DIVISORS ON ZARISKI-RIEMANN SPACES

Joseph Lipman*

References

[1]. ABHYANKAR, S.S. On the valuations centered in a local domain, Amer. J. Math. 78 (1956), 321-348.

[2] BOURBAKI, N. Algèbre Commutative, Hermann, Paris.

[3] SNAPPER, E. Higher-dimensional field theory, Compositio Math. 13 (1956), 1-46.

[4] v.d. WAERDEN, B. The invariant theory of linear sets on an algebraic variety. Proc. International Congress of Math. (1954), vol. III, 542-544. North-Holland, Amsterdam, 1956.

[5] ZARISKI, O., Polynomial ideals defined by infinitely near base points, Amer. J. Math. 60 (1938), 151-204.

[6] ZARISKI, O. and SAMUEL P. Commutative Algebra, vol. 2, Van Nostrand, Princeton, 1960.

* Written with the support, in part, of the National Science Foundation under GP-6388 at Purdue University.

Introduction. In this appendix we outline a theory of "invariant divisors" which includes as a special case the theory of linear systems with base conditions presented in the foregoing chapter two. Let us first sketch roughly the main features; precise definitions and statements appear in the indicated sections. Given a field K and a subring A, an invariant divisor of K/A is a divisor on the Zariski-Riemann space of K/A (cf. §2). If V is a model of K/A, say normal and noetherian with function field K, then all information about invariant divisors is contained in the monoid of complete fractionary ideals on V (§2).

Each invariant divisor D of K/A is represented in the form C_H, where C is a divisorial cycle on V and H is a "base divisor"; corresponding to the set of all positive invariant divisors linearly equivalent to D we have the set of all C'_H such that C' is linearly equivalent to C and C' "satisfies the base conditions imposed by H" (§§3-4). When V is locally factorial, the correspondence $D \longleftrightarrow C_H$ gives a decomposition of the group of invariant divisors into a direct sum of the group of divisorial cycles on V and the group of base divisors on V, so that $C_H + C'_{H'} = (C + C')_{H+H'}$.

Some additional information about the group of base divisors is obtained by introducing "virtual multiplicities at infinitely near local rings" (§ 5). This leads to complete results when V is a non-singular surface (Theorem 6.1); for higher dimensions little seems to be known.

The group of base divisors is generated by the submonoid consisting of those base divisors which appear "effectively" as base divisors of linear systems without fixed components on V (§4). In the case of non-singular surfaces, these are precisely the base divisors whose virtual multiplicities satisfy the "proximity inequalities" (Theorem 6.2).

The "invariant" aspect of our theory takes the following form: if V^* is another model of K/A, and if some invariant divisor D is represented by C_H on V and by $C^*_{H^*}$ on V^*, then we say that the pair (C^*, H^*) is the transform on V^* of (C, H); the properties of the pair (C, H) which are invariant under the transformation $(C, H) \rightarrow (C^*, H^*)$ are simply the properties belonging to D.

In order to keep prerequisites to a minimum, we have avoided explicit use of the language of sheafs and ringed spaces. Instead we give in §1 an ad hoc treatment of the basic facts required later on, all in the context of models (cf. [6; Ch. VI, §17]). Accordingly, some statements appear in somewhat less than full generality.

The idea of using valuation theory and complete ideals to simplify the discussion of linear systems with base conditions appears first in [5]. For further developments cf. [4], [3], and [6; Appendices 4 and 5].

1. Preliminaries on models and complete modules.

We fix once and for all a field K and a subring A of K. For any ring B of the type A[F] with F a finite subset of K, the set of (not necessarily noetherian) local rings

$$\mathscr{V}(B) = \{B_p \mid p \text{ runs through all prime ideals of } B\}$$

is called an affine model (over A); a model V (over A) is a finite union of affine models such that each valuation ring of K/A dominates at most one member of V;[1] such a V is proper (or complete) over A if each valuation ring of K/A does dominate a (necessarily unique) member of V. If $\{f_0, f_1, \ldots, f_r\}$ is a non-empty finite subset of K, with no $f_i = 0$, and if $V_i = \mathscr{V}(A[f_0/f_i, f_1/f_i, \ldots, f_r/f_i])$, then $\cup_{i=0}^{r} V_i$ is a proper model, called the projective model (over A) determined by $\{f_0, f_1, \ldots, f_r\}$. We topologize any model V by taking as a basis of open sets the collection of all those subsets of V which are affine models; V is then quasi-compact and irreducible (i.e. any two non-empty open subsets of V have non-empty intersection).

For any model V a (quasi-coherent) V-submodule (of K) is defined to be a family $\{J_R\}_{R \in V}$ where, for each local ring

(1) A valuation ring of K/A is a valuation ring which contains A and has field of fractions K. A local ring S dominates a local ring R if $R \subseteq S$ and every non-unit in R is a non-unit in S.

$R \in V$, J_R is an R-submodule of K, the family being subject to the following "cohesiveness" condition:

(C): There exists an open covering $\{U_i\}_{i \in I}$ of V, and a family $\{F_i\}_{i \in I}$ of subsets of K such that, for each $i \in I$, if $R \in U_i$ then $J_R = F_i R$ (= the R-module generated by F_i).

If the families $\{U_i\}$, $\{F_i\}$, in (C) can be chosen so that each F_i is a finite set, we say that the V-submodule is <u>of finite type</u>. One important finite type V-submodule is the family $\{R\}_{R \in V}$, which will always be denoted "$\mathcal{O}_V$". If F is a non-empty subset of K, we will write "$F\mathcal{O}_V$" for the V-submodule $\{FR\}_{R \in V}$, which is the V-submodule <u>generated by</u> F.

For any non-empty open subset U of V, and any V-submodule $\mathcal{J} = \{J_R\}$, set

$$\Gamma(U, \mathcal{J}) = \bigcap_{R \in U} J_R;$$

an element of $\Gamma(U, \mathcal{J})$ is called a <u>section of</u> $\mathcal{J}$ <u>over</u> U. $\mathcal{J}$ is said to be <u>generated by global sections</u> (over V) if $\mathcal{J}$ is of the form $F\mathcal{O}_V$. (Necessarily then $F \subseteq \Gamma(V, \mathcal{J})$ and $\mathcal{J} = \Gamma(V, \mathcal{J})\mathcal{O}_V$).

