COHOMOLOGY WITH SUPPORTS; IDEMPOTENT PAIRS

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Abstract. This chapter sets out preliminaries for the duality theory in later chapters. An underlying idea is that *local cohomology functors are higher derived functors of colocalizations* (*a.k.a. coreflections*).

Predominantly well-known facts about cohomology with supports—often under "finitary" conditions that obtain, e.g., under noetherian hypotheses—and its local and global interactions with quasi-coherence and with colimits, are reviewed from both the topological and scheme-theoretic perspectives. Some refinements of standard results are needed to accommodate certain features involving unbounded complexes and general systems of supports.

An important attribute of such cohomology is "$\otimes$-coreflectiveness", in its avatar—ultimately in the context of closed categories—as "idempotent pair," a notion which plays an important role in the sequel.

Some basic facts about linearly topologized noetherian rings and their maps, related to cohomology with supports, and subsumed under properties of idempotent pairs, are brought forth; and similarly for the less-familiar context of formal schemes.

Contents

1. Cohomology with supports; idempotent pairs

This chapter sets out some preliminaries for the duality theory in later chapters. An underlying idea is that *local cohomology functors are higher derived functors of colocalizations (a.k.a. coreflections).*

The first three sections review, from both the topological and scheme-theoretic perspectives (connected, as in 1.1.8 and 1.2.3), rudimentary facts about cohomology with supports—often under "finitary" conditions that obtain, in particular, under noetherian hypotheses—and its local and global interactions with quasi-coherence and with colimits (see e.g., 1.2.2 and 1.2.16). A basic attribute of such cohomology is "$\otimes$-coreflectiveness" (see 1.5.13). This property is elaborated on in the context of monoidal categories, as is its avatar "idempotent pair," a notion which plays an important role in the sequel (see Sections 1.3–1.6).

For instance, if X is a locally noetherian scheme and $Z \subset X$ is closed, then $\mathbf{R}\varGamma_Z\mathcal{O}_X$ and the natural map $\iota\colon \mathbf{R}\varGamma_Z\mathcal{O}_X \to \mathcal{O}_X$ form an idempotent $\mathbf{D}(X)$-pair, i.e., $\iota \underset{=}{\otimes} \mathbf{1}$ and $\mathbf{1} \underset{=}{\otimes} \iota$ are *equal isomorphisms* from $\mathbf{R}\varGamma_Z\mathcal{O}_X \underset{=}{\otimes} \mathbf{R}\varGamma_Z\mathcal{O}_X$ to $\mathbf{R}\varGamma_Z\mathcal{O}_X$; and the corresponding $\otimes$-coreflection is given by the functor $\mathbf{R}\varGamma_Z(-) := \mathbf{R}\varGamma_Z\mathcal{O}_X \underset{=}{\otimes} (-)$ together with the map $\iota \underset{=}{\otimes} \mathbf{1}\colon \mathbf{R}\varGamma_Z(-) \to (-)$. (See 1.5.7.)

The idempotent pairs in a monoidal category $\mathbf{D}$ are the objects of a strictly full monoidal subcategory $\mathbf{I}_{\mathbf{D}}$ of the slice category $\mathbf{D}/\mathcal{O}$ ($\mathcal{O} :=$ unit object of $\mathbf{D}$); $\mathbf{I}_{\mathbf{D}}$ is preordered, and the functor induced by the canonical functor $\mathbf{D}/\mathcal{O} \to \mathbf{D}$ is final in the category of all strong monoidal functors from preordered monoidal categories to $\mathbf{D}$ (see Remark 1.6.3).

Some basic facts about linearly topologized noetherian rings and their maps, related to cohomology with supports, are subsumed under properties of idempotent pairs (Sections 1.7 and 1.8); and similarly for formal schemes (Section 1.9). In the latter case, if $\mathbf{D}_{\mathrm{qct}}$ is the full subcategory of the derived category spanned by complexes with quasi-coherent torsion homology, then sending an idempotent pair in $\mathbf{D}_{\mathrm{qct}}$ to its support gives an equivalence of $\mathbf{I}_{\mathbf{D}_{\mathrm{qct}}}$ (modulo isomorphism) with the category of inclusion maps of specialization-stable subsets (see 1.9.21).

This material is predominantly well-known (cf. e.g., [GR2, Exposés I, II]), [Hg], [AJS2]; but some refinements of the standard results are needed to accommodate certain features involving unbounded complexes and general systems of supports. It is recommended to skim through these preliminaries, referring back as needed in the subsequent duality theory.

1.0. **Terminology and notation.** Let $\mathcal{A}$ be an abelian category.

An $\mathcal{A}$-*complex* $C = (C^\bullet, d^\bullet)$ is a sequence of $\mathcal{A}$-maps

$$\cdots \xrightarrow{d^{i-2}} C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots \qquad (i \in \mathbb{Z})$$

such that $d^i d^{i-1} = 0$ for all i. Homotopy equivalence of maps of $\mathcal{A}$-complexes is defined as usual [Hrt, p. 25]. The *i-th cohomology* $\mathrm{H}^i C := \ker(d^i)/\mathrm{im}(d^{i-1})$[1] is the object part of a natural $\mathcal{A}$-valued functor on the category $\mathbf{C}(\mathcal{A})$ of $\mathcal{A}$-complexes, or on the homotopy category $\mathbf{K}(\mathcal{A})$ whose objects are $\mathcal{A}$-complexes and whose morphisms are homotopy-equivalence classes of maps of $\mathcal{A}$-complexes, or on the derived category $\mathbf{D}(\mathcal{A})$ of $\mathbf{K}(\mathcal{A})$. (See e.g., [Lp1, §§1.1, 1.2].)

[1] Implicit here and elsewhere is the assumption that a *specific choice* has been made in $\mathcal{A}$ of the kernel and cokernel of each $\mathcal{A}$-map, of a 0-object, of a direct sum for any two objects, . . .

A *quasi-isomorphism* in $\mathbf{C}(\mathcal{A})$ (resp. $\mathbf{K}(\mathcal{A})$) is a map of $\mathcal{A}$-complexes $\phi\colon C \to C'$ which induces isomorphisms $\mathrm{H}^i C \xrightarrow{\sim} \mathrm{H}^i C'$ for all i (resp. the homotopy equivalence class $\bar\phi$ of such a map); or equivalently, with $q_{\mathcal{A}}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ the canonical functor, such that $q_{\mathcal{A}}\bar\phi$ is an isomorphism. When the context dictates what is meant, there will usually be no notational distinction among a map in $\mathbf{C}(\mathcal{A})$, its homotopy class in $\mathbf{K}(\mathcal{A})$, and the image of that class under $q_{\mathcal{A}}$.

With reference to maps or diagrams in $\mathcal{A}$, $\mathbf{K}(\mathcal{A})$ or $\mathbf{D}(\mathcal{A})$, "natural" means that unless otherwise specified, the maps involved are the obvious ones.

Any functor Γ between triangulated categories (such as $\mathbf{K}(\mathcal{A})$ or $\mathbf{D}(\mathcal{A})$) is understood to be *additive* and *triangle-preserving* (a Δ-functor, for short), i.e., equipped with a functorial isomorphism $\theta(E)\colon \Gamma(E[1]) \xrightarrow{\sim} (\Gamma E)[1]$ such that for any triangle $E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1]$, the sequence $\Gamma E \xrightarrow{\Gamma u} \Gamma F \xrightarrow{\Gamma v} \Gamma G \xrightarrow{\theta\circ\Gamma w} (\Gamma E)[1]$ is also a triangle. In each instance the natural definition of θ is left to the reader. (Sometimes there are sign considerations, see, e.g., [Lp1, §1.5] for more details, and for examples involving $\otimes$ and Hom.) By definition, maps of Δ-functors commute with the associated θs.

A *plump* subcategory (or *weak Serre subcategory*) $\mathcal{A}^\# \subset \mathcal{A}$ is a full subcategory containing 0 and such that for any exact $\mathcal{A}$-sequence $M_1 \to M_2 \to M \to M_3 \to M_4$, if $M_i \in \mathcal{A}^\#$ for $i = 1, 2, 3, 4$, then $M \in \mathcal{A}^\#$. The kernel and cokernel (in $\mathcal{A}$) of a map in such an $\mathcal{A}^\#$ both lie in $\mathcal{A}^\#$; so $\mathcal{A}^\#$ is abelian, and any object of $\mathcal{A}$ isomorphic to one in $\mathcal{A}^\#$ is itself in $\mathcal{A}^\#$.

An $\mathcal{A}$-complex I is *K-injective* (*q-injective* in the terminology of [Lp1], with "q" connoting "quasi-isomorphism") if any quasi-isomorphism $\psi\colon I \to I'$ has a left homotopy-inverse, that is, there exists an $\mathcal{A}$-homomorphism $\psi'\colon I' \to I$ such that $\psi'\psi$ is homotopic to the identity map of C. Numerous equivalent conditions can be found in [Spn, p. 129, Prop. 1.5] and in [Lp1, §2.3]. One such is that the functor $\mathrm{Hom}^\bullet(-, I)\colon \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$ preserves quasi-isomorphism. Another is that for every $\mathcal{A}$-complex F, the natural map is an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(F, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(A)}(q_{\mathcal{A}}F, q_{\mathcal{A}}I).$$

Any bounded-below injective (in every degree) complex is K-injective.

A *K-injective resolution* of $E \in \mathcal{A}$ is a quasi-isomorphism $\sigma_E\colon E \to I_E$ where I_E is K-injective and also injective.

All rings will be commutative.

For a commutative ring S, $\mathcal{A}(S)$ is the (abelian) category of small S-modules. We write $q_S\colon \mathbf{K}(S) \to \mathbf{D}(S)$ for $q_{\mathcal{A}(S)}\colon \mathbf{K}(\mathcal{A}(S)) \to \mathbf{D}(\mathcal{A}(S))$. Each S-complex E admits a K-injective resolution $\sigma_E\colon E \to I_E$, see [Spn, p. 133, Prop. 3.11].

Similar considerations hold for any ringed space $(X, \mathcal{O}_X)$ (X a topological space and $\mathcal{O}_X$ a sheaf of commutative rings on X), with $\mathcal{A}(X)$ the category of $\mathcal{O}_X$-modules, with $q_X\colon \mathbf{K}(X) \to \mathbf{D}(X)$ signifying $q_{\mathcal{A}(X)}\colon \mathbf{K}(\mathcal{A}(X)) \to \mathbf{D}(\mathcal{A}(X))$, etc. (See [Spn, p. 138, Thm. 4.5]).[2] $\mathbf{D}^+(X) \subset \mathbf{D}(X)$ is the full subcategory spanned by the locally cohomologically bounded-below $\mathcal{O}_X$-complexes (those $C \in \mathbf{D}(X)$ for which there is an open cover $(X_\alpha)_{\alpha\in A}$ of X and for each α an integer n_α such that the restriction $(H^iC)|_{X_\alpha}$ vanishes for all $i < n_\alpha$). For such $(X, \mathcal{O}_X)$, restriction to open subsets preserves K-injectivity of $\mathcal{O}_X$-complexes [Lp1, Lemma 2.4.5.2].

[2] Such assertions hold in any Grothendieck category, see [AJS1, p. 243, Thm. 5.4], or [Lu, Propositions 1.3.5.3 and 1.3.5.6], noting that K-injective $\Leftrightarrow$ homotopically equivalent to fibrant.

For example, any topological space X can be regarded as a ringed space, with $\mathcal{O}_X$ the sheaf $\mathbb{Z}_X$ of locally constant functions from X to $\mathbb{Z}$; and then $\mathcal{A}(X)$ is just the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups.

When $(X, \mathcal{O}_X)$ is a scheme, $\mathcal{A}_{\mathrm{qc}}(X) \subset \mathcal{A}(X)$ is the full subcategory of quasi-coherent $\mathcal{O}_X$-modules, and $\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}(X)$ is the full subcategory whose objects are the complexes with quasi-coherent homology.

1.1. **Finitary supports.**

1.1.1. A *system of supports* (s.o.s.) (a.k.a. *family of supports*) in a topological space X is a nonempty set Φ of closed subsets of X such that any closed subset of any finite union of members of Φ is a member of Φ.

For instance, if $Y \subset X$ then the set Φ_Y consisting of all subsets of Y that are closed in X is an s.o.s. An s.o.s. has this form if and only if it contains every X-closed subset of the union of all its members. In fact, there is a one-one correspondence between such s.o.s. and *specialization-stable* $Y \subset X$ (i.e., Y contains the X-closure of each of its points, or equivalently, Y is a union of closed subsets of X): to such an s.o.s. Φ corresponds the union of its members, and to a specialization-stable $Y \subset X$ corresponds Φ_Y.

If X is *noetherian,* i.e., every open subset is quasi-compact [Brb, II, §4.2],[3] then every closed subset of X is a finite union of irreducible closed subsets; and if, furthermore, every irreducible closed subset of X is the closure of one of its points (for instance, if X is the underlying space of a noetherian scheme), then every s.o.s. in X is Φ_Y for a unique specialization-stable Y.

An s.o.s. Φ in X is *finitary* if each member of Φ is contained in a member Z such that $X \setminus Z$ is *retrocompact* in X, i.e., for every quasi-compact open $U \subset X$, the open subset $U \setminus Z$ is quasi-compact.

For example, the s.o.s. Φ_X consisting of all closed subsets of X is finitary.

One checks that *every s.o.s. in X is finitary $\Leftrightarrow$ every quasi-compact open subset of X is noetherian $\Leftrightarrow$ every open subset of X is retrocompact in X.* (To see this, consider the s.o.s. Φ_Y for an arbitrary closed $Y \subset X \dots$).

These conditions on X have no substance if the only quasi-compact open subset of X is the empty one. More noteworthy is the situation where X is a union of quasi-compact open subsets (for instance, the underlying space of a scheme): then the conditions hold if and only if X is *locally noetherian,* i.e., every point of X has a noetherian neighborhood.

Lemma 1.1.2. *Let X be a quasi-compact topological space and let Φ be a finitary s.o.s. in X. If $(Z_\delta)_{\delta\in\mathcal{D}}$ is a family of closed subsets of X such that $\bigcap_{\delta\in\mathcal{D}} Z_\delta \in \Phi$ then there is a finite subset $\mathcal{D}_0 \subset \mathcal{D}$ such that $\bigcap_{\delta\in\mathcal{D}_0} Z_\delta \in \Phi$.*

Proof. Fix $Z \supset \bigcap_{\delta\in\mathcal{D}} Z_\delta$ such that $Z \in \Phi$ and $X \setminus Z$ is retrocompact in X, hence quasi-compact. The family $(Z_\delta \setminus Z)_{\delta\in\mathcal{D}}$ of closed subsets of $X \setminus Z$ has empty intersection, whence there is a finite $\mathcal{D}_0 \subset \mathcal{D}$ such that $\bigcap_{\delta\in\mathcal{D}_0}(Z_\delta \setminus Z)$ is empty, i.e., $\bigcap_{\delta\in\mathcal{D}_0} Z_\delta \subset Z$, so that $\bigcap_{\delta\in\mathcal{D}_0} Z_\delta \in \Phi$. □

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[3] A (possibly non-Hausdorff) topological space is *quasi-compact* if every open cover has a finite subcover.

1.1.3. (Inverse image of an s.o.s.) Let $f\colon W \to X$ be a continuous map of topological spaces, and Φ an s.o.s. in X. Set

$$\begin{aligned}\Phi_f &:= \{\, V \text{ closed in } W \mid \text{the closure of } f(V) \text{ belongs to } \Phi \,\}\\ &= \{\, V \text{ closed in } W \mid V \subset f^{-1}Z \text{ for some } Z \in \Phi \,\}\\ &= \bigcup_{Z\in\Phi} \Phi_{f^{-1}Z}\,,\end{aligned} \qquad (1.1.3.1)$$

the smallest s.o.s. in W that contains $f^{-1}Z$ for all $Z \in \Phi$.

For example, if $Y \subset X$ then $(\Phi_Y)_f \subset \Phi_{f^{-1}Y}$, with equality if Y is closed or if Y is specialization-stable, W is noetherian and every irreducible closed subset of W is the closure of one of its points.

