_0$ is admissible for $L$ if and only if $G_0$ and $L$ are linearly disjoint over $K$. Let $F$ be a field between $K$ and $L$ such that $F$ is purely transcendental over $K$ and $[L: F] < \infty$; since $G_0$ and $F$ are linearly disjoint over $K$, it will suffice to find $k_0$ such that $G_0F$ and $L$ are linearly disjoint over $F$.

For any field of the form $G_0$, set

$$\nu(G_0) = [G_0L: G_0F] \leqq [L: F] .$$

(Equality holds here if and only if $G_0F$ and $L$ are linearly disjoint over $F$.) Clearly there exists $k_0$ such that, for all fields $G_\alpha = k_\alpha^{p^{-1}}((U_1, \cdots, U_n))$ with a base field $k_\alpha \subseteqq k_0$ we have $\nu(G_\alpha) = \nu(G_0)$. *If we can show* that the intersection of all the fields $G_\alpha F$ is $F$, then it will follow that $\nu = \nu(G_0) = [L: F]$ (proving (4.11)!); for, if $\boldsymbol{\zeta} = (\zeta_1, \cdots, \zeta_\nu)$ ($\zeta_i \in L$) is a vector-space basis of $G_0L$ over $G_0F$, then $\boldsymbol{\zeta}$ is a basis for $G_\alpha L$ over $G_\alpha F$, whence any $\zeta \in L$ is of the form

$$\zeta = a_1\zeta_1 + \cdots + a_\nu\zeta_\nu \qquad (a_i \in \textstyle\bigcap_\alpha G_\alpha F = F) ,$$

and so $[L: F] \leqq \nu$.

Let us show then that $\bigcap_\alpha G_\alpha F = F$. Let $F = K(V_1, \cdots, V_t)$ ($V_i$-indeterminates), and let $a \in F^{p^{-1}} - F$:

$$a = f/g \qquad\qquad (f^p, g^p \in R[V_1, \cdots, V_t]) .$$

We want to find $G_\alpha$ with $a \notin G_\alpha F$. For this, we can replace $a$ by $ag^p$, so we may assume that $a^p \in R[V_1, \cdots V_t] = R[V]$. Note that $R[V]^{p^{-1}} \cap G_0F$ is integral over—hence equal to—

$$k_0^{p^{-1}}[[U_1, \cdots, U_n]][V_1, \cdots, V_t] = \text{(say) } R_0[V] .$$

So either $a \notin G_0F$, in which case we are done, or $a \in R_0[V]$.

If $a \in R_0[V]$, set $a = A(U_1, \cdots, U_n, V_1, \cdots, V_t)$ where $A$ is a power series in $U$ and $V$, at least one of whose coefficients, say $a_0$, does not lie in $k$ (since $a \notin F$). If $\{b_\beta\}$ is a $p$-basis of $k_0$ over $k^p(a_0^p)$, and $k_\alpha = k^p(\{b_\beta\})$, then $[k_0: k_\alpha] = p$, and $a_0 \notin k_\alpha^{p^{-1}}$, so

$$a \notin k_\alpha^{p^{-1}}[[U_1, \cdots, U_n]][V_1, \cdots, V_t] = R_0[V] \cap G_\alpha F . \qquad \text{Q.E.D.}$$

Let $X$ be a reduced scheme of finite type over a power series ring $k[[U_1, \cdots, U_n]]$. For any base field $k'$, there is a *coherent* $\mathcal{O}_X$-module $\Omega(X/k')$ whose stalk at any $x \in X$ is $\Omega(\mathcal{O}_{X,x}/k')$ (cf. (4.1)). We say that $k'$ is *admissible for* $X$ if $k'$ is admissible for $\mathcal{O}_{X,x}$ for each $x \in X$ such that $\mathcal{O}_{X,x}$ is regular (cf. (4.10)). For example, if $X$ is irreducible, then $k'$ is admissible for $X$ if and only if the restriction of $\Omega(X/k')$ to $X$—Sing$(X)$ (Sing$(X)$ = singular locus of $X$) is *locally free of rank* $n + t + e$, where

$$n = n(X) \quad (= n(\mathcal{O}_{X,x}) \text{ for any } x \in X) ,$$
$$t = t(X) \quad (= t(\mathcal{O}_{X,x}) \text{ for any } x \in X) ,$$
$$[k: k'] = p^e \quad (e = 0 \text{ if } p = 0) .$$

PROPOSITION (4.12). *Let $X$ be as above, and let $k_*$ be a base field. Then $k_*$ contains a base field $k_0$ such that $k_0$ is admissible for $X$.*

*Remark.* For $[k: k^p] < \infty$, we can take $k_0 = k^p$ ($=k$ when $p = 0$); cf. (4.15) below.