PROPOSITION 1.1. <u>If</u> $V = \mathcal{V}(B)$ <u>is an affine model, then every</u> V-<u>submodule</u> $\mathcal{J} = \{J_R\}$ <u>is generated by global sections. Thus if</u> $R = B_p \in V$ (p a prime ideal in B) <u>we have, with</u> $J = \Gamma(V, \mathcal{J})$,

$$J_R = JR = J_p = \{j/b \mid j \in J,\ b \in B-p\}.$$

<u>Moreover</u> $\mathcal{J}$ <u>is of finite type if and only if</u> J <u>is a finitely generated</u> B-<u>module.</u>

<u>Proof.</u> We must show that $\mathcal{J} = F\mathcal{O}_V$ with F finite if $\mathcal{J}$ is of finite type; the last assertion will follow since then

$$J = \bigcap_{R \in V} J_R = \bigcap_p FB_p = FB.$$

Now V has the basis of open sets $\{V_f\}$ $(0 \neq f \in B)$ where $V_f = \mathcal{V}(B[1/f]) = \{B_p \mid f \notin p\}$; hence in view of (C) and the quasi-compactness of V it suffices to prove:

LEMMA 1.2. V <u>and</u> $\mathcal{J}$ <u>being as in</u> 1.1, <u>if</u> $s \in \Gamma(V_f, \mathcal{J})$, <u>then</u> $f^n s \in \Gamma(V, \mathcal{J})$ <u>for every sufficiently large</u> n.

<u>Proof.</u> If $\mathcal{J}$ is generated by $J = \Gamma(V, \mathcal{J})$ then 1.2 is clear, since

$$\Gamma(V_f, \mathcal{J}) = \bigcap_{f \notin p} J_p = J_f = \{j/f^m \mid j \in J,\ m \geq 0\}.$$

But, by (C), each $R \in V$ has an open affine neighborhood U such that the U-submodule $\mathcal{J}_U = \{J_R\}_{R \in U}$ is generated by global sections (over U); since $s \in \Gamma(V_f, \mathcal{J}) \subseteq \Gamma(U_f, \mathcal{J}_U)$ we conclude that $f^n\, s \in \Gamma(U, \mathcal{J}_U) = \Gamma(U, \mathcal{J})$ for sufficiently large n; 1.2 follows because V, being quasi-compact, has a finite covering by such U.

For any model V, the <u>product</u> of two V-submodules $\mathcal{J} = \{J_R\}$, $\mathcal{L} = \{L_R\}$ is the family $\{J_R L_R\}$. ($J_R L_R$ is the R-module generated by all the products xy, $x \in J_R$, $y \in L_R$. The verification of (C) for $\{J_R L_R\}$ is straightforward). $\mathcal{J}$ is <u>invertible</u> if there exists $\mathcal{L}$ such that $\mathcal{J}\mathcal{L} = \mathcal{O}_V$. The invertible V-submodules form, under multiplication, an abelian group $\mathrm{Inv}_K(V)$ with identity element $\mathcal{O}_V$.

We can specify invertible V-submodules in the following way: let $\{V_i\}_{i \in I}$ be an open covering of V, and for each i let $f_i \neq 0$ be an element of K; assume that for each i,j and $R \in V_i \cap V_j$ we have $f_i R = f_j R$ (in other words for each i,j, $f_i/f_j \in \Gamma(V_i \cap V_j, \mathcal{O}_V)$); then for each R we can unambiguously set $J_R = f_i R$ for any i such that $R \in V_i$; the family $\{J_R\}$ is clearly an invertible V-submodule. Conversely if $\mathcal{L} = \{L_R\}$ is an invertible V-submodule, then one sees that $\mathcal{L}$ is of finite type and that for each local ring R, L_R is generated by a single element of K; it follows without difficulty that $\mathcal{L}$ is given by a family $\{V_i, f_i\}_{i \in I}$ as above. Since V is quasi-compact, we may even assume I to be finite.

A family $D = \{V_i, f_i\}$ with $f_i/f_j \in \Gamma(V_i \cap V_j, \mathcal{O}_V)$ for all i,j is called a "K-valued divisor" (or simply "K-divisor") on V, two such divisors being considered equal if they define (as above) the same invertible V-submodule. The invertible submodule corresponding to the divisor D is denoted "$\mathcal{J}_V(D)$";

also, we set $\mathcal{O}_V(D) = (\mathcal{J}_V(D))^{-1}$. The one-one correspondence $D \longleftrightarrow \mathcal{J}_V(D)$ allows us to endow the set $\mathrm{Div}_K(V)$ of K-divisors on V with the structure of an abelian group, isomorphic to $\mathrm{Inv}_K(V)$.

Now let

$$V = \bigcup_{i=0}^{r} V_i = \bigcup_{i=0}^{r} \mathcal{V}(A[f_0/f_i, f_1/f_i, \ldots, f_r/f_i])$$

be a projective model. It is clear that the family $\{V_i, f_i^n\}_{0 \leq i \leq r}$ is a K-divisor for each integer n; the corresponding invertible V-submodule is denoted "$\mathcal{O}_V(n)$". For $n \geq 0$, $\mathcal{O}_V(n)$ is generated by its global sections: indeed $f_i^n \in \Gamma(V, \mathcal{O}_V(n))$ for all i. Moreover:

PROPOSITION 1.3. <u>For any finite-type V-submodule</u> $\mathcal{J}$, <u>the V-submodule</u> $\mathcal{J}(n) = \mathcal{J}\mathcal{O}_V(n)$ <u>is generated by finitely many global sections for all sufficiently large</u> n.

<u>Proof.</u> Let $s \in \Gamma(V_i, \mathcal{J})$. By Lemma 1.2, applied to V_j and $V_j \cap V_i$ $(=(V_j)_{f_i/f_j})$, $s(f_i/f_j)^n \in \Gamma(V_j, \mathcal{J})$ for all $n \geq n_j(s)$; hence for $n \geq \max_j \{n_j(s)\}$ we have $sf_i^n \in \Gamma(V_j, \mathcal{J}(n))$ for all j, i.e. $sf_i^n \in \Gamma(V, \mathcal{J}(n))$. Proposition 1.3 now follows easily from Proposition 1.1 (with $V = V_i$, $i = 1,2,\ldots,r$).

We close this preliminary section with some remarks about <u>complete</u> V-submodules and finiteness conditions. If T is any subring of K, and J is a T-submodule of K, then the <u>completion</u> J' of J in K is defined to be the T-submodule

$$J' = \cap JR \qquad \text{(R runs through all valuation rings of } K/T\text{).}$$

J is <u>complete</u> if $J = J'$; clearly J' itself is complete. It is shown in [6; Theorem 1, p. 350] that an element **z** of K is in J' if and only if z satisfies a relation of the form

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0 \qquad a_i \in J^i.$$

From this it follows that if M is a multiplicatively closed subset of T, with $0 \notin M$, then the completion of the T_M-module J_M is $(J')_M$, i.e. <u>completion commutes with localization</u>.