For another example, if f is the inclusion map of a subspace $W \subset X$ then

$$\Phi_f = \Phi|_W := \{\, Z \cap W \mid Z \in \Phi \,\}. \qquad (1.1.3.2)$$

Moreover, every s.o.s. Φ_0 in W has the form Φ_f: let Φ consist of all closed subsets of X whose intersection with W is in Φ_0.

The pairs (X, Φ) with X a topological space and Φ an s.o.s. in X are the objects of a category in which a morphism $(W, \Psi) \to (X, \Phi)$ is a continuous map $f\colon W \to X$ such that $\Psi \subset \Phi_f$. Such a morphism is called *strict* if $\Psi = \Phi_f$.

Remark 1.1.4. Let $W \subset X$ be open. If Φ is finitary then so is $\Phi|_W$.

Remark 1.1.5. Suppose that Z is *locally* in Φ, i.e., $Z \subset \cup_{\alpha\in A} U_\alpha$ with each U_α an open subset of X, such that $Z \cap U_\alpha \in \Phi|_{U_\alpha}$ (i.e., $\overline{Z \cap U_\alpha} \in \Phi$). If A is finite, or if $\Phi = \Phi_Y$ $(Y \subset X)$, then $Z \in \Phi$.

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1.1.6. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-*base* is a nonempty set $\mathcal{I}$ of *quasi-coherent* $\mathcal{O}_X$-ideals such that:

(i) if $I \in \mathcal{I}$ and if J is a quasi-coherent $\mathcal{O}_X$-ideal such that $\sqrt{J} \supset I$, then $J \in \mathcal{I}$, and

(ii) if $I \in \mathcal{I}$ and $J \in \mathcal{I}$ then $I \cap J \in \mathcal{I}$.

Since $\sqrt{IJ} \supset (I \cap J) \supset IJ$, therefore if (i) holds then (ii) is equivalent to:

(ii)$'$ if $I \in \mathcal{I}$ and $J \in \mathcal{I}$ then $IJ \in \mathcal{I}$.

For example, if $\mathbf{I}$ is a nonempty set of $\mathcal{O}_X$-ideals, and $f\colon W \to X$ is a map of schemes, then the smallest $\mathcal{O}_W$-base containing $I\mathcal{O}_W$ for all $I \in \mathbf{I}$ is

$$\begin{aligned}\mathbf{I}_f := \{\, &\text{quasi-coherent } \mathcal{O}_W\text{-ideals } J \mid\\ &\sqrt{J} \supset I_1 \cdots I_n \mathcal{O}_W \text{ for some integer } n \ge 0 \text{ and } I_1, \dots, I_n \in \mathbf{I} \,\}.\end{aligned} \qquad (1.1.6.1)$$

When f is the inclusion map of a subspace $W \subset X$, $\mathbf{I}_f$ is denoted $\mathbf{I}|_W$.

If $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{O}_X$-bases, then so is $\mathcal{I} \cap \mathcal{J} = \{\, I + J \mid I \in \mathcal{I},\ J \in \mathcal{J} \,\}$.

An $\mathcal{O}_X$-base $\mathcal{I}$ is *finitary* if X is covered by open subsets U such that each member of $\mathcal{I}|_U$ contains a *finite-type* member of $\mathcal{I}|_U$.

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1.1.7. Again, let $(X, \mathcal{O}_X)$ be a scheme.

The *support of an $\mathcal{O}_X$-module M* is

$$\operatorname{Supp}(M) := \{\, x \in X \mid M_x \ne (0)\,\}. \tag{1.1.7.1}$$

For example, let $u\colon U \hookrightarrow X$ be the inclusion map of an open subscheme, and let N be an $\mathcal{O}_U$-module, with support (in U) $\operatorname{Supp}_U(N)$. Any point $x \in X$ lying outside the X-closure $\overline{\operatorname{Supp}_U(N)}$ has a neighborhood in which u_*N vanishes, and so $\operatorname{Supp}(u_*N) \subset \overline{\operatorname{Supp}_U(N)}$.

For any $s \in \Gamma(X, M)$, the *support of s* is the closed set

$$\operatorname{supp}(s) = \operatorname{supp}_X(s) := \{\, x \in X \mid s_x \ne 0\,\} = \operatorname{Supp}(s\mathcal{O}_X) \subset \operatorname{Supp}(M). \tag{1.1.7.2}$$

If M is of finite type then $\operatorname{Supp}(M)$ is locally the union of the supports of members of a finite generating set, and so $\operatorname{Supp}(M)$ is a closed subset of X.

For an $\mathcal{O}_X$-ideal I, the *zero-set of I* is the closed set

$$Z(I) := \operatorname{Supp}(\mathcal{O}_X/I).$$

Every closed subset of X is $Z(I)$ for some quasi-coherent $\mathcal{O}_X$-ideal I.

Clearly, $Z(I) = Z(\sqrt{I}\,)$ and $Z(I_1I_2) = Z(I_1) \cup Z(I_2)$.

If I_1 and I_2 are quasi-coherent, one checks locally that

$$Z(I_1) \supset Z(I_2) \iff \sqrt{I_1} \subset \sqrt{I_2}. \tag{1.1.7.3}$$

For $s \in \Gamma(X, M)$,

$$\operatorname{supp}(s) = Z(\operatorname{ann}(s))$$

where $\operatorname{ann}(s)$, the *annihilator of s*, is the kernel of the $\mathcal{O}_X$-homomorphism $\mathcal{O}_X \to M$ taking $1 \in \Gamma(X, \mathcal{O}_X)$ to s.

Proposition 1.1.8. *There is an inclusion-preserving bijection $\mathcal{S}$ from the set of $\mathcal{O}_X$-bases onto the set of systems of supports in a scheme X, such that for any quasi-coherent $\mathcal{O}_X$-ideal I, $\mathcal{O}_X$-base $\mathcal{I}$ and s.o.s. Φ,*

$$I \in \mathcal{I} \iff Z(I) \in \Phi_{\mathcal{I}} := \mathcal{S}(\mathcal{I}), \tag{1.1.8.1}$$

or equivalently,

$$I \in \mathcal{I}_\Phi := \mathcal{S}^{-1}(\Phi) \iff Z(I) \in \Phi. \tag{1.1.8.2}$$

Proof. Left to the reader. □

Example 1.1.9. Let $f\colon W \to X$ be a scheme-map, Φ an s.o.s. in X, Φ_f as in 1.1.3, $\mathcal{I}_\Phi$ and $\mathcal{I}_{\Phi_f}$ as in (1.1.8.2), and $(\mathcal{I}_\Phi)_f$ as in (1.1.6.1). For $I \in \mathcal{I}_\Phi$ and J a quasi-coherent $\mathcal{O}_W$-ideal, (1.1.7.3) gives

$$\{Z(J) \subset f^{-1}Z(I) = Z(I\mathcal{O}_W)\} \iff \{\sqrt{J} \supset I\mathcal{O}_W\},$$

whence $\mathcal{I}_{\Phi_f} = (\mathcal{I}_\Phi)_f$.

Corollary 1.1.10. *Let $\mathcal{I}$ be an $\mathcal{O}_X$-base, and I an $\mathcal{O}_X$-ideal locally in $\mathcal{I}$, i.e., X has an open covering $(U_\alpha)_{\alpha\in A}$ such that for each α, $I\mathcal{O}_{U_\alpha} \in \mathcal{I}|_{U_\alpha}$. If A is finite, or if $\Phi_{\mathcal{I}} = \Phi_Y$ for some $Y \subset X$ (see 1.1.8.1, 1.1.1), then $I \in \mathcal{I}$.*

Proof. For all $\alpha \in A$, the $\mathcal{O}_{U_\alpha}$-ideal $I\mathcal{O}_{U_\alpha}$ is quasi-coherent; so the $\mathcal{O}_X$-ideal I is quasi-coherent. Also, with $\Phi := \Phi_{\mathcal{J}}$, so that, by 1.1.8, $\mathcal{J} = \mathcal{J}_\Phi$,

$$I\mathcal{O}_{U_\alpha} \in \mathcal{J}|_{U_\alpha} \underset{1.1.9}{\Longrightarrow} I\mathcal{O}_{U_\alpha} \in \mathcal{J}_{(\Phi|_{U_\alpha})} \underset{1.1.8}{\Longrightarrow} Z(I\mathcal{O}_{U_\alpha}) = Z(I) \cap U_\alpha \in \Phi|_{U_\alpha}.$$

Remark 1.1.5 ensures then that $Z(I) \in \Phi$, that is, $I \in \mathcal{J}$. □

Recall that the scheme X is *quasi-separated* if the intersection of any two quasi-compact open subsets is quasi-compact, see [GD, p. 296, (6.1.12)].

Lemma 1.1.11. *Let X be a quasi-compact quasi-separated scheme, and $\mathcal{J}$ a finitary $\mathcal{O}_X$-base. Every member of $\mathcal{J}$ contains a finite-type member of $\mathcal{J}$.*

Proof. Let $I \in \mathcal{J}$. As $\mathcal{J}$ is finitary and X quasi-compact, there exists a covering (U_i) $(1 \le i \le n)$ of X by a finite family of affine open subsets, and for each i, a finite-type $J_i \in \mathcal{J}|_{U_i}$ with $J_i \subset I\mathcal{O}_{U_i}$. Let $\bar{J}_i \subset I$ be a finite-type $\mathcal{O}_X$-ideal whose restriction to U_i is J_i (see [GD, p. 318, Thm. (6.9.7)]). Then $\bar{J} := \sum_{i=1}^n \bar{J}_i \subset I$ is a quasi-coherent finite-type $\mathcal{O}_X$-ideal whose restriction to each U_i contains J_i, hence lies in $\mathcal{J}|_{U_i}$; so by 1.1.10, $\bar{J} \in \mathcal{J}$. □

Proposition 1.1.12. *Let $\mathcal{J}$ be an $\mathcal{O}_X$-base. If $\Phi_{\mathcal{J}}$ is finitary then $\mathcal{J}$ is finitary. The converse holds if X is quasi-compact and quasi-separated.*

Proof. Suppose $\Phi_{\mathcal{J}}$ finitary. For any open $U \subset X$, $\Phi|_U$ is finitary. Hence to show that $\mathcal{J}$ is finitary, one may assume that X is affine, say $X = \mathrm{Spec}(R)$. Let $I \in \mathcal{J}$. Then $Z(I) \subset Z(\bar{I})$ for some $\bar{I} \in \mathcal{J}$ such that $X \setminus Z(\bar{I})$ is quasi-compact and so covered by finitely many open subsets $X \setminus Z(f_iR)$ with $f_i \in \Gamma(X, \bar{I})$ $(i = 1, 2, \dots, n)$. Since $\bar{I} \subset \sqrt{I}$ (see (1.1.7.3)), one can, upon replacing each f_i by a suitable power, assume that every f_i is in $\Gamma(X, I)$; and then by 1.1.6(i), the ideal $(f_1, f_2, \dots, f_n)R$, whose radical contains $\Gamma(X, \bar{I})$, sheafifies to a finite-type ideal in $\mathcal{J}$ that is contained in I. Thus $\mathcal{J}$ is indeed finitary.

For the converse, suppose $\mathcal{J}$ finitary and X quasi-compact and quasi-separated. Let $Z \in \Phi_{\mathcal{J}}$, say $Z = Z(I)$ $(I \in \mathcal{J})$. Let (U_i) $(1 \le i \le n)$ and $\bar{J}$ be as in the proof of 1.1.11, so that $Z \subset Z(\bar{J}) \in \Phi_{\mathcal{J}}$. U_i being affine, $\bar{J}\mathcal{O}_{U_i}$ is generated by finitely many of its sections over U_i; so $U_i \setminus Z(\bar{J})$, being an intersection of finitely many quasi-compact open sets, is quasi-compact, whence $X \setminus Z(\bar{J}) = \bigcup_{i=1}^n \big(U_i \setminus Z(\bar{J})\big)$ is quasi-compact, hence retrocompact in X. Thus $\Phi_{\mathcal{J}}$ is finitary. □

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1.1.13. Let $(X, \mathcal{O}_X)$ be a ringed space, and $M \in \mathcal{A}(X)$. The support $\mathrm{supp}(s)$ of $s \in \Gamma(X, M)$ is closed in X (see (1.1.7.2)).

For any s.o.s. Φ in X, and open $U \subset X$, one has the $\Gamma(U, \mathcal{O}_X)$-module

$$\Gamma_\Phi(U, M) := \{\, s \in \Gamma(U, M) \mid \mathrm{supp}_U(s) \in \Phi|_U \}.$$

Let $\varGamma_\Phi$ be the *left-exact subfunctor of the identity functor on* $\mathcal{A}(X)$ such that for any $M \in \mathcal{A}(X)$, $\varGamma_\Phi(M)$ is the sheaf associated to the presheaf $U \mapsto \Gamma_\Phi(U, M)$ (U open in X), that is, the sheaf of sections of M whose support is locally in Φ. (See, apropos, Remark 1.1.5.)

Following [GR2, Exposé I, §1], for closed $Z \subset X$ we set $\Gamma_Z := \Gamma_{\Phi_Z}$ and $\varGamma_Z := \varGamma_{\Phi_Z}$ (Φ_Z consisting, as in 1.1.1, of all closed subsets of Z).

Clearly, for any s.o.s. Φ and for U, M as above,

$$\begin{aligned}\Gamma_\Phi(U,M) &= \bigcup_{Z\in\Phi}\Gamma_Z(U,M) = \varinjlim_{Z\in\Phi}\Gamma_Z(U,M),\\ \varGamma_\Phi(M) &= \bigcup_{Z\in\Phi}\varGamma_Z(M) = \varinjlim_{Z\in\Phi}\varGamma_Z(M).\end{aligned} \tag{1.1.13.1}$$

As in [GR2, Exposé I, 1.6], the functor $\Gamma_Z(U,M)$ (denoted there by $\Gamma_{Z\cap U}(M)$) is naturally isomorphic to $\mathrm{Hom}_{\mathfrak{Ab}(U)}(\mathbb{Z}_{Z\cap U,U}, M|_U)$, where $\mathbb{Z}_{Z\cap U,U}$ is the abelian sheaf on U which restricts over $Z\cap U$ to the locally constant sheaf of integers $\mathbb{Z}$ and vanishes elsewhere. Hence there is a functorial isomorphism of $\mathcal{O}_X$-modules

$$\varGamma_Z(M) \cong \mathcal{H}om_{\mathfrak{Ab}(X)}(\mathbb{Z}_{Z,X}, M).$$

The functor $\varGamma_\Phi$ is *idempotent*: $\varGamma_\Phi\varGamma_\Phi = \varGamma_\Phi$. In fact, if each of Φ and Ψ is an s.o.s. in X then so is $\Phi\cap\Psi$, and one checks that

$$\varGamma_\Phi\varGamma_\Psi = \varGamma_\Phi \cap \varGamma_\Psi = \varGamma_{\Phi\cap\Psi}. \tag{1.1.13.2}$$

And if U is an open subset of X such that every member of $\Psi|_U$ is quasi-compact (for instance, if U itself is quasi-compact), or if $\Psi = \Phi_Y$ for some $Y \subset X$, then using Remark 1.1.5 one checks that

$$\Gamma_\Phi(U, \varGamma_\Psi M) = \Gamma_{\Phi\cap\Psi}(U,M). \tag{1.1.13.3}$$

Let $f\colon W\to X$ be a continuous map of topological spaces, Φ an s.o.s. in X, and as in 1.1.3, $\Phi_f := \{\, V \text{ closed in } W \mid \text{the closure of } f(V) \text{ belongs to } \Phi\,\}$. In particular, if $Y\subset X$ is closed and $\Phi = \Phi_Y$ then $\Phi_f = \Phi_{f^{-1}Y}$.