*Proof.* If a base field is admissible for $X$, then, by (4.13) below, so is any smaller base field. Moreover it is clear that a base field which is admissible for each irreducible component of $X$ is also admissible for $X$. Hence we may assume that $X$ is *irreducible*.

For any base field $k'$ with $[k: k'] =$ (say) $p^e$ ($e = 0$ when $p = 0$), let $S_X(k')$ be the set of points of $X$ where $\Omega(X/k')$ is *not* locally free of rank $n(X) + t(X) + e$ (see above). Since $\Omega(X/k')$ is a coherent $\mathcal{O}_X$-module, $S_X(k')$ is *closed* in $X$. $X$ being *noetherian*, there is a base field $k_0 \subseteq k_*$ such that $S_X(k_0)$ is *minimal* among closed sets of the form $S_X(k')$, $k' \subseteq k_*$. From (4.13) below, it follows then that for any base field $k' \subseteq k_0$, we have $S_X(k') = S_X(k_0)$.

*This* $k_0$ *is admissible for* $X$; otherwise there would be a regular point $x$ of $X$ with $x \in S_X(k_0)$, and by (4.11), a base field $k' \subseteq k_0$ such that $k'$ is admissible for the residue field of $\mathcal{O}_{X,x}$; but then by (4.14) below, $k'$ would be admissible for $\mathcal{O}_{X,x}$, i.e.,

$$x \notin S_X(k') = S_X(k_0) ,$$

a contradiction.

It remains to prove (4.13) and (4.14).

LEMMA (4.13). *Let* $B \in \mathfrak{B}$ *be a local domain,* $n = n(B)$, $t = t(B)$. *Let* $k' \subseteq k_0$ *be two base fields with* $[k: k_0] = p^e$, $[k: k'] = p^{e+f}$ ($e = f = 0$ *when* $p = 0$). *If the B-module* $\Omega(B/k_0)$ *is free of rank* $n + t + e$, *then* $\Omega(B/k')$ *is free of rank* $n + t + e + f$. (*In particular, if B is regular and* $k_0$ *is admissible for B, then also* $k'$ *is admissible for B.*)

*Proof.* If $p = 0$ then $k' = k_0 = k$, so assume $p > 0$. Let $B$ be essentially of finite type over $k[[U_1, \cdots, U_n]] \subseteq B$. There exists a $p$-basis $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_f)$ of $k_0$ over $k'$, and $\boldsymbol{\xi}$ is also a $p$-basis of $k_0[[U_1^p, \cdots, U_n^p]]$ over $k'[[U_1^p, \cdots, U_n^p]]$; hence (cf. (4.2)) the kernel of the natural surjection $\Omega(B/k') \to \Omega(B/k_0)$ is generated by $d\xi_1, \cdots, d\xi_f$ (where $d: B \to \Omega(B/k')$ is the canonical map). Since $\Omega(B/k_0)$ is free of rank $n + t + e$, therefore $\Omega(B/k')$ has a generating set with $\leqq n + t + e + f$ members. But $\Omega(L/k') = L \otimes_B \Omega(B/k')$ ($L$ = fraction field of $B$) has dimension $\geqq n + t + e + f$ over $L$ (cf. (4.9)); hence $\Omega(B/k')$ is free of rank $n + t + e + f$. Q.E.D.

LEMMA (4.14). *Let* $B \in \mathfrak{B}$ *be a regular local ring, with residue field F. If a base field* $k_0$ *is admissible for F, then* $k_0$ *is also admissible for B.*

*Proof.* Let $B$ be essentially of finite type over $R = k[[U_1, \cdots, U_n]] \subseteq B$ ($n = n(B)$). Let $\mathfrak{m}$ be the maximal ideal of $B$, and let $\mathfrak{p} = \mathfrak{m} \cap R$. Then $n_0 = n(F)$ is the dimension of $R/\mathfrak{p}$, and $t_0 = t(F)$ is the transcendence degree of $F$ over $R/\mathfrak{p}$. By assumption, the $F$-vector space $\Omega(F/k_0)$ has dimension $n_0 + t_0 + e$ (where $p^e = [k: k_0]$ if $p > 0$, and $e = 0$ if $p = 0$).