Let V be a model and let $\mathcal{J} = \{J_R\}$ be a V-submodule. The preceding remarks imply that $\mathcal{J}' = \{(J_R)'\}$ is also a V-submodule; $\mathcal{J}'$ is called the <u>completion</u> of $\mathcal{J}$. $\mathcal{J}'$ may not be of finite type, even if $\mathcal{J}$ is. For this reason, we will need to consider integral domains T which have the following property:

(F): For any integral domain S which contains and is finitely generated over T, the integral closure of S in any finite algebraic extension of the field of fractions of S is a finitely generated S-module.[2]

---

(2) (F) is satisfied for example if T is a ring of fractions of a finitely generated extension of a field, of a complete local domain, or of a Dedekind domain of characteristic zero.

LEMMA 1.4. *Let* T *be a subring of* K *satisfying* (F) *and such that* K *is a finitely generated field extension of the fraction field* L *of* T. *Then for any finitely generated* T-*module* $J \subseteq K$, *the completion* J' *is also finitely generated over* T.

Proof. In the polynomial ring K[X], we have the graded subring S = T[JX] which is an integral domain, finitely generated over T. The integral closure $\bar{S}$ of S in K[X] is a graded ring [2; ch. V, §1, no. 8] and $\bar{S}$ is a *finitely generated* S-module, since K(X) is finitely generated over L so that the fraction field of $\bar{S}$ is a finite algebraic extension of that of S. Moreover, the above relation for z says that J'X is the set of all homogeneous elements of degree 1 in $\bar{S}$. The conclusion follows easily.(3)

COROLLARY 1.5. *Suppose that condition* (F) *holds with* T = A, *and that* K *is a finitely generated field extension of the fraction field of* A. *If* V *is a model over* A *and* $\mathcal{J}$ *is a finite type* V-*submodule, then the completion* $\mathcal{J}'$ *is also of finite type.*

As an application of 1.4, we have:

PROPOSITION 1.6. *If* A *is a noetherian ring such that* (F) *holds for* T = A, *if* V *is a proper model over* A, *and if* $\mathcal{J} = \{J_R\}$ *is a finite type* V-*submodule, then* $\Gamma(V, \mathcal{J})$ *is a finitely generated* A-*module.*

---

(3) If T is *noetherian* we can dispense with the remark that $\bar{S}$ is a graded ring. This simplification is of interest in connection with Proposition 1.6, for example.

Proof. Let $\{U_i, F_i\}_{i\epsilon I}$ be as in (C), with each F_i finite; since V is quasi-compact we may assume I to be finite. Let J be the A-module generated by the finite set $\cup_{i\epsilon I} F_i$, let K_1 be the common fraction field of all members of V, and let $K_2 = K_1(J)$. Then $J_R \subseteq JR$ for every $R \epsilon V$, whence

$$\Gamma(V, \mathcal{J}) \subseteq \bigcap_{R\epsilon V} JR \subseteq \bigcap_{S} JS$$

where S runs through all valuation rings of K_2/A. The result follows now from Lemma 1.4, with T replaced by A and K by K_2.

2. Invariant divisors. Let V, W be proper models such that W dominates V, i.e. each $S \epsilon W$ dominates a member d(S) (necessarily unique) of V; we have then the "domination map" $d = d_{W,V}$ which is a continuous map of W onto V. The inverse image $\mathcal{J}\mathcal{O}_W$ of a V-submodule $\mathcal{J} = \{J_R\}$ is the W-submodule $\{J_{d(S)}S\}_{S\epsilon W}$ (here condition (C) of §1 is straightforward). Clearly $(\mathcal{J}_1\mathcal{J}_2)\mathcal{O}_W = (\mathcal{J}_1\mathcal{O}_W)(\mathcal{J}_2\mathcal{O}_W)$; we deduce at once a homomorphism of groups $\mathrm{Inv}_K(V) \to \mathrm{Inv}_K(W)$ and correspondingly a homomorphism

$$d^{-1} : \mathrm{Div}_K(V) \to \mathrm{Div}_K(W).$$

The inverse image $d^{-1}D$ of a divisor $D = \{V_i, f_i\}$ is found to be $\{d^{-1}_{W,V}(V_i), f_i\}$.

If W is dominated in turn by a proper model X, then $d^{-1}_{X,W} \circ d^{-1}_{W,V} = d^{-1}_{X,V}$; moreover any two proper models are dominated by a third, namely their join [6; pp. 120-121]. Thus we have a filtered inductive system of groups and homomorphisms; we set

$$\mathrm{Div}(K/A) = \varinjlim_{V} \mathrm{Div}_K(V) \qquad (V \text{ runs through all proper models over } A).$$

We call the elements of the group Div(K/A) "invariant divisors".

Although one can proceed with the theory on the basis of the preceding definition, it is more satisfactory in some ways to reinterpret the notion of "invariant divisor" by means of the Zariski-Riemann space Z of K/A. Z is the topological space whose points are the valuation rings of K/A, with basis of open sets $\{U_F\}$, where F runs through all finite subsets of K (including the empty one) and U_F consists of all those valuation rings which contain F. Z-submodules (of K) are defined in the same way as V-submodules were in §1. Note however that since the local rings belonging to Z are <u>valuation rings</u> with field of fractions K, <u>every non-zero finite type</u> Z-<u>submodule</u> $\mathcal{J}$[4] <u>is invertible.</u> Indeed, by suitably refining the covering stipulated in condition (C) (§1), one sees that $\mathcal{J} = \{J_R\}_{R \in Z}$ is given by an open covering $\{V_i\}_{i \in I}$ of Z and a family $\{f_i\}_{i \in I}$ $(0 \neq f_i \in K)$ such that

(4) $\mathcal{J} = \{J_R\}$ is "non-zero" if $J_R \neq 0$ for some R; the set of all such R is open and closed, whence $J_R \neq 0$ for all $R \in Z$,

$J_R = f_i R$ for all $R \in V_i$. Here again since Z is quasi-compact [6; p. 113] we may assume I to be finite.

We define the group $\mathrm{Div}_K(Z)$ as we did the groups $\mathrm{Div}_K(V)$ (§1). For any proper model V, we have the domination map $d_V: Z \to V$ and, as above, the associated homomorphism $d_V^{-1}: \mathrm{Div}_K(V) \to \mathrm{Div}_K(Z)$. Passing to the limit, we obtain a canonical homomorphism

$$\tilde{d}: \mathrm{Div}(K/A) \to \mathrm{Div}_K(Z).$$

PROPOSITION 2.1. $\tilde{d}$ <u>is an isomorphism.</u>

To show that $\tilde{d}$ is injective, let $D = \{V_i, f_i\}_{i\in I} \in \mathrm{Div}_K(V)$ (I finite) be such that $d_V^{-1}(D) = 0$ (i.e. f_i is a unit in every valuation ring which dominates (an element of) V_i). Let W be the join of V and all the projective models $W^{(i)} = \mathscr{V}(A[f_i]) \cup \mathscr{V}(A[1/f_i])$. If $S \in W$, then S dominates V_i for some i, and S contains either f_i or $1/f_i$; since both f_i and $1/f_i$ are units in any valuation ring of K dominating S, f_i is also a unit in S. Hence $d_{W,V}^{-1}(D) = 0$, i.e. $\tilde{d}$ is injective. For the surjectivity let $D' = \{U_i, g_i\}_{i\in I'} \in \mathrm{Div}_K(Z)$ (I' finite); we may assume that each U_i is a basic open set, $U_i = U_{F_i}$, so that U_i is the inverse image of an open set on some proper model (for example the projective model over A determined by $(F_i - \{0\}) \cup \{1\}$); if W is the join of these models and all the models $W^{(jk)} = \mathscr{V}(A[g_j/g_k]) \cup \mathscr{V}(A[g_k/g_j])$ $(j,k \in I')$ then for every $i \in I'$, U_i is the inverse image of an open set V_i on W, and moreover for each $S \in W$ and for each pair j,k, S contains either g_j/g_k or g_k/g_j. It follows easily that the family $\{V_i, g_i\}$ defines a divisor on W whose inverse image on Z is D'. q.e.d.