It is straightforward to see that for any $N\in\mathcal{A}(W)$, the support of a global section of f_*N is the closure of the image under f of the support of the corresponding global section of N. It follows that

$$\Gamma_{\Phi_f}(W,N) = \Gamma_\Phi(X, f_*N) \qquad (N\in\mathcal{A}(W)). \tag{1.1.13.4}$$

If X has a base of open sets U such that $f^{-1}U$ is quasi-compact, or if $\Phi=\Phi_Y$ with $Y\subset X$ closed, then (1.1.13.3) (with (Φ, U, Ψ) replaced by $(\Phi_W, f^{-1}U, \Phi_f)$) and (1.1.13.4) (with X replaced by an arbitrary U and W by $f^{-1}U$) give

$$f_*\varGamma_{\Phi_f} = \varGamma_\Phi f_*. \tag{1.1.13.5}$$

* * * * *

1.1.14. Let X be a scheme, $U\subset X$ open, $\mathfrak{I}$ an $\mathcal{O}_X$-base, M an $\mathcal{O}_X$-module and

$$\Gamma_{\mathfrak{I}}(U,M) := \varinjlim_{I\in\mathfrak{I}} \mathrm{Hom}_{\mathcal{O}_U}\big(\mathcal{O}_U/(I|_U), M|_U\big).$$

There is a natural isomorphism

$$\Gamma_{\mathfrak{I}}(U,M) \xrightarrow{\ \sim\ } \{\, s\in\Gamma(U,M) \mid \mathrm{ann}_U(s)\supset I|_U \text{ for some } I\in\mathfrak{I}\,\}.$$

There is an obvious presheaf $U\mapsto\Gamma_{\mathfrak{I}}(U,M)$. The associated sheaf is

$$\varGamma_{\mathfrak{I}}(M) := \varinjlim_{I\in\mathfrak{I}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I, M) \subset M. \tag{1.1.14.1}$$

There results a *left-exact subfunctor* $\varGamma_{\mathfrak{I}}\colon \mathcal{A}(X)\to\mathcal{A}(X)$ *of the identity functor.*

For a quasi-coherent $\mathcal{O}_X$-ideal I, let $\mathfrak{I}_I$ be the $\mathcal{O}_X$-base consisting of all quasi-coherent $\mathcal{O}_X$-ideals whose radical contains I, i.e., the smallest $\mathcal{O}_X$-base containing I. (According to (1.1.6.1), this is $\{I\}_{\mathbf{1}_X}$.)

Set $\Gamma_I := \Gamma_{\mathcal{I}_I}$ and $\varGamma_I := \varGamma_{\mathcal{I}_I}$. Then for any $\mathcal{O}_X$-base $\mathcal{I}$,

$$\Gamma_{\mathcal{I}}(U, M) = \bigcup_{I\in\mathcal{I}} \Gamma_I(U, M) = \varinjlim_{I\in\mathcal{I}} \Gamma_I(U, M), \tag{1.1.14.2}$$

whence

$$\varGamma_{\mathcal{I}}(M) = \varinjlim_{I\in\mathcal{I}} \varGamma_I(M). \tag{1.1.14.3}$$

If the open $U \subset X$ is *quasi-compact* then for any $\mathcal{O}_X$-bases $\mathcal{I}$ and $\mathcal{J}$,

$$\Gamma_{\mathcal{I}}(U, \varGamma_{\mathcal{J}} M) = \Gamma_{\mathcal{I}\cap\mathcal{J}}(U, M). \tag{1.1.14.4}$$

Indeed, for any $s \in \Gamma_{\mathcal{I}}(U, \varGamma_{\mathcal{J}} M) \subset \Gamma(U, M)$ there is a finite open cover $U = \cup_{i=1}^n U_i$ such that for each i, the restriction $s|_{U_i}$ is annihilated by (the restriction of) some $J_i \in \mathcal{J}$; and then for some $I \in \mathcal{I}$, s is annihilated by $I + J_1 J_2 \cdots J_n \in \mathcal{I} \cap \mathcal{J}$. Thus $\Gamma_{\mathcal{I}}(U, \varGamma_{\mathcal{J}} M) \subset \Gamma_{\mathcal{I}\cap\mathcal{J}}(U, M)$; and the opposite inclusion is clear.

As X has a base of quasi-compact open sets, sheafifying shows then that

$$\varGamma_{\mathcal{I}} \varGamma_{\mathcal{J}} = \varGamma_{\mathcal{I}\cap\mathcal{J}}. \tag{1.1.14.5}$$

In particular (set $\mathcal{J} := \mathcal{I}$), the functor $\varGamma_{\mathcal{I}}$ is *idempotent*.

1.1.15. Let $f\colon (W, \mathcal{O}_W) \to (X, \mathcal{O}_X)$ be a map of schemes, and $N \in \mathcal{A}(W)$. A section $s \in \Gamma(X, f_*N) = \Gamma(W, N)$ can be regarded as the $\mathcal{O}_X$-homomorphism $\mathbf{s}\colon \mathcal{O}_X \to f_*N$ that takes $1 \in \Gamma(X, \mathcal{O}_X)$ to s, or as the natural composite $\mathcal{O}_W$-homomorphism $\bar{\mathbf{s}}\colon \mathcal{O}_W = f^*\mathcal{O}_X \xrightarrow{f^*\mathbf{s}} f^*f_*N \longrightarrow N$ (taking $1 \in \Gamma(W, \mathcal{O}_W)$ to s).

Let $\mathcal{I}$ be an $\mathcal{O}_X$-base. Complying with 1.1.6.1, set

$$\mathcal{I}_f := \{\text{quasi-coherent } \mathcal{O}_W\text{-ideals } J \mid \sqrt{J} \supset I\mathcal{O}_W \text{ for some } I \in \mathcal{I}\}.$$

If $I \subset \ker(\mathbf{s}) = \operatorname{ann}_X(s)$, then $I\mathcal{O}_W \subset \ker(\bar{\mathbf{s}}) = \operatorname{ann}_W(s)$, whence

$$\Gamma_{\mathcal{I}}(X, f_*N) \subset \Gamma_{\mathcal{I}_f}(W, N).$$

Furthermore, *if X is quasi-compact and quasi-separated and $\mathcal{I}$ is finitary* then by 1.1.11, one can assume that in the definition of $\mathcal{I}_f$, the ideal I is of finite type, so one can replace $\sqrt{J}$ by J. Thus if $s \in \Gamma_{\mathcal{I}_f}(W, N)$, then there is an $I \in \mathcal{I}$ such that $I\mathcal{O}_W \subset \ker(\bar{\mathbf{s}})$, whence the top row in the natural commutative diagram

$$\begin{array}{ccccc}
f_*f^*I & \longrightarrow & f_*\mathcal{O}_W & \xrightarrow{f_*\bar{\mathbf{s}}} & f_*N \\
\uparrow & & \uparrow & & \| \\
I & \longrightarrow & \mathcal{O}_X & \xrightarrow[\mathbf{s}]{} & f_*N
\end{array}$$

composes to 0, whence so does the bottom row, and so $s \in \Gamma_{\mathcal{I}}(X, f_*N)$. Hence

$$\Gamma_{\mathcal{I}}(X, f_*N) = \Gamma_{\mathcal{I}_f}(W, N).$$

From this plus (1.1.14.4), it follows—without X having to be quasi-compact and quasi-separated—that *if the map f is quasi-compact and $\mathcal{I}$ is finitary then*

$$\varGamma_{\mathcal{I}} f_* = f_* \varGamma_{\mathcal{I}_f}.$$

Proposition 1.1.16. *Let X be a scheme, $\mathcal{I}$ a finitary $\mathcal{O}_X$-base, and M a quasi-coherent $\mathcal{O}_X$-module. Then the $\mathcal{O}_X$-module $\varGamma_{\mathcal{I}} M$ is quasi-coherent.*

Proof. The assertion being local (see 1.1.4), X can be assumed affine, so that every member of $\mathcal{I}$ contains a finite-type member (see 1.1.11). The assertion follows then from (1.1.14.1) and [GD, p. 217, (2.2.2)]. □

Proposition 1.1.17. *Let X be a scheme, M an $\mathcal{O}_X$-module, Φ an s.o.s. in X and $\mathcal{I} := \mathcal{I}_\Phi$ (see 1.1.8). Then $\Gamma_{\mathcal{I}}(X, M) \subset \Gamma_\Phi(X, M)$ and $\varGamma_{\mathcal{I}} M \subset \varGamma_\Phi M$, with equality (in either case) if M is quasi-coherent.*[4]

Proof. Any $s \in \Gamma_{\mathcal{I}}(X, M)$ is annihilated by an $I \in \mathcal{I}$, whence $\mathrm{supp}(s) \subset Z(I) \in \Phi$, that is, $s \in \Gamma_\Phi(X, M)$. Thus $\Gamma_{\mathcal{I}}(X, M) \subset \Gamma_\Phi(X, M)$.

If, moreover, M is quasi-coherent, then so is $\mathrm{ann}(s)$ for any $s \in \Gamma(X, M)$, and $s \in \Gamma_\Phi(X, M) \iff \mathrm{supp}(s) = Z(\mathrm{ann}(s)) \in \Phi \iff \mathrm{ann}(s) \in \mathcal{I} \iff s \in \Gamma_{\mathcal{I}}(X, M)$, so that $\Gamma_{\mathcal{I}}(X, M) = \Gamma_\Phi(X, M)$.

Replacing X by an arbitrary open subset, one gets inclusion (resp. equality) for the resulting presheaves, and sheafification gives inclusion (resp. equality) for $\varGamma$. □

The next result is immediate from 1.1.16 and 1.1.17. (See also 1.2.2 below for an essentially well-known generalization.)

Corollary 1.1.18. *Let X be a scheme and Φ an s.o.s. in X. If M is a quasi-coherent $\mathcal{O}_X$-module then so is $\varGamma_\Phi M$.*

Remark. In [GS, p. 2293] there is an example in which X is the spectrum of a polynomial ring in countably many variables over a field, I is the sheafification of the ideal generated by the variables, and M is a certain quasi-coherent $\mathcal{O}_X$-module such that $\varGamma_I(M)$ is not quasi-coherent. (There, of course, $\mathcal{I}_I$ is not finitary.)

* * * * *

Proposition 1.1.19. (i) *Suppose that the topological space X has a base of quasi-compact open sets, and that the s.o.s. Φ in X is finitary. Then $\varGamma_\Phi$ commutes with small filtered colimits, hence with small direct sums.*

More exactly, if A is a small filtered category [Mc, p. 211] *and $\mathcal{M}\colon A \to \mathfrak{Ab}(X)$ is a functor, then the natural map is an isomorphism*

$$\lambda\colon \varinjlim_A (\varGamma_\Phi \circ \mathcal{M}) \xrightarrow{\ \sim\ } \varGamma_\Phi(\varinjlim_A \mathcal{M}).$$

(ii) *Let X be a scheme and $\mathcal{I}$ a finitary $\mathcal{O}_X$-base. Then $\varGamma_{\mathcal{I}}$ commutes with small filtered colimits* (*as in* (i)), *hence with small direct sums.*

Proof. (i). Since the composite map

$$\varinjlim_A (\varGamma_\Phi \circ \mathcal{M}) \xrightarrow{\lambda} \varGamma_\Phi(\varinjlim_A \mathcal{M}) \xrightarrow{\text{natural}} \varinjlim_A \mathcal{M}$$

is the natural injection, therefore λ is injective.

Surjectivity can be checked stalkwise. Fix $x \in X$. Any element of $(\varGamma_\Phi(\varinjlim \mathcal{M}))_x$ is the germ σ_x of a section σ of $\varinjlim \mathcal{M}$ over a quasi-compact open neighborhood V of x, such that σ is the natural image of a section $\sigma_a \in \Gamma(V, \mathcal{M}a)$ for some $a \in A$, and $\mathrm{supp}(\sigma) \in \Phi|_V$.

[4] For examples of inequality, with X noetherian and M injective, see the proof of 1.2.5.

For each A-morphism $\alpha\colon a \to b$, let σ_α be the image of σ_a under the induced map $\Gamma(V, \mathcal{M}\alpha) \to \Gamma(V, \mathcal{M}b)$. Then σ is the natural image of σ_α; and for all $y \in V$, $\sigma_y \neq 0 \iff (\sigma_\alpha)_y \neq 0$ for all α, i.e., $\bigcap_\alpha \operatorname{supp}(\sigma_\alpha) = \operatorname{supp}(\sigma) \in \Phi|_V$. Since V is quasi-compact and $\Phi|_V$ is finitary, and since A is filtered, Lemma 1.1.2 implies that there exists a single $\alpha\colon a \to b$ with $\operatorname{supp}(\sigma_\alpha) \in \Phi|_V$. For such an α, $(\sigma_\alpha)_x$ is an element of $(\mathit{\Gamma}_\Phi(\mathcal{M}b))_x$ whose natural image in $(\varinjlim(\mathit{\Gamma}_\Phi \circ \mathcal{M}))_x$ is taken by λ_x to σ_x. Thus λ_x is surjective for any $x \in X$, that is, λ is surjective.

The passage from filtered direct limits to direct sums is standard (cf. the last part of the proof of Proposition 1.2.15 below).

(ii) The assertion being locally verifiable, one can assume X affine. Lemma 1.1.11 shows then that every member of the finitary $\mathcal{O}_X$-base contains a finite-type one. Hence the assertion is given by the natural isomorphisms, with $\mathcal{I}_0$ consisting of all finite-type $I \in \mathcal{I}$,

$$\begin{aligned}\varinjlim_A (\mathit{\Gamma}_{\mathcal{I}} \circ \mathcal{M}) &= \varinjlim_A \varinjlim_{I\in\mathcal{I}_0} \big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I, -) \circ \mathcal{M}\big)\\ &\xrightarrow{\ \sim\ } \varinjlim_{I\in\mathcal{I}_0} \varinjlim_A \big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I, -) \circ \mathcal{M}\big)\\ &\xrightarrow{\ \sim\ } \varinjlim_{I\in\mathcal{I}_0} \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X/I, \varinjlim_A \mathcal{M}\big) \xrightarrow{\ \sim\ } \mathit{\Gamma}_{\mathcal{I}}(\varinjlim_A \mathcal{M}).\end{aligned}$$

□

As in [Kf, p. 640], a *quasi-noetherian* topological space is one that is quasi-compact and has a base of quasi-compact open subsets any two of which have quasi-compact intersection.

For example, the underlying space of a quasi-compact quasi-separated scheme or formal scheme is quasi-noetherian.

Corollary 1.1.20. (i) *Let X be a quasi-noetherian topological space, Φ a finitary s.o.s. in X, and $\mathcal{M}$ as in 1.1.19. The natural map is an isomorphism*

$$\varinjlim_A \big(\Gamma_\Phi(X, -) \circ \mathcal{M}\big) \xrightarrow{\ \sim\ } \Gamma_\Phi(X, \varinjlim_A \mathcal{M}).$$

In particular, $\Gamma_\Phi(X, -)$ commutes with direct sums.

(ii) *Let X be a quasi-compact quasi-separated scheme, $\mathcal{I}$ a finitary $\mathcal{O}_X$-base, and $\mathcal{M}$ as in 1.1.19. The natural map is an isomorphism*

$$\varinjlim_A \big(\Gamma_{\mathcal{I}}(X, -) \circ \mathcal{M}\big) \xrightarrow{\ \sim\ } \Gamma_{\mathcal{I}}(X, \varinjlim_A \mathcal{M}).$$

In particular, $\Gamma_{\mathcal{I}}(X, -)$ commutes with direct sums.

Proof. Let $\bullet$ denote one of Φ and $\mathcal{I}$. As $\varinjlim$ commutes with $\Gamma(X, -)$ (see [Kf, p. 641, Prop. 6]), one gets, by setting, in (1.1.13.3), $\Phi{:=}\{$all closed subsets of $X\}$, or by switching, in (1.1.14.4), $\mathcal{I}$ and $\mathcal{J}$ and then setting $\mathcal{J} :=\{$all quasi-coherent $\mathcal{O}_X$-ideals$\}$, natural isomorphisms

$$\begin{aligned}\varinjlim_A \big(\Gamma_\bullet(X, -) \circ \mathcal{M}\big) &\xrightarrow{\ \sim\ } \varinjlim_A \big(\Gamma(X, -) \circ \mathit{\Gamma}_\bullet \circ \mathcal{M}\big)\\ &\xrightarrow{\ \sim\ } \Gamma(X, -) \circ \varinjlim_A (\mathit{\Gamma}_\bullet \circ \mathcal{M})\\ &\underset{1.1.19}{\xrightarrow{\ \sim\ }} \Gamma(X, -) \circ \mathit{\Gamma}_\bullet(\varinjlim_A \mathcal{M}) \xrightarrow{\ \sim\ } \Gamma_\bullet(X, \varinjlim_A \mathcal{M}),\end{aligned}$$

whose composition is the map in question. □

1.2. **Cohomology with supports: topological spaces and schemes.** Next, the derived functors of those just considered. Notation remains as in Section 1.0.