$t = t(B)$ is the transcendence degree of $B$ over $R$. Let $\delta$ be the Krull dimension of $B$. Since $R$ is complete, hence universally catenary, we have the "dimension formula"

$$n - n_0 + t = \delta + t_0 ,$$

so that the dimension of $\Omega(F/k_0)$ is

$$n_0 + t_0 + e = n + t + e - \delta .$$

Now $\delta$ is the dimension of the $F$-vector space $\mathfrak{m}/\mathfrak{m}^2$, and we have a canonical exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega(B/k_0) \otimes_B F \longrightarrow \Omega(F/k_0) \longrightarrow 0$$

(use (4.1) and the similar exact sequence for $\Omega_{k_0}$). Thus $\Omega(B/k_0) \otimes_B F$ has dimension $\leqq n + t + e$, whence (Nakayama's lemma) $\Omega(B/k_0)$ is generated by $\leqq n + t + e$ elements. But if $L$ is the fraction field of $B$, then the $L$-vector space $\Omega(L/k_0) = L \otimes_B \Omega(L/k_0)$ has dimension $\geqq n + t + e$ (cf. (4.9)); it follows that $\Omega(B/k_0)$ is a free $B$-module of rank $n + t + e$, i.e., $k_0$ is admissible for $B$. Q.E.D.

From (4.9) and its proof, we deduce:

COROLLARY (4.15). *With $B$, $F$ as above*:

(a) *If $[k: k^p] < \infty$ (where $k^p = k$ if $p = 0$) then $k^p$ is admissible for $B$.*

(b) *If $F$ is* separable *over the fraction field of some $k$-subalgebra $R_0 \cong k[[V_1, \cdots, V_{n_0}]]$ ($V_i$-indeterminates; $n_0 = n(F)$), then $k$ is admissible for $B$.*

PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA

REFERENCES


[1] S. S. ABHYANKAR, On the valuations centered in a local domain, Amer. J. Math. **78** (1956), 321–348.

[2] ———, Resolution of singularities of algebraic surfaces, *Algebraic Geometry*, 1–11, Oxford Univ. Press, London, 1969.

[3] R. BERGER, R. KIEHL, E. KUNZ and H.-J. NASTOLD, *Differentialrechnung in der analytischen Geometrie*, Lecture Notes in Math. **38**, Springer-Verlag, Heidelberg, 1967.

[4] N. BOURBAKI, *Algèbre Commutative*, Chap. 7, Act. Sci. et Ind. **1314**, Hermann, Paris, 1965.

[5] H. GRAUERT and O. RIEMENSCHNEIDER, Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen, Inven. Math. **11** (1970), 263–292.

[EGA] A. GROTHENDIECK and J. DIEUDONNÉ, *Éléments de Géométrie Algébrique*:

[6] [EGA II] Publ. Math. I.H.E.S, **8**, 1961.

[7] [EGA III] Publ. Math. I.H.E.S., **11**, 1961.

[8] [EGA $0_{IV}$] Publ. Math. I.H.E.S., **20**, 1964.

[9] [EGA IV] Publ. Math. I.H.E.S., **24**, 1965; **32**, 1967.

[10] A. GROTHENDIECK, *Local Cohomology*, Lecture Notes in Math. **41**, Springer-Verlag, Heidelberg, 1967.

[11] R. HARTSHORNE, *Residues and Duality*, Lecture Notes in Math. **20**, Springer-Verlag, Heidelberg, 1966.

[11a] ———, On the de Rham cohomology of algebraic varieties, Publ. Math. I.H.E.S., **45** (1976), 5–99.

[12] H. HIRONAKA, Desingularization of excellent surfaces, Advanced Science Seminar in Algebraic Geometry, Bowdoin College, Summer 1967 (notes by Bruce Bennett on Hironaka's lectures).