Thus we may identify "invariant divisors" with "K-divisors on Z", or as we shall say from now on, "Z-divisors".

Let V, Z be as above (with V proper). For each finite type non-zero V-submodule $\mathcal{J}$, the Z-submodule $\mathcal{J}\mathcal{O}_Z$ is of finite type, hence invertible; thus $\mathcal{J}\mathcal{O}_Z = \mathcal{O}_Z(D)$ for some Z-divisor D, which we denote "$\mathrm{div}_Z(\mathcal{J})$". It follows immediately from the definitions that $\mathrm{div}_Z(\mathcal{J}_1) = \mathrm{div}_Z(\mathcal{J}_2)$ if and only if $\mathcal{J}_1$ and $\mathcal{J}_2$ have the same completion. It is also clear that

$$\mathrm{div}_Z(\mathcal{J}_3\mathcal{J}_4) = \mathrm{div}_Z(\mathcal{J}_3) + \mathrm{div}_Z(\mathcal{J}_4).$$

Moreover, any Z-divisor D is of the form

$$D = \mathrm{div}_Z(\mathcal{J}_5) - \mathrm{div}_Z(\mathcal{J}_6).$$

where $\mathcal{J}_5 = F\mathcal{O}_V$, $\mathcal{J}_6 = G\mathcal{O}_V$, with finite sets F, G. For, by Proposition 2.1 and its proof, D is the inverse image on Z of a divisor D_W on some projective model W; if $\mathcal{L} = \mathcal{O}_W(D_W)$ then, by Proposition 1.3, for some suitably large

n we have $\mathcal{L}(n) = F\mathcal{O}_W$, $\mathcal{O}_W(n) = G\mathcal{O}_W$ (F, G finite), and the assertion follows since $\mathcal{L} = \mathcal{L}(n).\mathcal{O}_W(n)^{-1}$.

In summary:

PROPOSITION 2.2. Let $\underset{\sim}{C}_V$ be the monoid whose elements are the completions of finite type non-zero V-submodules of K, with product $\mathfrak{m} * \mathfrak{n}$ = completion of $\mathfrak{m}\mathfrak{n}$. The preceding considerations give rise to an injective homomorphism of monoids $\underset{\sim}{C}_V \to \mathrm{Div}_K(Z)$ such that the group $\mathrm{Div}_K(Z)$ is generated by the image of $\underset{\sim}{C}_V$.

We see then that the group of invariant divisors is completely determined by $\underset{\sim}{C}_V$ (for any V). It may be observed that under the conditions of Corollary 1.5 the members of $\underset{\sim}{C}_V$ are themselves finite-type V-submodules, and that the homomorphism of 2.2 takes any $\mathcal{J} \in \underset{\sim}{C}_V$ to $\mathrm{div}_Z(\mathcal{J})$; in this case, moreover, $\underset{\sim}{C}_V = \underset{\sim}{C}_W$ where W is the derived normal model (=integral closure) of V in K [6; ch. VI, §18].

3. Linear systems. We assume henceforth that A is a noetherian ring such that condition (F) of §1 holds for T = A and that K is finitely generated over the fraction field of A. Replacing A by its integral closure in K we may further assume, without loss of generality, that A is integrally closed in K. We have then:

PROPOSITION 3.1. If $\mathcal{J}$ is an invertible Z-submodule, then $\Gamma(Z, \mathcal{J})$ is a complete finitely generated A-module.

(Completeness is straightforward; for the rest, the proof of 1.6 applies mutatis mutandis).

A relatively principal Z-divisor is one of the form $\mathrm{div}_Z(\mathcal{J})$, where $\mathcal{J}$ is an invertible $\mathcal{V}(A)$-submodule. The relatively principal Z-divisors form a subgroup P of $\mathrm{Div}_K(Z)$. Two Z-divisors D_1, D_2 are relatively linearly equivalent (notation: $D_1 \equiv D_2$) if their difference is relatively principal, i.e. if they lie in the same coset of P.

For any Z-divisor D and non-zero finite type $\mathcal{V}(A)$-submodule $\mathfrak{m}$, we define a non-empty set $\Lambda(D, \mathfrak{m}) \subseteq D + P$,

$$\Lambda(D, \mathfrak{m}) = \{D - \mathrm{div}_Z(\mathcal{J}) \mid \mathcal{J} \text{ an invertible } \mathcal{V}(A)\text{-submodule, } \mathcal{J} \subseteq \mathfrak{m}\}^{(5)}.$$

A set of the form $\Lambda(D, \mathfrak{m})$ is called a virtual linear system. Note that the members of $\Lambda(D, \mathfrak{m})$ correspond one-one with the invertible $\mathcal{J} \subseteq \mathfrak{m}$: this follows at once from the fact that, A being integrally closed in K, every invertible $\mathcal{V}(A)$-submodule is complete. Note also that these $\mathcal{J}$ correspond one-one with projective rank one A-submodules of the finitely generated A-module $\Gamma(\mathcal{V}(A), \mathfrak{m})$ (Proposition 1.1, and cf. [2; ch. II, §5, no. 3]).

(5) For two V-submodules (or Z-submodules) $\mathcal{J} = \{J_R\}$, $\mathcal{L} = \{L_R\}$, "$\mathcal{J} \subseteq \mathcal{L}$" means "$J_R \subseteq L_R$ for all $R \in V$ (resp. $R \in Z$)".