Let $(X, \mathcal{O}_X)$ be a ringed space. An additive functor $\mathcal{G}\colon \mathcal{A}(X) \to \mathcal{A}$ extends naturally to a functor $\overline{\mathcal{G}}\colon \mathbf{K}(X) \to \mathbf{K}(\mathcal{A})$. Given, for each $\mathcal{O}_X$-complex E, a K-injective resolution $\sigma_E\colon E \to I_E$, with homotopy class $\tilde\sigma_E$, there exists a right-derived functor $\mathbf{R}\mathcal{G}\colon \mathbf{D}(X) \to \mathbf{D}(\mathcal{A})$ and a functorial map $\zeta_{\mathcal{G}}\colon q_{\mathcal{A}}\overline{\mathcal{G}} \to \mathbf{R}\mathcal{G}q_X$ such that for all E, $\mathbf{R}\mathcal{G}q_X E = q_{\mathcal{A}}\overline{\mathcal{G}}I_E$ and $\zeta_{\mathcal{G}}(E) = q_{\mathcal{A}}\overline{\mathcal{G}}\tilde\sigma_E$. (See, e.g., [Lp1, §2.3].) To a functorial map $\lambda\colon \mathcal{G} \to \mathcal{G}'$, with natural extension $\bar\lambda\colon \overline{\mathcal{G}} \to \overline{\mathcal{G}'}$, there is associated a unique functorial map $\mathbf{R}\lambda\colon \mathbf{R}\mathcal{G} \to \mathbf{R}\mathcal{G}'$ such that $\mathbf{R}\lambda \circ \zeta_{\mathcal{G}} = \zeta_{\mathcal{G}'} \circ \bar\lambda$.

For instance, with Φ an s.o.s. in X and $U \subset X$ open, let $\mathcal{G}$ be the functor

$$\Gamma_\Phi(U, -)\colon \mathcal{A}(X) \to \mathcal{A}(\mathrm{H}^0(U, \mathcal{O}_U)).$$

Let $u\colon U \hookrightarrow X$ be the inclusion. Then u^* takes K-injective resolutions to K-injective resolutions, and therefore one has, with $\Phi|_U$ as in (1.1.3.2), a natural isomorphism of functors (from $\mathbf{D}(X)$ to $\mathbf{D}\big(\mathrm{H}^0(U, \mathcal{O}_U)\big)$): $\mathbf{R}\Gamma_\Phi(U, -) \xrightarrow{\sim} \mathbf{R}\Gamma_{\Phi|_U}(U, -) \circ u^*$.

Likewise, if $(X, \mathcal{O}_X)$ is a scheme and $\mathcal{I}$ an $\mathcal{O}_X$-base then one has, with $\mathcal{I}|_U$ as in the line following (1.1.6.1), a natural isomorphism $\mathbf{R}\Gamma_{\mathcal{I}}(U, -) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathcal{I}|_U}(U, -) \circ u^*$.

The ordered set Φ (resp. $\mathcal{I}$) will always be regarded as a filtered category, with inclusions (resp. containments) as morphisms.

1.2.1. With preceding notation, set $H^n_\Phi := H^n\mathbf{R}\varGamma_\Phi$ $(n \in \mathbb{Z})$, and $H^n_Z := H^n\mathbf{R}\varGamma_{\Phi_Z}$. One has natural functorial isomorphisms

$$H^n_\Phi E \cong H^n\varGamma_\Phi I_E \underset{(1.1.13.1)}{\cong} H^n \varinjlim_{Z\in\Phi} \varGamma_Z I_E \cong \varinjlim_{Z\in\Phi} H^n\varGamma_Z I_E \cong \varinjlim_{Z\in\Phi} H^n_Z E. \tag{1.2.1.1}$$

Similarly, if $(X, \mathcal{O}_X)$ is a scheme and $\mathcal{I}$ an $\mathcal{O}_X$-base, then with

$$H^n_{\mathcal{I}} E := H^n\mathbf{R}\varGamma_{\mathcal{I}} E = \varinjlim_{I\in\mathcal{I}} \mathcal{E}xt^n(\mathcal{O}_X/I, E) \qquad (n \in \mathbb{Z})$$

(set $M := I_E$ in (1.1.14.1)), with $\varGamma_I$ as in the lines preceding (1.1.14.2), and $H^n_I E := H^n\mathbf{R}\varGamma_I E$, one has natural functorial isomorphisms

$$H^n_{\mathcal{I}} E \cong H^n\varGamma_{\mathcal{I}} I_E \underset{(1.1.14.3)}{\cong} H^n \varinjlim_{I\in\mathcal{I}} \varGamma_I I_E \cong \varinjlim_{I\in\mathcal{I}} H^n\varGamma_I I_E \cong \varinjlim_{I\in\mathcal{I}} H^n_I E. \tag{1.2.1.2}$$

Ditto, via (1.1.13.1) or (1.1.14.2), with $\bullet := \Phi$ or $\mathcal{I}$, for $\mathrm{H}^n_\bullet(U, E) := \mathrm{H}^n\mathbf{R}\Gamma_\bullet(U, E)$ (U open in X).

Proposition 1.2.2. *If Φ is a finitary s.o.s. in a scheme X, then*

$$\mathbf{R}\varGamma_\Phi\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}_{\mathrm{qc}}(X).$$

Proof. In view of (1.2.1.1), one may assume $\Phi = \Phi_Z$ with $Z \subset X$ closed and the inclusion map $i\colon (X \setminus Z) \hookrightarrow X$ quasi-compact. The assertion is given then by [AJL1, p. 25, (3.2.5)(iii)]. $\square$

Proposition 1.2.3. *Let X be a locally noetherian scheme, $E \in \mathbf{D}_{\mathrm{qc}}(X)$, Φ an s.o.s. in X and $\mathcal{I} := \mathcal{I}_\Phi$* (see 1.1.8)*. Deriving the inclusion $\varGamma_{\mathcal{I}} \hookrightarrow \varGamma_\Phi$ from 1.1.17 gives an* isomorphism *$\mathbf{R}\varGamma_{\mathcal{I}}E \xrightarrow{\sim} \mathbf{R}\varGamma_\Phi E$.*

Proof. One needs the natural maps $H^n_{\mathcal{I}}E \xrightarrow{\sim} H^n_\Phi E$ $(n \in \mathbb{Z})$ to be isomorphisms. By (1.1.8.2), (1.2.1.1) and (1.2.1.2), and the fact that for $I \in \mathcal{I}$, $\mathcal{I}_{\Phi_{Z(I)}} = \mathcal{I}_I$ (see (1.1.7.3) with $I_1 := I$), one reduces to where $\Phi = \Phi_{Z(I)}$ for some quasi-coherent $\mathcal{O}_X$-ideal I, and $\mathcal{I} = \mathcal{I}_I$. As $Z(I)$ is proregularly embedded in X ([AJL1, p. 16, Example (a)] and the lines before it), the assertion is given by [AJL1, p. 25, (3.2.4)]. $\square$

From 1.2.2 and 1.2.3 (or from [AJL1, p. 21, (3.1.4)(iii)]) one gets:

Proposition 1.2.4. *For any locally noetherian scheme X and $\mathcal{O}_X$-base $\mathcal{I}$,*

$$\mathbf{R}\varGamma_{\mathcal{I}}\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}_{\mathrm{qc}}(X). \qquad \square$$

* * * * *

Next, the stage is set for subsequent propositions.

Lemma 1.2.5. *Let X be a locally noetherian scheme, $\mathcal{J}$ an $\mathcal{O}_X$-base, Ψ an s.o.s in X and E a (degreewise) injective $\mathcal{O}_X$-complex. Then both $\varGamma_{\mathcal{J}}E$ and $\varGamma_{\Psi}E$ are injective.*

Proof. Using the results about injective $\mathcal{O}_X$-modules on p. 127 of [Hrt], and the fact that $\varGamma_{\mathcal{J}}$ commutes with direct sums (see 1.1.19), one reduces to checking that if $x \in X$ specializes to $x' \in X$ and $J(x, x')$ is the direct image on X of the constant sheaf on the closure $\overline{x'}$ of x' whose stalk at x' is the injective hull J_x of the residue field of $\mathcal{O}_{X,x}$, then

$$\varGamma_{\mathcal{J}}J(x,x') = \begin{cases} J(x,x') & \text{if } \overline{x} \in \Psi_{\mathcal{J}}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \varGamma_{\Psi}J(x,x') = \begin{cases} J(x,x') & \text{if } \overline{x'} \in \Psi, \\ 0 & \text{otherwise.} \end{cases}$$

This checking is left to the reader, with the reminders that for any specialization x'' of x', the $\mathcal{O}_{X,x''}$-module structure on the stalk $J(x, x')_{x''}$ is induced by the natural homomorphism $\mathcal{O}_{X,x''} \to \mathcal{O}_{X,x}$, and that every element of J_x is annihilated by a power of the maximal ideal of $\mathcal{O}_{X,x}$. $\square$

Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathbf{E}$ an additive category and $\phi\colon \mathbf{D}(X) \to \mathbf{E}$ an additive functor. An $\mathcal{O}_X$-complex F is *(right-)ϕ-acyclic* if the natural map $\phi F \to \mathbf{R}\phi F$ is a $\mathbf{D}(X)$-isomorphism. (See [Lp1, p. 50, Proposition 2.2.6]).

Lemma 1.2.6. *Let X be a scheme, $E^\bullet$ an $\mathcal{O}_X$-complex, Φ a finitary s.o.s. in X, and $\mathcal{I}$ an $\mathcal{O}_X$-base.*

(i) *If $E \in \mathbf{D}_{\mathrm{qc}}(X)$ and every E^i is $\varGamma_\Phi$-acyclic then the complex $E^\bullet$ is $\varGamma_\Phi$-acyclic.*

(ii) *If X is locally noetherian and every E^i is $\varGamma_{\mathcal{I}}$-acyclic, then $E^\bullet$ is $\varGamma_{\mathcal{I}}$-acyclic.*

Proof. Using Remark 1.1.4 and the fact that K-injectivity is preserved under restriction to open subsets—whence $\mathbf{R}\varGamma_\Phi$ "commutes" with such restriction—one finds that the assertions are local on X, so that X may be assumed affine.

In view of (1.1.13.1) and since Φ is finitary, one can also assume that $\Phi = \Phi_{Z(I)}$ with I generated by a finite sequence $\mathbf{t} = (t_1, \dots, t_d)$ of global sections. With $\mathcal{K}^\bullet_\infty(t_i)$ the complex which vanishes everywhere except in degrees 0 and 1, where it is

$$\mathcal{O}_X^{(0)} \xrightarrow{\text{natural}} \varinjlim \big(\mathcal{O}_X^{(0)} \xrightarrow{t_i} \mathcal{O}_X^{(1)} \xrightarrow{t_i} \mathcal{O}_X^{(2)} \xrightarrow{t_i} \cdots\big) \qquad \big(\mathcal{O}_X^{(n)} := \mathcal{O}_X \ \forall\, n \ge 0\big),$$

and with $\mathcal{K}^\bullet_\infty(\mathbf{t})$ the bounded flat complex $\mathcal{K}^\bullet_\infty(\mathbf{t}) := \otimes_{i=1}^d \mathcal{K}^\bullet_\infty(t_i)$, [AJL1, (3.2.3)] gives an isomorphism $\mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes E \xrightarrow{\sim} \mathbf{R}\varGamma_\Phi E$, which implies that the functor $\varGamma_\Phi$ is such that [Lp1, p. 77, (a)] (dualized) applies, giving (i).

As for (ii), it's enough that the maps induced by a K-injective resolution $E \to L$ be isomorphisms $H^n\varGamma_{\mathcal{I}}E \xrightarrow{\sim} H^n\varGamma_{\mathcal{I}}L$ ($n \in \mathbb{Z}$). For this, (1.1.14.3) allows one to replace $\mathcal{I}$ by a quasi-coherent $\mathcal{O}_X$-ideal I generated by a sequence $\mathbf{t} = (t_1, \dots, t_d)$ of global sections. One has then an isomorphism $\mathbf{R}\varGamma_I E \xrightarrow{\sim} \mathcal{K}^\bullet_\infty(\mathbf{t}) \otimes E$, (see proof of [AJL1, (3.1.1)(2)$'$ $\Rightarrow$ (3.1.1)(2)]), so [Lp1, p. 77, (a)] (dualized) applies. $\square$

1.2.7. Again, $(X, \mathcal{O}_X)$ is a ringed space.

An $\mathcal{O}_X$-complex $E \in \mathfrak{Ab}(X)$ is *flabby* (or *flasque*) if for every open $U \subset X$ the restriction map $\Gamma(X, E) \to \Gamma(U, E)$ is surjective; and *quasi-flabby* if the same holds for every *quasi-compact* open $U \subset X$. An $\mathcal{O}_X$-complex E is *K-flabby* (or *K-flasque*) if for every s.o.s. Φ in X and for every open $U \subset X$, the natural $\mathbf{D}(\mathrm{H}^0(U, \mathcal{O}_U))$-map $\Gamma_\Phi(U, E) \to \mathbf{R}\Gamma_\Phi(U, E)$ is an *isomorphism*—see [Spn, p. 144, 5.19]. In other words, E is K-flabby $\Leftrightarrow$ E is ϕ-acyclic for all functors ϕ of the form $\Gamma_\Phi(U, -)$.

For example, if E is K-injective then for any $\mathcal{O}_X$-complex C, the $\mathcal{O}_X$-complex $\mathcal{H}om_{\mathcal{O}_X}(C, E)$ is K-flabby [Spn, p. 142, 5.14 and p. 141, 5.12].

If $E \to E'$ is a $\mathbf{K}(X)$-isomorphism then E is K-flabby $\Leftrightarrow$ E' is K-flabby; and if $E \to E'$ is a quasi-isomorphism of K-flabby complexes then for all Φ and U as above, the induced map $\Gamma_\Phi(U, E) \to \Gamma_\Phi(U, E')$ is a quasi-isomorphism.

If $E \to I$ is a K-injective resolution, then E is K-flabby if and only if for all Φ, all U and all $n \in \mathbb{Z}$, the induced map $\mathrm{H}^n\Gamma_\Phi(U, E) \to \mathrm{H}^n\Gamma_\Phi(U, I)$ is an isomorphism. For any $\mathcal{O}_X$-complex C and $x \in X$, the stalk $(H^n\varGamma_\Phi C)_x$ satisfies

$$(H^n\varGamma_\Phi C)_x = \mathrm{H}^n \varinjlim_{x \in U} \Gamma_\Phi(U, C) = \varinjlim_{x \in U} \mathrm{H}^n\Gamma_\Phi(U, C).$$

Hence if E is K-flabby then the induced map $\varGamma_\Phi E \to \varGamma_\Phi I$ is a $\mathbf{D}(X)$-isomorphism, so that K-flabby $\Rightarrow$ $\varGamma_\Phi$-acyclic.

If $f \colon W \to X$ is a map of ringed spaces, then any K-flabby $\mathcal{O}_W$-complex is f_*-acyclic [Spn, p. 147, 6.7(a) and p. 141, 5.12]; and the functor f_* preserves K-flabbiness [Spn, p. 143, 5.15(b)]. Upon replacing E by a K-injective resolution, it follows then from (1.1.13.4) that there is a natural functorial isomorphism

$$\mathbf{R}\Gamma_{\Phi_f}(W, E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_\Phi(X, \mathbf{R}f_*E) \qquad (E \in \mathbf{D}(W)). \tag{1.2.7.1}$$

Also, taking f to be the natural map $(X, \mathcal{O}_X) \to (X, \mathbb{Z}_X)$, one gets that any K-flabby $\mathcal{O}_X$-complex is K-flabby as a complex of abelian sheaves. Hence for any integer n and $E \in \mathbf{D}(X)$, $\mathrm{H}^n_\Phi(E)$ depends, as an abelian group, only on X (not on $\mathcal{O}_X$)—and likewise for open $U \subset X$, whence for the abelian sheaves $H^n_\Phi(E)$.