[13] ———, Characteristic polyhedra of singularities, J. Math. Kyoto Univ. **7** (1967), 251–293.

[14] ———, Certain numerical characters of singularities, J. Math. Kyoto Univ. **10** (1970), 151–187.

[15] ———, Additive groups associated with points of a projective space, Annals of Math. **92** (1970), 327–334.

[16] R. KIEHL, Ausgezeichnete Ringe in der nichtarchimedischen analytischen Geometrie, J. Reine Angew. Math. **234** (1969), 89-98.

[17] K. KIYEK, Kohomologiegruppen und Konstantenreduktion in Funktionenkörpern, Inven. Math. **9** (1970), 99-120.

[18] S. L. KLEIMAN, Toward a numerical theory of ampleness, Annals of Math. **84** (1966), 293-344.

[19] E. KUNZ, Differentialformen inseparabler algebraischer Funktionenkörper, Math. Z. **76** (1961), 56-74.

[20] ———, Über die kanonische Klasse eines vollständigen Modells eines algebraischen Funktionenkörpers, J. Reine Angew. Math. **209** (1962), 17-28.

[20a] ———, Arithmetische Anwendungen der Differentialalgebren, J. Reine Angew. Math. **214/215** (1964), 276-320.

[21] ———, Differentialformen auf Algebraischen Varietäten mit Singularitäten II, Abh. Math. Sem. Univ. Hamburg, forthcoming.

[22] H. LAUFER, On rational singularities, Amer. J. Math. **94** (1972), 597-608.

**[RS]** J. LIPMAN, Rational singularities..., Publ. Math. I.H.E.S., **36** (1969), 195-279.

[24] ———, Local duality and rational singularities (preliminary report), Notices A.M.S. **20** (1973), A-385.

[25] ———, Introduction to resolution of singularities, Proc. Sympos. Pure Math. A.M.S. **29** (1975), 187-230.

[26] E. MATLIS, Injective modules over noetherian rings, Pacific J. Math. **8** (1958), 511-528.

[27] D. MUMFORD, The topology of normal singularities of an algebraic surface, Publ. Math. I.H.E.S. **9** (1961), 5-22.

[28] D. REES, A note on analytically unramified local rings, J. London Math. Soc. **36** (1961), 24-28.

[29] ———, Local birational geometry, *Proc. Internat. Colloq. Algebraic Geometry* (*Madrid*, 1965), 17-22, Inst. Jorge Juan, Madrid, 1966.

[30] D. REES, D. G. NORTHCOTT, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc. **50** (1954), 145-158.

[31] J. SALLY, On the associated graded ring of a local Cohen-Macaulay ring, J. Math. Kyoto Univ., to appear.

[32] J. WAHL, Vanishing theorems for resolutions of surface singularities, Inven. Math. **31** (1975), 17-41.

[32a] ———, Equisingular deformations of normal surface singularities, I, Annals of Math. **104** (1976), 325-356.

[33] ———, Equations defining rational singularities, Ann. Sci. École Norm. Sup., 4e série, t. **10** (1977), 231-264.

[34] O. ZARISKI, The reduction of the singularities of an algebraic surface, Annals of Math. **40** (1939), 639-689. (Reprinted in: *Collected Papers, vol.* 1, 325-375, M.I.T. Press, Cambridge, Mass., 1972.)

[34a] ———, The concept of a simple point of an abstract algebraic variety, Trans. A.M.S. **62** (1947), 1-52. (Reprinted in: *Collected Papers, vol.* 1, 253-304, loc. cit.)

[35] ———, Postulation et genre arithmétique, Colloque d'Algèbre et Théorie des Nombres, Paris, 1949, 115-116. (Reprinted in: *Collected Papers, vol.* 2, p. 208, M.I.T. Press, Cambridge, Mass., 1973.)

[36] ———, *Introduction to the problem of minimal models in the theory of algebraic surfaces*, Publ. Math. Soc. Japan, no. 4 (1958) (Reprinted in: *Collected Papers, vol.* 2, 277-369, loc. cit.)

[37] O. ZARISKI and P. SAMUEL, *Commutative Algebra, vol.* 1, Van Nostrand, Princeton, N. J., 1958.

[38] ———, *Commutative Algebra, vol.* 2, Van Nostrand, Princeton, N. J., 1960.

(Received February 23, 1977)