It is readily seen that $\Lambda(D_1, \mathfrak{m}_1) = \Lambda(D_2, \mathfrak{m}_2)$ if and only if there is an invertible $\mathcal{V}(A)$-submodule $\mathcal{J}$ such that $D_2 = D_1 + \mathrm{div}_Z(\mathcal{J})$ and $\mathfrak{m}_2 = \mathfrak{m}_1\mathcal{J}$ (use Proposition 1.1). Thus the set $\Lambda(D, \mathfrak{m})$ uniquely determines (and, conversely, is determined by) two things:

(i) the Z-divisor $D\text{-}\mathrm{div}_Z(\mathfrak{m})$, which we call the <u>fixed part</u> (or <u>excess</u>) of $\Lambda(D, \mathfrak{m})$;

(ii) the equivalence class of $\mathfrak{m}$ for the following equivalence relation -

$\mathfrak{m}_1 \equiv \mathfrak{m}_2 \Leftrightarrow \mathfrak{m}_2 = \mathfrak{m}_1\mathcal{J}$ for some invertible $\mathcal{V}(A)$-submodule $\mathcal{J}$.

Since, for invertible $\mathcal{V}(A)$-submodules $\mathcal{J}$,

$$\mathcal{J} \subseteq \mathfrak{m} \Longleftrightarrow \mathcal{O}_{\mathcal{V}(A)} \subseteq \mathfrak{m}\mathcal{J}^{-1}$$

we see that the elements of $\Lambda(D, \mathfrak{m})$ correspond one-one with those members of the equivalence class of $\mathfrak{m}$ which contain $\mathcal{O}_{\mathcal{V}(A)}$.

We say that $\Lambda(D, \mathfrak{m})$ is <u>reduced</u> if it has no fixed part (i.e. $D = \mathrm{div}_Z(\mathfrak{m})$). The linear system $\Lambda(\mathrm{div}_Z(\mathfrak{m}), \mathfrak{m})$ (which depends only on the equivalence class of $\mathfrak{m}$) is called <u>the reduced linear system associated with</u> $\Lambda(D, \mathfrak{m})$.

The group $\mathrm{Div}_K(Z)$ is (partially) ordered by the relation

$$D_1 \geq D_2 \Leftrightarrow \mathcal{O}_Z(D_1) \supseteq \mathcal{O}_Z(D_2).$$

We say that a linear system $\Lambda(D,\mathfrak{m})$ is <u>real</u> if all its members are positive (i.e. ≥ 0). For this to be so, it is necessary and sufficient that the fixed part $D - \mathrm{div}_Z(\mathfrak{m})$ be positive; in particular any <u>reduced</u> linear system is real. Another necessary and sufficient condition for reality is that $\mathfrak{m} \subseteq \mathfrak{m}_D$, where $\mathfrak{m}_D$ is the $\mathfrak{r}(A)$-submodule generated by $\Gamma(Z,\mathcal{O}_Z(D))$.

For given D, $\mathfrak{m}_D$ is complete and of finite type (Proposition 3.1). If $\mathfrak{m}_D \neq (0)$, then the set $|D|$ of all positive divisors which are relatively linearly equivalent to D is a real linear system, namely

$$|D| = \Lambda(D,\mathfrak{m}_D) .$$

$|D|$ is called the <u>complete linear system determined by</u> D. We emphasize that $|D|$ is a real linear system if and only if $\mathfrak{m}_D \neq 0$ (otherwise $|D|$ is empty).

The connection between complete linear systems and complete modules is expressed in the following remark:

Let V be a proper model over A and let $\mathcal{J}$ be a finite type V-submodule, with completion $\mathcal{J}'$. Then the elements of $|\mathrm{div}_Z(\mathcal{J})|$ correspond one-one with the projective rank one A-submodules of $\Gamma(V,\mathcal{J}')$.

(In view of the preceding discussion, it is sufficient to note that, with $D = \mathrm{div}_Z(\mathcal{I})$,

$$\Gamma(\mathcal{V}(A), \mathcal{m}_D) = \Gamma(Z, \mathcal{I}\mathcal{O}_Z) = \Gamma(V, \mathcal{I}')).$$

4. Divisorial cycles and base divisors. With A and K as in §3, we can fix a proper model V such that V is noetherian and normal, with function field K (i.e. $K \in V$, or, equivalently, K is the common fraction field of the members of V). In this case, finite type V-submodules are usually called fractionary $\mathcal{O}_V$-ideals. For any non-zero fractionary $\mathcal{O}_V$-ideal $\mathcal{J} = \{J_R\}_{R\in V}$ we define $\mathcal{J}^{-1}$ to be the family $\{R:J_R\}_{R\in V}$ where $R:J_R = \{x \in K | x\, J_R \subseteq R\}$. (If $\mathcal{J}$ happens to be invertible there is no conflict with earlier notation, i.e. $J_R(R:J_R) = R$ for all $R \in V$). One checks that $\mathcal{J}^{-1}$ is a complete fractionary $\mathcal{O}_V$-ideal; a fractionary $\mathcal{O}_V$-ideal of the form $\mathcal{J}^{-1}$ is called divisorial. Every invertible fractionary $\mathcal{O}_V$-ideal is divisorial.

A divisorial cycle on V is a formal sum $\sum n_T\, T$, where T runs through all discrete valuation rings which are members of V, and for each T, n_T is a rational integer, with $n_T = 0$ for all but a finite number of T. For each non-zero fractionary $\mathcal{O}_V$-ideal $\mathcal{I} = \{I_R\}_{R\in V}$ we define the divisorial cycle

$$\mathrm{cyc}_V(\mathcal{I}) = \sum - v_T(I_T).T$$

(where v_T is the normalized discrete valuation associated with T, and $v_T(I_T) = \min\{v_T(x) | x \in I_T\}$). For each divisorial cycle C, there is then precisely one divisorial ideal $\mathcal{O}_V(C)$ such that $\mathrm{cyc}_V(\mathcal{O}_V(C)) = C$ (cf. [2; ch. VII, §1]). For example if $C = \mathrm{cyc}_V(\mathcal{J})$, then $\mathcal{O}_V(-C) = \mathcal{J}^{-1}$ and $\mathcal{O}_V(C) = (\mathcal{J}^{-1})^{-1}$. We will find it convenient to associate with C the Z-divisor

$$\mathrm{div}_Z(C) = -\mathrm{div}_Z(\mathcal{O}_V(-C)).$$

If $\mathcal{O}_V(C)$ is <u>invertible</u>, we have simply $\mathrm{div}_Z(C) = \mathrm{div}_Z(\mathcal{O}_V(C))$.

For any Z-divisor $D = \{V_i, f_i\}$ and any valuation v of K/A we define $v(D)$ to be $v(f_i)$ for any i such that V_i contains the valuation ring R_v; clearly $v(f_i)$ does not depend on the choice of i, and $v(D)$ does not depend on the choice of the family $\{V_i, f_i\}$ defining D. With D we associate a divisorial cycle on V, namely

$$\mathrm{cyc}_V(D) = \sum v_T(D).T$$

For example, if $\mathcal{J}$ is a non-zero fractionary $\mathcal{O}_V$-ideal then $\mathrm{cyc}_V(\mathrm{div}_Z(\mathcal{J})) = \mathrm{cyc}_V(\mathcal{J})$.