1.2.8. An $\mathcal{O}_X$-module E—as a complex $E^\bullet$ vanishing in all nonzero degrees—is flabby if $\mathrm{H}^1_Z(X, E) = 0$ for all closed $Z \subset X$ (see [GR2, I, Corollaire 2.12]), and only if $\mathrm{H}^n_\Phi(X, E) = 0$ for every s.o.s. Φ and $n > 0$ (see [Gdm, p. 174, 4.4.3(a)]). In particular, any injective $\mathcal{O}_X$-module is flabby.

The restriction of a flabby $\mathcal{O}_X$-module E to any open $U \subset X$ is (clearly) a flabby $\mathcal{O}_U$-module; it follows that *a flabby $\mathcal{O}_X$-module E is K-flabby.*

Conversely, *if E is K-flabby* then $\mathrm{H}^1_Z(X, E) \cong \mathrm{H}^1\Gamma_Z(X, E^\bullet) = 0$, and therefore *$E$ is flabby.* (Alternatively, see [Spn, 5.13(a)].)

Actually, *any bounded-below quasi-flabby $\mathcal{O}_X$-complex is K-flabby.* To prove this, use the dual version of [Lp1, Proposition 2.7.2], whose hypotheses hold for the class of flabby $\mathcal{O}_X$-modules by virtue of the second paragraph on page 147 and Théorème 3.1.2 + Corollaire on page 148 in [Gdm]. (Alternatively, see [Spn, 2.2(c) and 5.15(c)].)

Likewise, *if X is quasi-noetherian then for every s.o.s. Φ in X and quasi-compact open $U \subset X$, any bounded-below quasi-flabby $\mathcal{O}_X$-complex is $\Gamma_\Phi(U, -)$-acyclic and $\varGamma_\Phi$-acyclic.* (Use [Kf, Proposition 4] instead of [Gdm, Théorème 3.1.2].)

Lemma 1.2.9. *Let X be a topological space, Ψ an s.o.s. in X, E an $\mathfrak{Ab}(X)$-complex.*

(i) *Suppose that $\Psi = \Phi_Y$ for some $Y \subset X$, or that every $Z \in \Psi$ is noetherian. If E is flabby then so is $\mathit{\Gamma}_\Psi E$.*

(ii) *Suppose X quasi-compact and Ψ finitary. If E is quasi-flabby then so is $\mathit{\Gamma}_\Psi E$.*

Proof. (i). Let $U \subset X$ be open. By (1.1.13.3), any $s \in \Gamma(U, \mathit{\Gamma}_\Psi E)$ vanishes on $U \setminus Z$ for some $Z \in \Psi$, hence extends to an $s' \in \Gamma(U \cup (X \setminus Z), E)$ that vanishes on $X \setminus Z$. Since E is flabby, therefore s' extends to an $s'' \in \Gamma(X, E)$. This s'' is an extension of s to $\Gamma(X, \mathit{\Gamma}_\Psi E)$.

(ii). Let $U \subset X$ be open and quasi-compact. By (1.1.13.3), any $s \in \Gamma(U, \mathit{\Gamma}_\Psi E)$ vanishes on $U \setminus Z$ for some $Z \in \Psi$, and since Ψ is finitary and X quasi-compact, one may assume that $X \setminus Z$ quasi-compact. The section s extends to a section $s' \in \Gamma(U \cup (X \setminus Z), E)$ that vanishes on $X \setminus Z$. Since E is quasi-flabby and $U \cup (X \setminus Z)$ is quasi-compact, therefore s' extends to an $s'' \in \Gamma(X, E)$. This s'' is an extension of s to $\Gamma(X, \mathit{\Gamma}_\Psi E)$. □

With $\phi :=$ empty set, the (*Krull*) *dimension* $\dim X$ of a topological space $X \neq \phi$ is the supremum ($\leq \infty$) of the set of those integers n such that there exists a strictly increasing sequence $\phi \neq Z_0 < Z_1 < \cdots < Z_n$ of irreducible closed subsets of X; and $\dim. \phi := -1$.

Lemma 1.2.10. *If X is a finite-dimensional noetherian topological space, then any flabby $\mathfrak{Ab}(X)$-complex is K-flabby.*

Proof. Any open $U \subset X$ is noetherian; and $\dim U \leq \dim X$, since the X-closure $\overline{Z}$ of an irreducible Z closed in U is irreducible and such that $\overline{Z} \cap U = Z$.

For any s.o.s. Φ in X, it holds then that

$$\mathrm{H}^p_\Phi(U, F) = 0 \text{ for all } F \in \mathfrak{Ab}(U) \text{ and integers } p > \dim X,$$

see [St, Tag 02UZ], whose proof works with "H" replaced by "H_Φ." (Use (1.2.7.1), and 1.2.15 below. Note too that if X is irreducible then any constant sheaf in $\mathfrak{Ab}(X)$ is flabby; moreover, if $\dim X = 0$ then the only nonempty open subset of X is X itself, whence every $E \in \mathfrak{Ab}(X)$ is flabby.)

Since every abelian sheaf embeds into a flabby one [Gdm, p. 147, 2nd paragraph], it results as in the proof of (ii)⇒(iii)⇒(a) in [Lp1, pp. 76–77, (2.7.5)] (dualized) that any flabby $\mathfrak{Ab}(X)$-complex is $\Gamma_\Phi(U, -)$-acyclic, thus K-flabby. □

* * * * *

Proposition 1.2.11. *Let X be a ringed space, E an $\mathcal{O}_X$-complex, and each of Φ and Ψ an s.o.s. in X. Suppose one of the following holds.*

(i) *$E \in \mathbf{D}^+(X)$ and Ψ is as in 1.2.9(i).*

(ii) *X is quasi-noetherian, $E \in \mathbf{D}^+(X)$, and Ψ is finitary.*

(iii) *X is noetherian and finite-dimensional.*

Then the natural map (*arising from* (1.1.13.2) *is an isomorphism*

$$\gamma_{\Phi,\Psi} \colon \mathbf{R}\mathit{\Gamma}_{\Phi\cap\Psi} E \xrightarrow{\ \sim\ } \mathbf{R}\mathit{\Gamma}_\Phi \mathbf{R}\mathit{\Gamma}_\Psi E.$$

If, moreover, E is cohomologically bounded-below then the natural map (*arising from* (1.1.13.3)) *is an isomorphism*

$$\overline{\gamma}_{\Phi,\Psi} \colon \mathbf{R}\Gamma_{\Phi\cap\Psi}(X, E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_\Phi(X, \mathbf{R}\mathit{\Gamma}_\Psi E).$$

Proof. One can assume E to be injective; and since $\gamma_{\Phi,\Psi}$ is an isomorphism if it is so locally, therefore, if $E \in \mathbf{D}^+(X)$ then one can also assume E bounded-below. As in 1.2.8, bounded-below plus flabby implies K-flabby; so if (i) holds then by 1.2.9(i), $\varGamma_\Psi E$ is K-flabby, hence $\Gamma_\Phi(X,-)$- and $\varGamma_\Phi$-acyclic; if (ii) holds, argue similarly, replacing 1.2.9(i) with 1.2.9(ii); and if (iii) holds, reach the same conclusion via 1.2.10. That $\gamma_{\Phi,\Psi}$ and $\overline{\gamma}_{\Phi,\Psi}$ are isomorphisms follows. □

Proposition 1.2.12. *Let X be a scheme, E an $\mathcal{O}_X$-complex, and each of Φ and Ψ an s.o.s. in X.*

(i) *If $E \in \mathbf{D}_{\mathrm{qc}}(X)$ and both Φ and Ψ are finitary, then the natural map* (*arising from* (1.1.13.2)) *is an* isomorphism

$$\gamma_{\Phi,\Psi}\colon \mathbf{R}\varGamma_{\Phi\cap\Psi}E \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_\Phi\mathbf{R}\varGamma_\Psi E.$$

(ii) *If X is locally noetherian and E is cohomologically bounded-below, then the natural map* (*from* (1.1.13.3)) *is an isomorphism*

$$\overline{\gamma}_{\Phi,\Psi}\colon \mathbf{R}\Gamma_{\Phi\cap\Psi}(X,E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_\Phi(X,\mathbf{R}\varGamma_\Psi E).$$

Proof. The complex E can be assumed K-injective, and furthermore, bounded-below if E is cohomologically so.

(i). One checks that for $n \in \mathbb{Z}$, the cohomology map $H^n\gamma_{\Phi,\Psi}$ factors as the following sequence of isomorphisms, in which V, W are such that $X\setminus V$ and $X\setminus W$ are retrocompact in X:

$$\begin{aligned} H^n\varGamma_{\Phi\cap\Psi}E &\xrightarrow{\ \sim\ } \varinjlim_{V\in\Phi}\varinjlim_{W\in\Psi} H^n\varGamma_{V\cap W}E && (1.1.13.1)\\ &\xrightarrow{\ \sim\ } \varinjlim_{V\in\Phi}\varinjlim_{W\in\Psi} H^n\mathbf{R}\varGamma_V\varGamma_W E && \text{[AJL1, p. 25, 3.2.5(ii)]}\\ &\xrightarrow{\ \sim\ } \varinjlim_{V\in\Phi} H^n\mathbf{R}\varGamma_V \varinjlim_{W\in\Psi}\varGamma_W E && \text{1.2.2, 1.2.16 (below)}\\ &\xrightarrow{\ \sim\ } \varinjlim_{V\in\Phi} H^n\mathbf{R}\varGamma_V\varGamma_\Psi E \xrightarrow{\ \sim\ } H^n\mathbf{R}\varGamma_\Phi\varGamma_\Psi E && (1.1.13.1),\ (1.2.1.1). \end{aligned}$$

Hence $\gamma_{\Phi,\Psi}$ is an isomorphism.

(ii). By 1.2.5 the bounded-below $\mathcal{O}_X$-complex $\varGamma_\Psi E$ is injective, hence K-injective, hence $\Gamma_\Phi(X,-)$-acyclic, so that $\overline{\gamma}_{\Phi,\Psi}$ is indeed an isomorphism. □

Proposition 1.2.13. *Let X be a locally noetherian scheme, E an $\mathcal{O}_X$-complex and each of $\mathcal{I}$ and $\mathcal{J}$ an $\mathcal{O}_X$-base. The natural map* (*from* (1.1.14.5)) *is an isomorphism*

$$\gamma_{\mathcal{I},\mathcal{J}}\colon \mathbf{R}\varGamma_{\mathcal{I}\cap\mathcal{J}}E \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathcal{I}}\mathbf{R}\varGamma_{\mathcal{J}}E.$$

If X is noetherian and finite-dimensional, then the natural map (*from* (1.1.14.4)) *is an isomorphism*

$$\overline{\gamma}_{\mathcal{I},\mathcal{J}}\colon \mathbf{R}\Gamma_{\mathcal{I}\cap\mathcal{J}}(X,E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_{\mathcal{I}}(X,\mathbf{R}\varGamma_{\mathcal{J}}E).$$

Proof. One can assume E to be K-injective and injective.

By 1.2.5 and 1.2.6(ii), $\varGamma_{\mathcal{J}}E$ is $\varGamma_{\mathcal{I}}$-acyclic, and so $\gamma_{\mathcal{I},\mathcal{J}}$ is an isomorphism.

Next, $\varGamma_{\mathcal{J}}E$ is injective (1.2.5), hence flabby, so one has natural isomorphisms

$$\mathbf{R}\Gamma_{\mathcal{I}}(X,E) \xrightarrow{\ \sim\ } \Gamma_{\mathcal{I}}(X,E) \underset{(1.1.14.4)}{\xrightarrow{\ \sim\ }} \Gamma(X,\varGamma_{\mathcal{I}}E) \underset{1.2.10}{\xrightarrow{\ \sim\ }} \mathbf{R}\Gamma(X,\varGamma_{\mathcal{I}}E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(X,\mathbf{R}\varGamma_{\mathcal{I}}E),$$

via which, one checks, $\overline{\gamma}_{\mathcal{I},\mathcal{J}}$ factors as the sequence of natural isomorphisms

$$\mathbf{R}\Gamma_{\mathcal{I}\cap\mathcal{J}}(X,E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(X,\mathbf{R}\varGamma_{\mathcal{I}\cap\mathcal{J}}E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma(X,\mathbf{R}\varGamma_{\mathcal{I}}\mathbf{R}\varGamma_{\mathcal{J}}E) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_{\mathcal{I}}(X,\mathbf{R}\Gamma_{\mathcal{J}}E). \quad \square$$

* * * * *

1.2.14. For a topological space X, bounded-below complexes $E \in \mathfrak{Ab}(X)$ have canonical (Godement) flabby resolutions $E \to G(E)$, with $G(E)$ bounded below and varying functorially with E (see [Lp1, proof of 3.9.3.1]).

If X is quasi-noetherian, then the functor G, with "flabby" replaced by "quasi-flabby," extends to unbounded E: with $E^{\ge -n}$ the complex obtained from E by replacing E^m with 0 for all $m < -n$, and with $E^{\ge -n} \to E^{\ge -(n+1)}$ the obvious map, one has $E = \varinjlim E^{\ge -n}$; and since a filtered direct limit of flabby (hence quasi-flabby) sheaves is quasi-flabby [Kf, p. 641, Corollary 7], one can set $G(E) := \varinjlim_{n\in\mathbb{Z}} G(E^{\ge -n})$.

Proposition 1.2.15. *Let X be a quasi-noetherian topological space, Φ a finitary s.o.s. in X, A a small filtered category, and $\mathcal{M}$ a functor from A to the category of $\mathfrak{Ab}(X)$-complexes. If $\varinjlim_A \mathcal{M}$ is bounded-below, or if X is noetherian and of finite dimension, then for every $n \in \mathbb{Z}$, the natural maps are isomorphisms*

$$\varinjlim_A (H^n_\Phi \circ \mathcal{M}) \xrightarrow{\ \sim\ } H^n_\Phi \varinjlim_A \mathcal{M},$$
$$\varinjlim_A (\mathrm{H}^n_\Phi(X,-)\circ \mathcal{M}) \xrightarrow{\ \sim\ } \mathrm{H}^n_\Phi(X, \varinjlim_A \mathcal{M}).$$

In particular, $\mathbf{R}\varGamma_\Phi$ and $\mathbf{R}\Gamma_\Phi(X,-)$ commute with small direct sums.

Proof. With G as in 1.2.14, $\varinjlim_A \mathcal{M} \to \varinjlim_A (G\circ\mathcal{M})$ is a quasi-isomorphism whose target is, by [Kf, Corollary 7], a flabby, hence as in 1.2.8 or by 1.2.10, K-flabby, hence $\varGamma_\Phi$-acyclic, complex.

The first isomorphism is then the natural composite isomorphism

$$\begin{aligned}\varinjlim_A (H^n_\Phi \circ \mathcal{M}) \xrightarrow{\ \sim\ } \varinjlim_A (H^n \circ \varGamma_\Phi \circ G \circ \mathcal{M}) &\xrightarrow{\ \sim\ } H^n \varinjlim_A (\varGamma_\Phi \circ G \circ \mathcal{M}) \\ &\underset{1.1.19}{\xrightarrow{\ \sim\ }} H^n \varGamma_\Phi \varinjlim_A (G\circ\mathcal{M}) \xrightarrow{\ \sim\ } H^n_\Phi \varinjlim_A \mathcal{M}.\end{aligned}$$

The second is obtained similarly, via 1.2.10 and 1.1.20.

As for direct sums, the standard argument associates to any set I the ordered (by inclusion) set A of finite subsets of I, regards A in the usual way as a filtered category, and uses commutativity of the additive functor H^n_Φ with finite direct sums to get, for any family $(M_i)_{i\in I}$ of $\mathfrak{Ab}(X)$-complexes, any $n \in \mathbb{Z}$, and $M_\alpha := \oplus_{i\in\alpha} M_i$ ($\alpha \in A$), natural isomorphisms:

$$\begin{aligned}H^n\big(\oplus_{i\in I} \mathbf{R}\varGamma_\Phi M_i\big) \xrightarrow{\ \sim\ } \oplus_{i\in I} H^n_\Phi M_i \xrightarrow{\ \sim\ } \varinjlim_{\alpha\in A} H^n_\Phi M_\alpha \\ \xrightarrow{\ \sim\ } H^n_\Phi(\varinjlim_{\alpha\in A} M_\alpha) \xrightarrow{\ \sim\ } H^n_\Phi\big(\oplus_{i\in I} M_i\big) = H^n\mathbf{R}\varGamma_\Phi\big(\oplus_{i\in I} M_i\big).\end{aligned}$$

Thus the natural map is an isomorphism

$$\oplus_{i\in I} \mathbf{R}\varGamma_\Phi M_i \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_\Phi(\oplus_{i\in I} M_i).$$

Similar considerations hold with $\Gamma_\Phi(X,-)$ in place of $\varGamma_\Phi$. ☐

Proposition 1.2.16. *Let X be a scheme, Φ a finitary s.o.s. in X, A a small filtered category, $\mathcal{M}$ a functor from A to the category of $\mathcal{O}_X$-complexes with quasi-coherent homology, and $n \in \mathbb{Z}$. The natural map is an isomorphism*

$$\varinjlim_A (H^n_\Phi \circ \mathcal{M}) \xrightarrow{\ \sim\ } H^n_\Phi \varinjlim_A \mathcal{M}.$$

In particular, $\mathbf{R}\varGamma_\Phi$ commutes with small direct sums in $\mathbf{D}_{\mathrm{qc}}(X)$.

Proof. Using Remark 1.1.4 and the fact that K-injectivity is preserved under restriction to open subsets—whence $\mathbf{R}\varGamma_{\Phi}$ "commutes" with such restriction—one finds that the first assertion is local on X, so that X may be assumed affine.

From (1.2.1.1) it follows that it's enough to treat the case $\Phi = \Phi_Z$, with $Z \subset X$ closed and such that $X \setminus Z$ is retrocompact in X; therefore it may be assumed that $Z = \mathrm{Supp}(\mathcal{O}_X/\mathbf{t}\mathcal{O}_X)$ with $\mathbf{t}$ a finite sequence in $\Gamma(X, \mathcal{O}_X)$. Then the assertion is a simple consequence of the fact that for complexes with quasi-coherent homology, applying $\mathbf{R}\varGamma_Z$ is the same as tensoring with the complex $\mathcal{K}^{\bullet}_{\infty}(\mathbf{t})$ (see proof of 1.2.6).

The argument for direct sums is as in the proof of 1.2.15. $\square$

Proposition 1.2.17. *Let X be a locally noetherian scheme, $\mathcal{I}$ an $\mathcal{O}_X$-base, A a small filtered category, $n \in \mathbb{Z}$, and $\mathcal{M}$ a functor from A to the category of $\mathcal{O}_X$-complexes. The natural map is an isomorphism*

$$\varinjlim_{A}\,(H^n_{\mathcal{I}} \circ \mathcal{M}) \xrightarrow{\ \sim\ } H^n_{\mathcal{I}} \varinjlim_{A} \mathcal{M}.$$

In particular, $\mathbf{R}\varGamma_{\mathcal{I}}$ commutes with small direct sums in $\mathbf{D}(X)$.

Proof. Imitate the proof of 1.2.16, replacing [AJL1, (3.2.3)] in the proof of 1.2.6 by [AJL1, (3.1.1)(2)]. (Alternatively, using 1.2.3 deduce the result from 1.2.16.) $\square$

Remark. More generally, 1.2.15–1.2.17 hold when A is a *pseudo-filtered* category [Mc, p. 216, Exercise 2].

1.3. **Coreflections.** This section expands on *coreflectiveness,* both abstractly and in the context of ringed spaces. In the following section there is a discussion of $\otimes$-*compatible coreflectiveness* in the context of symmetric monoidal categories, leading to the subsequently important notion of *idempotent pairs* in such categories.

Definition 1.3.1. Let $\mathbf{D}$ be a category, with identity functor $\mathbf{1_D}$, let $\Gamma\colon \mathbf{D} \to \mathbf{D}$ be a functor and $\iota\colon \Gamma \to \mathbf{1_D}$ a functorial map. The pair (Γ, ι) is a *coreflection of* $\mathbf{D}$ (or *coreflecting in* $\mathbf{D}$, or *colocalizing in* $\mathbf{D}$) if for all $E \in \mathbf{D}$, the functorial maps $\Gamma\big(\iota(E)\big)$ and $\iota(\Gamma E)$ are equal isomorphisms from $\Gamma\Gamma E$ to ΓE.

The functor Γ is a *coreflector* if there exists an ι such that (Γ, ι) is a coreflection.

Lemma 1.3.2 (well-known)**.** *The pair (Γ, ι) is coreflecting in $\mathbf{D}$ $\iff$ for all $F, G \in \mathbf{D}$ the map induced by $\iota(G)$ is an* isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(\Gamma F, \Gamma G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}}(\Gamma F, G). \tag{1.3.2.1}$$

Proof. For $\Rightarrow$, one checks, using $\Gamma\big(\iota(G)\big) = \iota(\Gamma G)$ and the functoriality of ι, that the natural composite map

$$\mathrm{Hom}_{\mathbf{D}}(\Gamma F, G) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(\Gamma\Gamma F, \Gamma G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}}(\Gamma F, \Gamma G)$$

is inverse to the map in (1.3.2.1).

For $\Leftarrow$, simple considerations applied to (1.3.2.1) with ΓG in place of G show that $\iota(\Gamma G)$ is an isomorphism; and functoriality of ι implies

$$\iota(G) \circ \iota(\Gamma G) = \iota(G) \circ \Gamma(\iota(G)),$$

and so (1.3.2.1) with $F = \Gamma G$ gives that $\iota(\Gamma G) = \Gamma(\iota(G))$. $\square$

Examples 1.3.3. (a) Let (Γ, ι) be coreflecting in $\mathbf{D}$, and let $\mathbf{D}' \subset \mathbf{D}$ be a full subcategory such that $\Gamma\mathbf{D}' \subset \mathbf{D}'$. Let $\Gamma'\colon \mathbf{D}' \to \mathbf{D}'$ be the restriction $\Gamma|_{\mathbf{D}'}$. Then ι induces a functorial map $\Gamma' \to \mathbf{1}_{\mathbf{D}'}$, and (Γ', ι') is coreflecting in $\mathbf{D}'$.

(b) Setting $\Psi = \Phi$ in (1.1.13.2), one gets that for any s.o.s. Φ on a ringed space X, the functor $\varGamma_\Phi$ and its inclusion into $\mathbf{1}_{\mathcal{A}(X)}$ constitute a coreflection of $\mathcal{A}(X)$; and likewise, via (1.1.14.5), for any functor $\varGamma_{\mathcal{I}}$ with $\mathcal{I}$ an $\mathcal{O}_X$-base on a scheme X.

(c) Let X be a ringed space, E an $\mathcal{O}_X$-complex, and Φ an s.o.s. in X. Propositions 1.2.11 and 1.2.12 give that if one of the following conditions (i)–(v) holds, then there exists a natural isomorphism

$$\gamma_{\Phi,\Phi}\colon \mathbf{R}\varGamma_\Phi E \xrightarrow{\sim} \mathbf{R}\varGamma_\Phi \mathbf{R}\varGamma_\Phi E. \qquad (1.3.3.1)$$

(i) $E \in \mathbf{D}^+(X)$, and $\Phi = \Phi_Y$ for some $Y \subset X$.
(ii) $E \in \mathbf{D}^+(X)$, and every member of Φ is quasi-compact.
(iii) X is quasi-noetherian, $E \in \mathbf{D}^+(X)$, and Φ is finitary.
(iv) X is noetherian and finite-dimensional.
(v) X is a scheme, $E \in \mathbf{D}_{\mathrm{qc}}(X)$, and Φ is finitary.

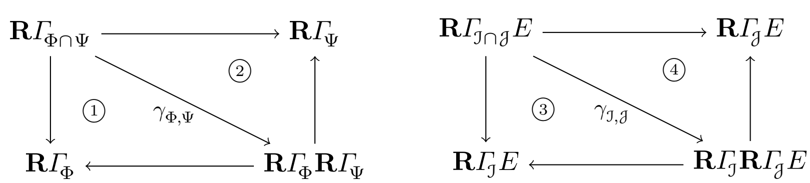
The next lemma implies that with $\iota_\Phi\colon \mathbf{R}\varGamma_\Phi \to \mathbf{1}$ the natural map, both $\mathbf{R}\varGamma_\Phi\iota_\Phi$ and $\iota_\Phi(\mathbf{R}\varGamma_\Phi)$ are inverse to (1.3.3.1), so they are equal isomorphisms from $\mathbf{R}\varGamma_\Phi\mathbf{R}\varGamma_\Phi$ to $\mathbf{R}\varGamma_\Phi$. Since $\mathbf{R}\varGamma_\Phi\mathbf{D}^+(X) \subset \mathbf{D}^+(X)$ (locally verifiable, so one need only consider bounded-below complexes...), and by 1.2.2, $\mathbf{R}\varGamma_\Phi\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}_{\mathrm{qc}}(X)$, therefore:

If (i), (ii) or (iii) *holds, then* $(\mathbf{R}\varGamma_\Phi, \iota_\Phi)$ *is coreflecting in* $\mathbf{D}^+(X)$; *if* (iv) *holds, then* $(\mathbf{R}\varGamma_\Phi, \iota_\Phi)$ *is coreflecting in* $\mathbf{D}(X)$*; and if* (v) *holds,* $(\mathbf{R}\varGamma_\Phi, \iota_\Phi)$ *is coreflecting in* $\mathbf{D}_{\mathrm{qc}}(X)$.

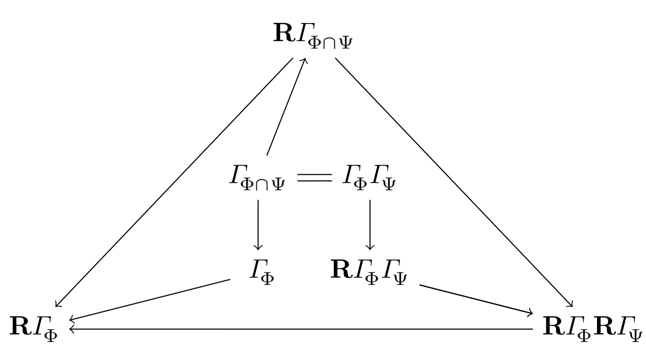
Similarly, using 1.2.13 one gets:

If X is a locally noetherian scheme and $\mathcal{I}$ is an $\mathcal{O}_X$-base, then $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ *is coreflecting in* $\mathbf{D}(X)$*—and also, by* 1.2.4*, in* $\mathbf{D}_{\mathrm{qc}}(X)$.

Lemma 1.3.4. *For systems of supports Φ, Ψ in a topological space, and bases $\mathcal{I}$, $\mathcal{J}$ over a scheme, the subtriangles in the following natural diagrams commute.*



Proof. For commutativity of ① it's enough (by the universal property of derived functors) to check after composing with the natural map $\varGamma_{\Phi\cap\Psi} \to \mathbf{R}\varGamma_{\Phi\cap\Psi}$, for which purpose it's enough to have commutativity of the subdiagrams in the following natural expansion of ①, commutativities that result directly from definitions.



That ②, ③ and ④ commute is shown similarly. (Details left to the reader.) $\square$

(d) Variants of the foregoing examples will emerge in the contexts of topological rings and of noetherian formal schemes (Propositions 1.7.4, 1.9.9 and 1.9.13).

* * * * *

1.3.5. The *essential image* $\mathbf{D}_\Gamma$ of a functor $\Gamma\colon \mathbf{D}\to\mathbf{D}$ is the strictly full subcategory of $\mathbf{D}$ spanned by the objects ΓF $(F\in\mathbf{D})$. The functor Γ factors as $\mathbf{D}\xrightarrow{\Gamma^0}\mathbf{D}_\Gamma\xhookrightarrow{j}\mathbf{D}$ where j is the inclusion functor.

It is easy to see that if (Γ,ι) is coreflecting in $\mathbf{D}$ then an object $E\in\mathbf{D}$ lies in $\mathbf{D}_\Gamma$ if and only if $\iota(E)$ is an isomorphism $\Gamma E\xrightarrow{\sim}E$.

Lemma 1.3.6. *The pair (Γ,ι) is coreflecting in $\mathbf{D}$ $\iff$ there is an adjunction $j\dashv\Gamma^0$ with counit ι. Thus Γ is a coreflector if and only if Γ^0 is right-adjoint to j (that is, if and only if $\mathbf{D}_\Gamma$ is a* coreflective subcategory *of $\mathbf{D}$* [Mc, p. 91, bottom]*).*

Proof. Let $E\in\mathbf{D}_\Gamma$, so that there is a $\mathbf{D}$-isomorphism $\alpha\colon jE\xrightarrow{\sim}\Gamma F$ $(F\in\mathbf{D})$. For any $G\in\mathbf{D}$, the square in the following diagram clearly commutes.

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{D}}(\Gamma F,\Gamma G) & \xrightarrow[\text{via }\alpha]{\sim} & \mathrm{Hom}_{\mathbf{D}}(jE,\Gamma G) & = & \mathrm{Hom}_{\mathbf{D}}(jE,j\Gamma^0G)\\
{\scriptstyle\text{via }\iota G}\big\downarrow{\scriptstyle a} & & {\scriptstyle b}\big\downarrow{\scriptstyle\text{via }\iota G} & & \big\|\\
\mathrm{Hom}_{\mathbf{D}}(\Gamma F,G) & \xrightarrow[\text{via }\alpha]{\sim} & \mathrm{Hom}_{\mathbf{D}}(jE,G) & & \mathrm{Hom}_{\mathbf{D}_\Gamma}(E,\Gamma^0G)
\end{array}$$

Hence (Γ,ι) is coreflecting $\Leftrightarrow$ a is an isomorphism (see 1.3.2) $\Leftrightarrow$ b is an isomorphism $\Leftrightarrow$ b gives an adjunction $j\dashv\Gamma^0$ whose counit (the image under b of the identity map of $j\Gamma^0G=\Gamma G$) is $\iota(G)$. $\square$

1.3.7. To illustrate, let $(X,\mathcal{O}_X)$ be a ringed space, let Φ be an s.o.s. in X, and let $\mathcal{A}_\Phi(X)\subset\mathcal{A}(X)$ be the full subcategory spanned by the Φ*-torsion* $\mathcal{O}_X$-modules, that is, those M such that $\mathit{\Gamma}_{\!\Phi}M=M$. Then $\mathcal{A}_\Phi(X)$ is the essential image of $\mathit{\Gamma}_{\!\Phi}$, since for any $\mathcal{O}_X$-isomorphism $M\xrightarrow{\sim}\mathit{\Gamma}_{\!\Phi}N$, (1.1.13.2) shows that M is Φ-torsion.

If X is a scheme, then in the preceding paragraph one can replace "$\mathcal{A}$" by "$\mathcal{A}_{\mathrm{qc}}$" (see 1.1.18); and if $\mathcal{I}$ is an $\mathcal{O}_X$-base, one can replace "Φ" by "$\mathcal{I}$." If $\Phi=\Phi_{\mathcal{I}}$ then $\mathcal{A}_{\mathcal{I}}(X)\subset\mathcal{A}_\Phi(X)$ and $\mathcal{A}_{\mathrm{qc}\mathcal{I}}(X)=\mathcal{A}_{\mathrm{qc}\Phi}(X)$ (see 1.1.16, 1.1.17).

The next lemma gives conditions on the ringed space X and the s.o.s. Φ ensuring that $\mathcal{A}_\Phi(X)$ is a *Serre subcategory* of $\mathcal{A}(X)$ [St, Tag 02MN], so that $\mathcal{A}_\Phi(X)$ *is plump in* $\mathcal{A}(X)$ (see section 1.0). Similarly, when X is a scheme and $\mathcal{I}$ a finitary $\mathcal{O}_X$-base, then $\mathcal{A}_{\mathcal{I}}(X)$ (resp. $\mathcal{A}_{\mathrm{qc}\mathcal{I}}(X)$) is a Serre—hence plump—subcategory of $\mathcal{A}(X)$ (resp. $\mathcal{A}_{\mathrm{qc}}(X)$).