Now clearly

$$\mathrm{cyc}_V(D_1 + D_2) = \mathrm{cyc}_V(D_1) + \mathrm{cyc}_V(D_2)$$

i.e. "cyc_V" is a homomorphism from the group $\text{Div}_K(Z)$ into the group Δ_V of divisorial cycles on V. This homomorphism is surjective: in fact for any divisorial cycle C we have

$$\text{cyc}_V(\text{div}_Z(C)) = C. \qquad \ldots(*)$$

Thus, since Δ_V is a free abelian group, $\text{Div}_K(Z)$ <u>is the direct sum of the kernel of</u> cyc_V <u>and another subgroup isomorphic to</u> Δ_V.

We call the elements of the kernel of cyc_V <u>base divisors</u> (with respect to V).

Since the group of base divisors is a direct summand of $\text{Div}_K(Z)$, there exist <u>group-theoretic</u> right inverses for the map cyc_V; but (in spite of (*)) div_Z may not be such an inverse, since $\text{div}_Z(C + C') \neq \text{div}_Z(C) + \text{div}_Z(C')$ in general. However, the product of a divisorial fractionary ideal with an <u>invertible</u> fractionary $\mathcal{O}_V$-ideal is again divisorial, and it follows easily that if, say, $C = \text{cyc}_V(\mathcal{J})$ where $\mathcal{J}$ is <u>invertible</u> then indeed, for any C', $\text{div}_Z(C + C') = \text{div}_Z(C) + \text{div}_Z(C')$.

In case V is <u>locally factorial</u> (i.e. every local ring belonging to V is a unique factorization domain) then <u>every</u> divisorial ideal is invertible. Thus:

PROPOSITION 4.1. <u>If</u> V <u>is locally factorial</u> (and noetherian, with function field K) then $\operatorname{div}_Z : \Delta_V \to \operatorname{Div}_K(Z)$ <u>is a group homomorphism such that</u> $\operatorname{cyc}_V \circ \operatorname{div}_Z = \text{identity}$.

Even if V is not locally factorial, (*) says that div_Z is a <u>set-theoretic</u> right inverse of cyc_V, so we can still associate to each pair (C, H) (C a divisorial cycle on V, H a base divisor w.r.t. V) the divisor

$$C_H = \operatorname{div}_Z(C) + H$$

and obtain a one-one correspondence between such pairs and Z-divisors. [The Z-divisor D corresponds to the pair $(\operatorname{cyc}_V(D), D-\operatorname{div}_Z(\operatorname{cyc}_V(D)))$]. When V is locally factorial (but not necessarily otherwise!) $C_H + C'_{H'} = (C + C')_{H+H'}$ (for all C, C', H, H').

PROPOSITION 4.2. <u>For any non-zero fractionary</u> $\mathcal{O}_V$-<u>ideal</u> $\mathcal{I}$, <u>we have</u> $\operatorname{div}_Z(\mathcal{I}) = C_H$, <u>where</u>

$$C = \operatorname{cyc}_V(\mathcal{I}), \qquad H = \operatorname{div}_Z(\mathcal{I}\mathcal{I}^{-1}).$$

(For, $\operatorname{div}_Z(\mathcal{I}\mathcal{I}^{-1})$ is clearly a base divisor, and by definition

$$\operatorname{div}_Z(C) + H = -\operatorname{div}_Z(\mathcal{I}^{-1}) + \operatorname{div}_Z(\mathcal{I}) + \operatorname{div}_Z(\mathcal{I}^{-1}) = \operatorname{div}_Z(\mathcal{I})).$$

We shall say that a base divisor H is effective (relative to V) if it is of the form $\mathrm{div}_Z(\mathcal{I})$ for some fractionary $\mathcal{O}_V$-ideal $\mathcal{I}$; for such an $\mathcal{I}$, 4.2 shows that $\mathrm{cyc}_V(\mathcal{I}) = 0$, i.e. $\mathcal{I}^{-1} = \mathcal{O}_V$ (so that $\mathcal{I} = \mathcal{I}\mathcal{I}^{-1} \subseteq \mathcal{O}_V$).

PROPOSITION 4.3. With notation as in 2.2, the complete fractionary ideals $\mathcal{J}$ such that $\mathcal{J}^{-1} = \mathcal{O}_V$ form a submonoid of $\underset{\sim}{C}_V$. The injective map defined in 2.2, namely $\mathcal{J} \longrightarrow \mathrm{div}_Z(\mathcal{J})$, takes this submonoid onto the set of all effective base divisors, and every base divisor is a difference of two effective base divisors.

Proof. In view of 1.5 and the preceding remarks, the first two assertions are straightforward. Because of 2.2 and 4.2 any base divisor B is of the form

$$B = \mathrm{div}_Z(C_1) + H_1 - \mathrm{div}_Z(C_2) - H_2$$

where C_1 and C_2 are divisorial cycles, and H_1 and H_2 are effective base divisors. Since $\mathrm{cyc}_V(B) = 0$, (*) shows that $C_1 = C_2$. q.e.d.

Effective base divisors play a special role in the theory of linear systems for reasons given below (Proposition 4.4 and subsequent remark). First recall that two Z-divisors C_H, $C'_{H'}$ are relatively linearly equivalent if and only if

$$\mathrm{div}_Z(C) + H = \mathrm{div}_Z(C') + H' + \mathrm{div}_Z(\mathcal{I})$$

($\mathcal{J}$ an invertible $\mathcal{V}(A)$-submodule), i.e.

$$C = C' + \mathrm{cyc}_V(\mathcal{J}\mathcal{O}_V); \quad H = H'.$$

Hence it makes sense to speak of <u>the</u> base divisor (w.r.t. V) of a linear system of Z-divisors.

PROPOSITION 4.4. <u>The base divisor of any</u> reduced <u>linear system is effective. Conversely, if</u> V <u>is a</u> projective <u>model, then any effective base divisor</u> G <u>is the base divisor of some reduced linear system.</u>

<u>Proof.</u> A typical member of a reduced linear system is of the form $\mathrm{div}_Z(\mathfrak{m})$, $\mathfrak{m}$ a finite $\mathcal{V}(A)$-submodule; but by 4.2,

$$\mathrm{div}_Z(\mathfrak{m}) = \mathrm{div}_Z(\mathfrak{m}\mathcal{O}_V) = C_H$$

with an <u>effective</u> H. Conversely, if G is effective then for some fractionary $\mathcal{O}_V$-ideal $\mathcal{J}$

$$G = \mathrm{div}_Z(\mathcal{J}\mathcal{J}^{-1}) = \mathrm{div}_Z(\mathcal{J}(n)\mathcal{J}(n)^{-1});$$

and for large n, $\mathcal{J}(n) = \mathfrak{m}\,\mathcal{O}_V$ for suitable $\mathfrak{m}$ (Proposition 1.3), so that G is the base of $|\mathrm{div}_Z(\mathfrak{m})|$.

It can be shown that for any base divisor H there is an effective base divisor $H^e \leq H$ such that for any effective base divisor $G \leq H$ we have also $G \leq H^e$. Moreover, for any divisorial cycle C on V, we have $C_H \geq 0$ if and only if $C_{H^e} \geq 0$. When V is projective over A, this last property also characterizes the effective divisor H^e.

By definition $C_H \geq 0$ if and only if $v(\mathrm{div}_Z(C)) \geq -v(H)$ for all valuations v of K/A. In classical terminology, this relation would be expressed as: "the divisorial cycle C satisfies the base conditions imposed by H."

We conclude this section by relating the linear systems of §3 to "linear systems of divisorial cycles, with base conditions". Linear systems of divisorial cycles are defined and discussed in a manner entirely similar to that of §3. Expressions like "$\mathrm{div}_Z(\mathscr{I})$", "$\mathrm{div}_Z(\mathscr{J})$", "$\mathrm{div}_Z(\mathfrak{m})$" are replaced everywhere by "$\mathrm{cyc}_V(\mathscr{I}\mathcal{O}_V)$", "$\mathrm{cyc}_V(\mathscr{J}\mathcal{O}_V)$", "$\mathrm{cyc}_V(\mathfrak{m}\mathcal{O}_V)$" (respectively). The order relation for divisorial cycles is:

$$\Sigma\ m_T\, T \geq \Sigma\ n_T\, T \Leftrightarrow m_T \geq n_T \quad \text{for all} \quad T.$$

The only other remark is that for a divisorial cycle C, the $\mathscr{V}(A)$-submodule $\mathfrak{m}_C = \Gamma(V, \mathcal{O}_V(C))$ need not be complete.

PROPOSITION 4.5. <u>Let</u> $|C|_H$ <u>be the set of all divisorial cycles</u> C' <u>such that</u>: $C' \equiv C$ <u>and</u> C' <u>satisfies the base conditions imposed by</u> H. <u>Then</u> $|C|_H$ <u>is a real linear system, and the complete linear system</u> $|C_H|$ <u>consists of all</u> Z-<u>divisors</u> C'_H <u>with</u> $C' \in |C|_H$.

<u>Proof.</u> The remarks preceding 4.4 show that $C'_{H'} \equiv C_H$ if and only if $C' \equiv C$ and $H' = H$; this shows that the members of $|C_H|$ are as asserted. We see then that $|C|_H$ is a linear system:

$$|C|_H = \Lambda(C, \mathfrak{m}_{C_H}).$$

Finally, the members of $|C|_H$ are <u>positive</u> divisorial cycles because $v(H) = 0$ for every v whose valuation ring is a member of V.

5. <u>Virtual multiplicity and infinitely near local rings.</u> Let A, K be as in §3 and let $R \supseteq A$ be a <u>regular</u> local ring of dimension ≥ 2, with maximal ideal m and field of fractions K. Let ord_R be the <u>order valuation</u> determined by R, i.e. the discrete valuation such that for $0 \neq x \in R$, $\text{ord}_R(x) = \max\{n | x \in m^n\}$. Let $Z_R \subseteq Z$ be the Zariski-Riemann space of K/R; topologically, Z_R is a subspace of Z. Any Z-divisor D defines, by restriction, a Z_R-divisor D_R; the operation of restriction is a group homomorphism, and two Z-divisors which are linearly equivalent relative to A

restrict to Z_R-divisors which are equivalent relative to R. As in the preceding section, we may associate to each Z_R-divisor its base divisor w.r.t $\mathscr{V}(R)$ (for this it is not essential that R satisfy condition (F) of §1). We define the <u>virtual multiplicity at</u> R of a Z-divisor D, $\nu_R(D)$ is symbol, by

$$\nu_R(D) = -\text{ord}_R \text{ (base divisor w.r.t. } \mathscr{V}(R) \text{ of } D_R).$$

Since regular local rings are factorial, we find, from the above discussion and the remarks preceding 4.2 and 4.4:

PROPOSITION 5.1. <u>Let</u> D, D' <u>be two</u> Z-<u>divisors</u>.

(i) If $D \equiv D'$ (relative to A) then $\nu_R(D) = \nu_R(D')$.

(ii) $\nu_R(D + D') = \nu_R(D) + \nu_R(D')$.

The above definition is closely related to the notion of "weak transform" of an ideal, namely: let V be a proper model over A such that R dominates some $T \in V$, and let $D = \text{div}_Z(\mathscr{I})$, $\mathscr{I} = \{I_S\}_{S\in V}$ being a non-zero finite type V-submodule; let $I_T R = xI_R$, where $x \in K$, and I_R is an ideal in R whose elements have no non-unit common divisor; if $\mathscr{I}_R$ is the $\mathscr{V}(R)$-submodule generated by I_R, then one sees that <u>the base of</u> D_R <u>w.r.t.</u> $\mathscr{V}(R)$ is $\text{div}_Z(\mathscr{I}_R)$.

Next, we recall that a quadratic transform of R is a local ring of the form $R[mx^{-1}]_p$, where $x \in m$, $x \notin m^2$, and p is a prime ideal in the ring $R[mx^{-1}]$ such that $m \subseteq p$. We say that a local ring S is <u>infinitely near</u> to V if $\dim S > 1$ and if there exists a sequence

$$R = R_0 < R_1 < \ldots < R_n = S$$

such that, for each $i = 0, 1, \ldots, n-1$, R_{i+1} is a quadratic transform of R_i. Such an S is necessarily regular, and the residue field of S has transcendence degree $\dim. R - \dim. S$ over that of R (cf. [1; Lemma 10, p. 334]). There is a unique one-dimensional quadratic transform of S, namely the valuation ring of ord_S.

We say that a valuation of K/R is a <u>prime divisor with respect to</u> R if v dominates R and the residue field of v has transcendence degree $\dim.R - 1$ over that of R. Such a v is necessarily discrete, of rank one [1; Th. 1, p. 330]. If S is infinitely near to R then ord_S is a prime divisor w.r.t. R. Conversely, any prime divisor v is of the form ord_S, where S is found as follows: let $R_0 = R$, and having defined R_i for some $i \geq 0$, with $\dim. R_i \geq 2$, let R_{i+1} be the unique quadratic transform of R_i dominated by v; then this process stops after a finite number of steps, i.e. for some integer N, R_{N+1} will be the valuation ring of v (cf. [1; Prop. 3, p. 336]); finally, let $S = R_N$. We obtain in this way a one-one correspondence between infinitely near local rings and prime divisors.

LEMMA 5.2. *Let* D_1, D_2 *be two* Z_R*-divisors. If* $v(D_1) = v(D_2)$ *for all prime divisors* v (w.r.t. R) *then* $D_1 = D_2$.