Lemma 1.3.8. *Let X be a ringed space, $M'\xrightarrow{f}M\xrightarrow{g}M''$ an exact sequence of $\mathcal{O}_X$-modules, Φ an s.o.s. in X, and when X is a scheme, $\mathcal{I}$ a finitary $\mathcal{O}_X$-base.*

(i) *Suppose X has a base of quasi-compact open sets, and either that Φ is finitary or that $\Phi=\Phi_Y$ $(Y\subset X)$. If M' and M'' are in $\mathcal{A}_\Phi(X)$ then $M\in\mathcal{A}_\Phi(X)$.*

(ii) *When X is a scheme, if M' and M'' are in $\mathcal{A}_{\mathcal{I}}(X)$ then $M\in\mathcal{A}_{\mathcal{I}}(X)$. Hence if M' and M'' are in $\mathcal{A}_{\mathrm{qc}\mathcal{I}}(X)$ and $M\in\mathcal{A}_{\mathrm{qc}}(X)$ then $M\in\mathcal{A}_{\mathrm{qc}\mathcal{I}}(X)$.*

Proof. (i). Fix an open $U\subset X$ and $m\in\Gamma(U,M)$. One needs that any $x\in U$ has an open neighborhood $V\subset U$ such that $\mathrm{supp}_V(m)\in\Phi|_V$. By assumption, $x\in V\subset U$ with V quasi-compact and open, and such that $\mathrm{supp}_V(g(m))\subset Z'\cap V$ for some $Z'\in\Phi$.

If Φ is finitary, then one can assume that $X \setminus Z'$ is retrocompact in X, so that $V \setminus Z'$ is quasi-compact. Over $V \setminus Z'$, $g(m) = 0$, so $m \in \operatorname{im}(f)$, whence by 1.1.5,

$$\operatorname{supp}_{V\setminus Z'}(m) = Z'' \cap (V \setminus Z') \text{ for some } Z'' \in \Phi.$$

Thus $\operatorname{supp}_V(m) \subset (Z' \cup Z'')$, and so $\operatorname{supp}_V(m) \in \Phi|_V$.

(ii). Fix an open $U \subset X$ and $m \in \Gamma(U, M)$. One needs, first, that each $x \in U$ has an open neighborhood $V \subset U$ over which m is annihilated by some $I \in \mathcal{I}$. By assumption, x has an open neighborhood $V \subset U$ over which $g(m)$ is annihilated by some $I' \in \mathcal{I}$ with $I'|_V$ generated by finitely many of its sections over V. Hence, over V, $I'm \subset \operatorname{im} f$, so over some open neighborhood $V' \subset V$, $I'm$ is annihilated by some $I'' \in \mathcal{I}$. One can then take $V := V'$, $I := I''I'$.

The last assertion follows at once. □

Upgrading to the derived level, let $\mathbf{D}_\Phi(X) \subset \mathbf{D}(X)$ be the full subcategory spanned by the complexes whose homology modules are all in $\mathcal{A}_\Phi(X)$.

Under the hypotheses of 1.3.8(i), $\mathcal{A}_\Phi(X)$ is plump in $\mathcal{A}(X)$, so the exact homology sequence of a triangle entails that $\mathbf{D}_\Phi(X)$ is a *triangulated subcategory* of $\mathbf{D}(X)$: if two vertices of a $\mathbf{D}(X)$-triangle lie in $\mathbf{D}_\Phi(X)$ then so does the third.

Furthermore, if $\varGamma_\Phi$ commutes with direct sums (see, e.g., Proposition 1.1.19(i)), then $\mathbf{D}_\Phi(X)$ is a *localizing subcategory* of $\mathbf{D}(X)$, meaning here a triangulated subcategory closed under small direct sums in $\mathbf{D}(X)$.

Plumpness of $\mathcal{A}_\Phi(X)$ also implies that any complex in $\mathcal{A}_\Phi(X)$ is in $\mathbf{D}_\Phi(X)$.

Similar statements hold, with $\mathcal{I}$ in place of Φ, when X is a scheme and $\mathcal{I}$ is a finitary $\mathcal{O}_X$-base.

Proposition 1.3.9. (i) *If* $(\mathbf{R}\varGamma_\Phi, \iota_\Phi)$ *is coreflecting in* $\mathbf{D} := \mathbf{D}(X)$ *or* $\mathbf{D}^+(X)$ (see 1.3.3(c)), *then* $\mathbf{D}_\Phi(X) \cap \mathbf{D}$ *is the essential image of* $\mathbf{R}\varGamma_\Phi \colon \mathbf{D} \to \mathbf{D}$.

(ii) *Similarly, if* X *is a locally noetherian scheme and* $\mathcal{I}$ *an* $\mathcal{O}_X$*-base, then the essential image of* $\mathbf{R}\varGamma_{\mathcal{I}} \colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\mathrm{qc}}(X)$ *is* $\mathbf{D}_{\mathcal{I}}(X) \cap \mathbf{D}_{\mathrm{qc}}(X)$.

Proof. By 1.2.3, (ii) follows from (i). Application of the sentence preceding 1.3.6 to the coreflecting pair $(\mathbf{R}\varGamma_\Phi, \iota_\Phi)$ shows that (i) results from the next lemma. □

Lemma 1.3.10. *For any s.o.s.* Φ *in a ringed space* X*, an* $\mathcal{O}_X$*-complex* E *lies in* $\mathbf{D}_\Phi(X)$ *if and only if* $\iota_\Phi(E)$ *is an isomorphism* $\mathbf{R}\varGamma_\Phi E \xrightarrow{\sim} E$.

Proof. Let $E \to I$ be a K-injective resolution. By (1.1.13.2), $\varGamma_\Phi I$ is a complex in $\mathcal{A}_\Phi(X)$, so as noted above, $\mathbf{R}\varGamma_\Phi E \cong \varGamma_\Phi I \in \mathbf{D}_\Phi(X)$, whence the essential image of $\mathbf{R}\varGamma_\Phi$ is contained in $\mathbf{D}_\Phi(X)$.

For the opposite inclusion it suffices to show that *if* E*—hence* I*—is in* $\mathbf{D}_\Phi(X)$*, then the natural map is an isomorphism* $\mathbf{R}\varGamma_\Phi E \xrightarrow{\sim} E$*, that is, for every* $n \in \mathbb{Z}$*, the natural map is an isomorphism* $H^n\varGamma_\Phi I \xrightarrow{\sim} H^n I$.

For any closed $Z \subset X$, set $\varGamma_Z := \varGamma_{\Phi_Z}$ and let $u_Z \colon (X \setminus Z) \hookrightarrow X$ be the inclusion. Since I is flabby, there is a natural exact sequence

$$0 \to \varGamma_Z I \to I \to u_{Z*}u_Z^* I \to 0$$

whence an exact cohomology sequence

$$(1.3.10.1) \qquad \cdots \to H^n\varGamma_Z I \to H^n I \to H^n u_{Z*}u_Z^* I \to H^{n+1}\varGamma_Z I \to H^{n+1} I \to \cdots$$

to which, by(1.2.1.1), application of the exact functor $\varinjlim_{Z\in\Phi}$ brings the problem down to proving the next Lemma. □

Lemma 1.3.11. *If* $J = (J^\bullet, d^\bullet) \in \mathbf{D}_\Phi(X)$ *is flabby, then* $\varinjlim_{Z\in\Phi} H^n u_{Z*}u_Z^* J = 0$.

Proof. For $Z \in \Phi$, since $\mathcal{A}_\Phi(X)$ is plump in $\mathcal{A}(X)$, therefore $\varGamma_Z J \in \mathbf{D}_\Phi(X)$; and the exactness of (1.3.10.1) with J in place of I shows that $u_{Z*}u_Z^* J \in \mathbf{D}_\Phi(X)$.

Let x be any point in X, V an open neighborhood of x, and $h \in \Gamma(V, u_{Z*}u_Z^* J^n)$ such that $d^n h = 0$. Since $u_{Z*}u_Z^* J \in \mathbf{D}_\Phi(X)$, x has an open neighborhood $U \subset V$ where the element $\bar h \in \Gamma(U, H^n u_{Z*}u_Z^* J)$ given by h is supported in a subset $Z' \cap U$ with $Z' \in \Phi$. Therefore, if $Z_1 := Z \cup Z'$ then the natural map

$$\Gamma(U, H^n u_{Z*}u_Z^* J) \to \Gamma(U, H^n u_{Z_1*}u_{Z_1}^* J)$$

annihilates $\bar h$. Thus the stalk at x of $\varinjlim_{Z\in\Phi} H^n(u_{Z*}u_Z^* J)$ vanishes. □

The derived functor $\mathbf{R}(\varGamma_\Phi^0)$ is right-adjoint to the derived functor

$$\boldsymbol{j} := \mathbf{R}j\colon \mathbf{D}(\mathcal{A}_\Phi(X)) \to \mathbf{D}(X),$$

see [AJL2, p. 49, 5.2.2] (in whose second line "$\boldsymbol{j}$ be the" should follow "let"). And $\mathbf{R}\varGamma_\Phi = \boldsymbol{j}\mathbf{R}(\varGamma_\Phi^0)$. From 1.3.10 and *loc.cit.* (2)⇒(1), one gets:

Corollary 1.3.12. $\mathbf{R}(\varGamma_\Phi^0)$ *restricts to an* equivalence of categories

$$\mathbf{D}_\Phi(X) \xrightarrow{\approx} \mathbf{D}(\mathcal{A}_\Phi(X)),$$

with quasi-inverse given by $\boldsymbol{j}$.

* * * * *

The *support* $\mathrm{Supp}(E)$ of an $\mathcal{O}_X$-complex E is the set of points at which E is not exact, that is, the union of the supports of all the homology sheaves of E.

Lemma 1.3.13. *For* $Y \subset X$, Φ_Y *as in* 1.1.1, *and* $E \in \mathbf{D}(X)$,

$$\mathrm{Supp}(E) \subset Y \iff E \in \mathbf{D}_{\Phi_Y}(X).$$

Proof. This is a statement about the homology modules of E, so it suffices to note that for an $\mathcal{O}_X$-module M, it follows directly from definitions that

$$\mathrm{Supp}(M) \subset Y \iff M \in \mathcal{A}_{\Phi_Y}(X). \qquad \square$$

Lemma 1.3.14. *For any s.o.s.* Φ *in a ringed space* X, *and* $E \in \mathbf{D}(X)$,

$$\mathrm{Supp}(\mathbf{R}\varGamma_\Phi E) \subset \bigcup_{Z\in\Phi} Z.$$

Proof. Since E can be assumed to be K-injective, it suffices to note that since $\varGamma_Z E$ vanishes outside Z, therefore $\varGamma_\Phi E = \varinjlim_{Z\in\Phi} \varGamma_Z E$ vanishes outside $\bigcup_{Z\in\Phi} Z$. □

1.4. Idempotent pairs in symmetric monoidal categories.

Part of the "basic formal setup," a category-theoretic framework for duality, local and global, to be built on in subsequent chapters, is the notion of *idempotent pair* in a symmetric monoidal category $\mathbf{D}$—more precisely, in the slice category $\mathbf{D}/\mathcal{O}$ with $\mathcal{O}$ the unit object (Definition 1.4.3).[5] This notion is equivalent to that of ⊗*-coreflection,* that is, coreflection (Γ, ι) with Γ isomorphic to a functor $\Gamma_A(-) := A \otimes -$ where A is a fixed object and $\otimes$ is the monoidal product (Proposition 1.5.7). This section and the following two review some basics about such pairs.

[5] $\mathbf{D}/\mathcal{O}$ has as objects the pairs (C, γ) with C an object of $\mathbf{D}$ and $\gamma\colon C \to \mathcal{O}$ a $\mathbf{D}$-map, and as morphisms $\lambda\colon (B, \beta) \to (A, \alpha)$ the $\mathbf{D}$-morphisms $\lambda_0\colon B \to A$ such that $\beta = \alpha\lambda_0$. (Henceforth, absent potential for confusion we will not differentiate notationally between λ and λ_0.)

Definition 1.4.1 ([Mc, p. 251ff])**.** A (*symmetric*) *monoidal category*

$$\mathbf{D} = (\mathbf{D_0}, \otimes, \mathcal{O}, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{s})$$

consists of a category $\mathbf{D_0}$, a "product" functor $\otimes\colon \mathbf{D_0} \times \mathbf{D_0} \to \mathbf{D_0}$, an object $\mathcal{O}$ of $\mathbf{D_0}$, and functorial isomorphisms (for A, B, C in $\mathbf{D_0}$)

(associativity) $\quad \mathsf{a} = \mathsf{a}_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$

(units) $\quad \mathsf{l} = \mathsf{l}_A\colon \mathcal{O} \otimes A \xrightarrow{\sim} A \qquad \mathsf{r} = \mathsf{r}_A\colon A \otimes \mathcal{O} \xrightarrow{\sim} A$

(symmetry) $\quad \mathsf{s} = \mathsf{s}_{A,B}\colon A \otimes B \xrightarrow{\sim} B \otimes A$

such that $\mathsf{s} \circ \mathsf{s} = \mathbf{1}$ (identity map) and the following diagrams commute:

(1.4.1.1)
$$\begin{array}{ccc} (A \otimes \mathcal{O}) \otimes B & \xrightarrow{\mathsf{a}} & A \otimes (\mathcal{O} \otimes B) \\ & {}_{\mathsf{r}\otimes\mathbf{1}}\searrow \quad \swarrow_{\mathbf{1}\otimes\mathsf{l}} & \\ & A \otimes B & \end{array}$$

(1.4.1.2)
$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\mathsf{a}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\mathsf{a}} & A \otimes (B \otimes (C \otimes D)) \\ {\scriptstyle \mathsf{a}\otimes\mathbf{1}}\downarrow & & & & \downarrow{\scriptstyle \mathbf{1}\otimes\mathsf{a}} \\ (A \otimes (B \otimes C)) \otimes D & & \xrightarrow[\mathsf{a}]{\qquad\qquad} & & A \otimes ((B \otimes C) \otimes D) \end{array}$$

(1.4.1.3)
$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\mathsf{a}} & A \otimes (B \otimes C) & \xrightarrow{\mathsf{s}} & (B \otimes C) \otimes A \\ {\scriptstyle \mathsf{s}\otimes\mathbf{1}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{a}} \\ (B \otimes A) \otimes C & \xrightarrow[\mathsf{a}]{} & B \otimes (A \otimes C) & \xrightarrow[\mathbf{1}\otimes\mathsf{s}]{} & B \otimes (C \otimes A) \end{array}$$

(1.4.1.4)
$$\begin{array}{ccc} A \otimes \mathcal{O} & \xrightarrow{\mathsf{s}} & \mathcal{O} \otimes A \\ & {}_{\mathsf{r}}\searrow \quad \swarrow_{\mathsf{l}} & \\ & A & \end{array}$$

Necessarily, the following diagrams commute too [Mc, p. 165, Exercise 1].

(1.4.1.5)
$$\begin{array}{ccc} (A \otimes B) \otimes \mathcal{O} & \xrightarrow{\mathsf{a}} & A \otimes (B \otimes \mathcal{O}) \\ & {}_{\mathsf{r}}\searrow \quad \swarrow_{\mathbf{1}\otimes\mathsf{r}} & \\ & A \otimes B & \end{array} \qquad \begin{array}{ccc} (\mathcal{O} \otimes A) \otimes B & \xrightarrow{\mathsf{a}} & \mathcal{O} \otimes (A \otimes B) \\ & {}_{\mathsf{l}\otimes\mathbf{1}}\searrow \quad \swarrow_{\mathsf{l}} & \\ & A \otimes B & \end{array}$$

Examples 1.4.2. (a) Let X be a ringed space. Derived tensor product makes $\mathbf{D}(X)$ into a monoidal category, with unit object $\mathcal{O}_X$ (1.5.12 below); and similarly for $\mathbf{D}_{\mathrm{qc}}(X)$ (resp. $\mathbf{D}_{\mathrm{qct}}(X)$) when X is a scheme (resp. noetherian formal scheme), see 1.9.28.