Proof. We may assume $D_2 = 0$. D_1 is the inverse image on Z of a divisor D_W on some proper model W over R (Proposition 2.1). Every valuation v which dominates a one-dimensional member of W is a prime divisor (theorem of Krull-Akizuki [2; ch. VII, §2, no. 5] and because regular local rings satisfy the "dimension formula", cf. [6; p. 326]). Now if $v(D_1) = v(D_W) = 0$ for all such v, then by Krull's principal ideal theorem $D_W = 0$, i.e. $D_1 = 0$.

PROPOSITION 5.3. *Let* H *be a base divisor on* Z_R (w.r.t. R), *and let* S *be infinitely near to* R, *with "quadratic sequence"*

$$R = R_0 < R_1 < \ldots < R_t = S.$$

Let m_i *be the maximal ideal of* R_i $(0 \leq i \leq t)$ *and let* ν_i *be the virtual multiplicity of* H *at* R_i. *Then*

$$\mathrm{ord}_S(H) = -\sum_{i=0}^{t} \nu_i \, \mathrm{ord}_S(m_i).$$

Using 5.2 we get:

COROLLARY 5.4. *If* H_1 *and* H_2 *are two base divisors on* Z_R, *then* $H_1 = H_2$ *if and only if* H_1 *and* H_2 *have the same virtual multiplicity at every local ring infinitely near to* R.

Proof of 5.3. Because of 4.3, we may assume that $H = \mathrm{div}_{Z_R}(\mathscr{I})$, where $\mathscr{I}$ is generated over $\mathscr{V}(R)$ by an ideal I_0 in R such that I_0 is not contained in any height one prime of R. For $0 \leq i \leq t$ let $I_i = x_i^{-1} I_0 R_i$, where x_i is the greatest common divisor in R_i of the elements of $I_0 R_i$. We see then that $\nu_i = \mathrm{ord}_{R_i}(I_i)$, and that

$$I_0 R_t = y_0^{\nu_0} y_1^{\nu_1} \cdots y_{t-1}^{\nu_{t-1}} I_t$$

where each y_i is such that $m_i R_{i+1} = y_i R_{i+1}$ $(0 \leq i < t)$. The conclusion follows.

We conclude with some remarks about the effective multiplicity $\nu_R^e(D)$ at R of any Z-divisor D such that $|D|$ is non-empty. By definition $\nu_R^e(D)$ is the virtual multiplicity of any member of the reduced linear system associated with $|D|$ (cf. §3). The additivity property (ii) of Proposition 5.1 need not hold for effective multiplicities. The following remark (which can be made precise by the introduction of sufficiently many transcendentals, a la Kronecker), gives a "geometric" interpretation of effective multiplicity:

Let V be as in §4, and such that R dominates some member of V. Let $|C_H|$ be a reduced linear system of Z-divisors. Then the virtual multiplicity at R of any member of $|C_H|$ is equal to the multiplicity at R of the "proper transform" of the "generic" member of $|C|_H$ (cf. Proposition 4.5.)

6. Base divisors on non-singular surfaces.

Suppose now that the model V of §4 is a non-singular surface, i.e. that the members of V are regular local rings of dimension ≤ 2, with equality for at least one member. In this case, if $\mathcal{J} = \{I_R\}$ is a fractionary $\mathcal{O}_V$-ideal with $\mathcal{J}^{-1} = \mathcal{O}_V$, then $I_R = R$ for almost all (i.e. all but a finite number of) $R \in V$. Hence, and in view of 4.3, the study of base divisors on V is reduced to that of complete ideals in two-dimensional regular local rings. The theory of such ideals is due to Zariski [5; and 6, Appendix 5].

From now on R will be a two-dimensional regular local ring which is a member of V. Then any two-dimensional regular local ring S with $R \subseteq S \subseteq K$ is infinitely near to R [1; Th. 3, p. 343]. For any ideal J in S such that $S:J = S$ we can define a Z-divisor $D = \mathrm{div}_Z(J)$ by the condition that $\mathcal{O}_Z(D) = \{J_Q\}_{Q\in Z}$ where $J_Q = JQ$ if Q dominates S and $J_Q = Q$ otherwise. If I is an ideal in R with $R:I = R$, then the transform I_S of I in S is defined by

$$IS = xI_S$$

where x is the greatest common divisor in S of the elements of IS. As indicated in the remarks following 5.1, the integer $\mathrm{ord}_S(I_S)$ is then the virtual multiplicity at S of the divisor $\mathrm{div}_Z(I)$.

It can be seen that $I_S = S$ (i.e. IS is a principal ideal in S) for almost all S. Consequently, every base divisor w.r.t. V has virtual multiplicity zero at almost all S. Moreover, for any S, if m_S is the maximal ideal of S, then the base divisor $div_Z(m_S)$ is found to have virtual multiplicity one at S and zero at all other local rings infinitely near to R. With 5.1, this gives us the structure of the group of base divisors w.r.t. V, namely:

THEOREM 6.1. <u>By associating to each base divisor its virtual multiplicities at all local rings infinitely near to (some member of)</u> V, <u>we obtain an isomorphism between the group of base divisors w.r.t.</u> V <u>and the free abelian group on the set of all such infinitely near local rings.</u>

We say that an effective base divisor (w.r.t. V) is <u>simple</u> if it is not a sum of two other non-zero effective base divisors. There is a one-one correspondence $S \longleftrightarrow H_S$ between local rings infinitely near to V and simple base divisors, such that H_S <u>is the largest effective base divisor whose virtual multiplcity at</u> S <u>is</u> one <u>and at any other local ring infinitely near to</u> S <u>is zero.</u> (In view of the relation between base divisors and complete ideals, this can be deduced from the considerations of [6; p. 391]). It follows that <u>the simple base divisors form a free basis for the group of base divisors.</u>

Our final result, given without proof, provides more information about this situation. For any two <u>distinct</u> two-dimensional regular local rings $S \subseteq T$ with fraction field K we say that T <u>is proximate to</u> S if the valuation ord_S is non-negative on T. For such a pair the residue field of T is a finite algebraic extension of that of S, and we denote the degree of this field extension by $[T:S]$. If S dominates some member of V, and H is a base divisor w.r.t. V, we define an integer

$$e_S(H) = \nu_S(H) - \sum_T [T:S]\ \nu_T(H)$$

where T runs through all local rings <u>proximate to</u> S, and $\nu_S(H)$, $\nu_T(H)$ are the virtual multiplicities of H at S, T respectively.

THEOREM 6.2. *For any base divisor* H *and infinitely near* (to V) *local ring* S *let* H_S, $e_S(H)$, *be as above. Then* $e_S(H) = 0$ *for almost all* S, *and*

$$H = \sum_S e_S(H).H_S.$$

Moreover, H *is an effective base divisor if and only if* $e_S(H) \geq 0$ *for all* S (i.e. *the virtual multiplicities of* H *satisfy the* "proximity inequalities").