(b) For a monoidal category $\mathbf{D}$, the slice category $\mathbf{D}/\mathcal{O}$ has a monoidal structure with unit object $(\mathcal{O}, \mathbf{1}_{\mathcal{O}})$, product $(A,\alpha)\underline{\otimes}(B,\beta) := (A\otimes B, \mu\circ(\alpha\otimes\beta))$ where $\mu = \mathsf{r}_{\mathcal{O}} = \mathsf{l}_{\mathcal{O}}$ (see proof of 1.4.6), and isomorphisms a, l, r and s whose images under the functor $(A,\alpha)\mapsto A$ are the the corresponding isomorphisms in $\mathbf{D}$. (Details are left to the reader.)

Until otherwise indicated, $\mathbf{D}$ will be a fixed monoidal category. Sometimes, for simplicity, $A\otimes\mathcal{O}$ and $\mathcal{O}\otimes A$ will be identified—harmlessly—with the object $A\in\mathbf{D}$.

Definition 1.4.3. A $\mathbf{D}$-*idempotent pair* (A,α) is a $\mathbf{D}$-map $\alpha\colon A\to\mathcal{O}$ such that the composite maps $A\otimes A \xrightarrow{\mathbf{1}\otimes\alpha} A\otimes\mathcal{O} \xrightarrow{\mathsf{r}} A$ and $A\otimes A\xrightarrow{\alpha\otimes\mathbf{1}}\mathcal{O}\otimes A\xrightarrow{\mathsf{l}} A$ are *equal isomorphisms*. An object $A\in\mathbf{D}$ is idempotent if such an α exists.

Examples 1.4.4. (a) Let X be a locally noetherian scheme, and $\mathcal{I}$ an $\mathcal{O}_X$-base. The pair $(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_X, \iota_{\mathcal{I}}(\mathcal{O}_X))$ is $\mathbf{D}_{\mathrm{qc}}(X)$- and $\mathbf{D}(X)$-idempotent (see 1.5.14).

(b) Let X be a scheme, and Φ a finitary s.o.s. in X. The pair $(\mathbf{R}\varGamma_{\Phi}\mathcal{O}_X, \iota_{\Phi}(\mathcal{O}_X))$ is $\mathbf{D}_{\mathrm{qc}}(X)$- and $\mathbf{D}(X)$-idempotent (see 1.5.14).

For additional such examples, involving topological rings, or noetherian formal schemes, see Corollary 1.7.10, Proposition 1.9.20 and Corollary 1.9.22.

(c) Let $\mathbf{D}$ be a category with a terminal object $\mathcal{O}$, and such that any two objects $A,B\in\mathbf{D}$ have a product, denoted $A\otimes B$. With this $\mathcal{O}$, $\otimes$ (made into a functor), and obvious choices for a, l, r and s, one gets a monoidal category. One verifies, for any $A\in\mathbf{D}$ with $\alpha\colon A\to\mathcal{O}$ the unique map, that (A,α) is idempotent if and only if α is a monomorphism (that is, for any $B\in\mathbf{D}$ there is at most one map $B\to A$).

(d) In particular, let $(\mathbf{D},\le)$ be a preordered set (set with a reflexive, transitive binary relation $\le$), considered as a category in the usual way: the objects are the elements of $\mathbf{D}$, there is a unique map $A\to B$ if $A\le B$, and otherwise no such map at all. Assume that $\mathbf{D}$ has a largest object $\mathcal{O}$, and that any two objects $A,B\in\mathbf{D}$ have a greatest lower bound (= product), denoted $A\otimes B$. With the obviously unique a, l, r and s, $\mathbf{D}$ is a monoidal category in which for *any* object A with $\alpha\colon A\to\mathcal{O}$ the unique map, (A,α) is idempotent.

Note that $B\le A \iff \alpha\otimes\mathbf{1}\colon A\otimes B\to\mathcal{O}\otimes B\cong B$ *is an isomorphism.*

Also, $B\cong A \iff B\le A$ *and* $A\le B$.

Such categories will be called *preordered* monoidal categories. They can be viewed as small monoidal categories in which for any objects A and B, there exists at most one map $A\to B$, and exactly one if $B=\mathcal{O}$ or if $B=A\otimes A$.

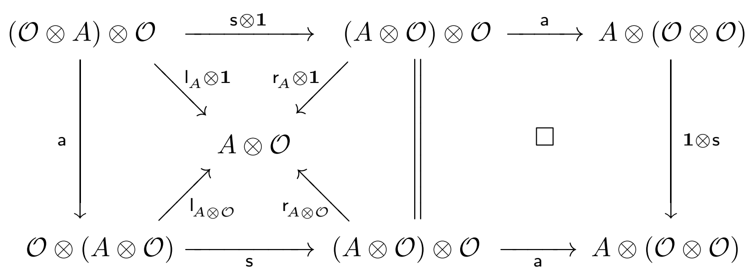
Remark 1.4.5. The full subcategory $\mathbf{I}_{\mathbf{D}}$ of $\mathbf{D}/\mathcal{O}$ spanned by the $\mathbf{D}$-idempotent pairs is *strictly* full: use the fact that if (A,α) is idempotent and $\lambda\colon B\xrightarrow{\sim} A$ is a $\mathbf{D}$-isomorphism, then $(B,\alpha\lambda)$ is idempotent. Moreover, $\mathbf{I}_{\mathbf{D}}$ is a *preordered monoidal* subcategory of $\mathbf{D}/\mathcal{O}$, see 1.4.6, 1.5.11 and 1.6.1 below.

Note that (A,α) is $\mathbf{D}$-idempotent $\Leftrightarrow$ $((A,\alpha),\,\alpha)$ is $(\mathbf{D}/\mathcal{O})$-idempotent.

Lemma 1.4.6. *The pair* $(\mathcal{O},\mathbf{1}_{\mathcal{O}})$ *is* $\mathbf{D}$*-idempotent.*

Proof. The assertion means that the unit isomorphisms $\mathsf{l}=\mathsf{l}_{\mathcal{O}}\colon\mathcal{O}\otimes\mathcal{O}\xrightarrow{\sim}\mathcal{O}$ and $\mathsf{r}=\mathsf{r}_{\mathcal{O}}\colon\mathcal{O}\otimes\mathcal{O}\xrightarrow{\sim}\mathcal{O}$ are the same, or, by (1.4.1.4), that *the automorphism* $\mathsf{s}=\mathsf{s}_{\mathcal{O},\mathcal{O}}\colon\mathcal{O}\otimes\mathcal{O}\xrightarrow{\sim}\mathcal{O}\otimes\mathcal{O}$ *is the identity map.*

By (1.4.1.3) (with (A,B,C) replaced by $(\mathcal{O},A,\mathcal{O})$), the border of the following diagram of isomorphisms commutes, for any $A\in\mathbf{D}$:



The subtriangle on the left commutes by (1.4.1.5) (second diagram). The ones at the top and bottom commute by (1.4.1.4). Furthermore, $\mathsf{r}_A\otimes\mathbf{1}=\mathsf{r}_{A\otimes\mathcal{O}}$, as shown by the next diagram, which commutes because r is functorial:

$$\begin{array}{ccc}
(A\otimes\mathcal{O})\otimes\mathcal{O} & \xrightarrow{\mathsf{r}_A\otimes\mathbf{1}} & A\otimes\mathcal{O}\\
{\scriptstyle \mathsf{r}_{A\otimes\mathcal{O}}}\big\downarrow & & \simeq\big\downarrow{\scriptstyle \mathsf{r}_A}\\
A\otimes\mathcal{O} & \underset{\mathsf{r}_A}{\xrightarrow{\ \sim\ }} & A
\end{array}$$

It follows that subrectangle $\square$ commutes, whence $\mathbf{1}\otimes\mathsf{s}$ is the identity map, whence so is s, as one sees by taking $A=\mathcal{O}$ and applying $\mathsf{l}_{\mathcal{O}\otimes\mathcal{O}}$. $\square$

Remark 1.4.7. *For any idempotent pair (A,α), the symmetry automorphism $\mathsf{s}_{A,A}\colon A\otimes A\xrightarrow{\sim}A\otimes A$ is the identity map.* Indeed, (1.4.1.4) shows that the following diagram—whose rows compose to the same isomorphism—commutes:

$$\begin{array}{ccccc}
A\otimes A & \xrightarrow{\mathbf{1}\otimes\alpha} & A\otimes\mathcal{O} & \xrightarrow{\mathsf{r}_A} & A\\
{\scriptstyle \mathsf{s}_{A,A}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{s}_{A,\mathcal{O}}} & & \big\|\\
A\otimes A & \underset{\alpha\otimes\mathbf{1}}{\longrightarrow} & \mathcal{O}\otimes A & \underset{\mathsf{l}_A}{\longrightarrow} & A
\end{array}$$

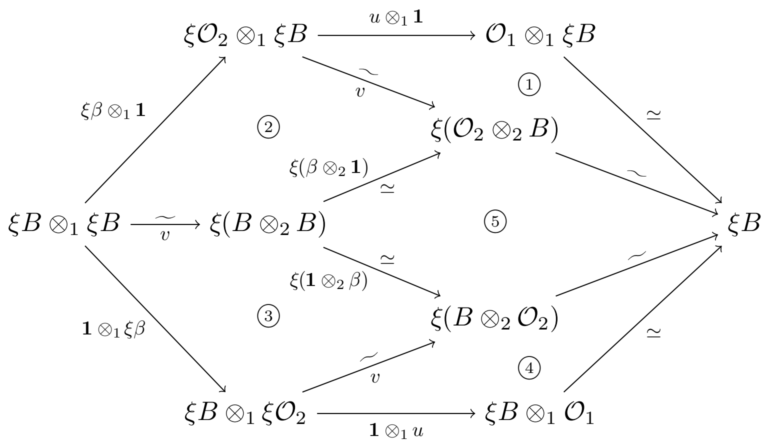
More generally, by 1.4.5 and 1.5.11 below, $(A,\alpha)\underline{\otimes}(A,\alpha)$ is $(\mathbf{D}/\mathcal{O})$-idempotent, and so by 1.6.1, its only $(\mathbf{D}/\mathcal{O})$-endomorphism is the identity map.

Lemma 1.4.8. *Let $\xi\colon\mathbf{D}_2\to\mathbf{D}_1$ be a functor between monoidal categories having respective units $\mathcal{O}_2$, $\mathcal{O}_1$ and product functors $\otimes_2$, $\otimes_1$. Let (B,β) be $\mathbf{D}_2$-idempotent.*

Suppose there exists a $\mathbf{D}_1$-map $u\colon\xi\mathcal{O}_2\to\mathcal{O}_1$ and a bifunctorial $\mathbf{D}_2$-isomorphism

$$v(E,F)\colon\xi E\otimes_1\xi F\xrightarrow{\sim}\xi(E\otimes_2 F)\qquad(E,F\in\mathbf{D}_2)$$

such that subdiagrams ① and ④ of the following natural diagram commute.



Then $(\xi B, u\circ\xi\beta)$ *is* $\mathbf{D}_1$*-idempotent.*

Proof. The commutativity of subdiagrams ② and ③ is given by the functoriality of v; and that of ⑤ holds by the idempotence of (B,β). These commutativities, plus those of ① and ④, imply that the border of the diagram commutes, and consists entirely of isomorphisms, whence the conclusion. □

Remark 1.4.9. The hypotheses in 1.4.8 are satisfied if, $f\colon X_1\to X_2$ being a map of ringed spaces, ξ is $\mathbf{L}f^*\colon \mathbf{D}(X_2)\to\mathbf{D}(X_1)$ and u, v are the natural isomorphisms: commutativity of subdiagram ① follows by the *duality principle* [Lp1, p. 106] from that of the first diagram in [Lp1, p. 103, (3.4.2.2)], and that of ④ is shown similarly.

For another instance, see 1.5.10 below.

1.5. **Idempotent pairs and ⊗-coreflections.** The main results in this section are 1.5.7 and 1.5.13, whose corollary, 1.5.14, motivates much of the subsequent approach to duality theory.

Fix a symmetric monoidal category $\mathbf{D}=(\mathbf{D_0},\otimes,\mathcal{O},\mathsf{a},\mathsf{l},\mathsf{r},\mathsf{s})$.

Sending an object $A\in\mathbf{D}$ to the natural functor $\Gamma_{\!A}\colon\mathbf{D}\to\mathbf{D}$ taking F to $A\otimes F$ gives an *equivalence* from the category $\mathbf{D}$ to the category of $\otimes$*-endofunctors of* $\mathbf{D}$, that is, those functors $\Gamma\colon\mathbf{D}\to\mathbf{D}$ such that there exists a functorial isomorphism $\Gamma\mathcal{O}\otimes F\xrightarrow{\,\sim\,}\Gamma F$. There is a quasi-inverse equivalence taking Γ to $\Gamma\mathcal{O}$.

These quasi-inverse equivalences lift to quasi-inverse equivalences between $\mathbf{D}/\mathcal{O}$ and the category $\mathbf{E}^{\otimes}$ of pairs (Γ,ι) with $\iota\colon\Gamma\to\mathbf{1_D}$ a map of endofunctors of $\mathbf{D}$ such that there exists a functorial isomorphism

(1.5.1) $$\psi(F)\colon\Gamma\mathcal{O}\otimes F\xrightarrow{\,\sim\,}\Gamma F\qquad(F\in\mathbf{D})$$

making the following diagram commute:

(1.5.2) $$\begin{array}{ccc}\Gamma\mathcal{O}\otimes F & \xrightarrow[\psi(F)]{\sim} & \Gamma F\\ {\scriptstyle\iota(\mathcal{O})\otimes\mathbf{1}}\big\downarrow & & \big\downarrow{\scriptstyle\iota(F)}\\ \mathcal{O}\otimes F & \xrightarrow[\mathsf{l}_F]{\sim} & F\end{array}$$

The lifted quasi-inverse equivalences act (objectwise) as follows:

(1.5.3) $$\begin{aligned}&(A,\alpha)\mapsto\big(\Gamma_{\!A},\iota_\alpha\colon\Gamma_{\!A}E=A\otimes E\xrightarrow{\alpha\otimes\mathbf{1}}\mathcal{O}\otimes E\xrightarrow{\mathsf{l}_E}E\big)\qquad(E\in\mathbf{D});\\ &(\Gamma,\iota)\mapsto\big(\Gamma\mathcal{O},\iota(\mathcal{O})\big).\end{aligned}$$

(Note that if $\psi(F)$ is the natural functorial isomorphism $(A\otimes\mathcal{O})\otimes F \xrightarrow{\sim} A\otimes F$, then with $\Gamma := \Gamma_A$ and $\iota := \iota_\alpha$, the diagram (1.5.2) commutes.)

As $\mathsf{l}_{\mathcal{O}} = \mathsf{r}_{\mathcal{O}}$ (see 1.4.6), it is straightforward to verify that (1.5.3) does give rise naturally to quasi-inverse functors, that is, there are functorial isomorphisms

$$(\Gamma_{\Gamma\mathcal{O}}, \iota_{\iota(\mathcal{O})}) \xrightarrow{\sim} (\Gamma,\iota), \qquad \big(\Gamma_A\mathcal{O}, \iota_\alpha(\mathcal{O})\big) \xrightarrow{\sim} (A,\alpha).$$

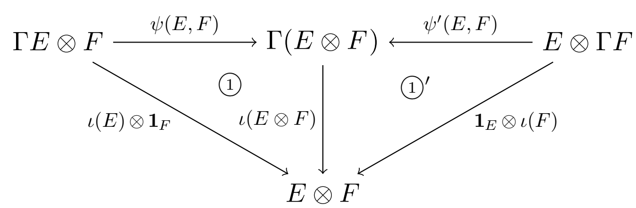
The monoidal structure on $\mathbf{E}^{\otimes}$ corresponding under this lifted equivalence to the one on $\mathbf{D}/\mathcal{O}$ mentioned in 1.4.2(b) has product $\bar{\otimes}$ such that

$$(\Gamma,\iota)\,\bar{\otimes}\,(\Gamma',\iota') = (\Gamma\circ\Gamma',\, \iota\circ\Gamma(\iota')).$$

Proposition 1.5.4. *For any* $(\Gamma,\iota)\in\mathbf{E}^{\otimes}$ *and* $E,F\in\mathbf{D}$ *there are isomorphisms*

$$\Gamma E\otimes F \underset{\psi(E,F)}{\xrightarrow{\sim}} \Gamma(E\otimes F) \underset{\psi'(E,F)}{\xleftarrow{\sim}} E\otimes\Gamma F$$

making the following diagram commute:



If, moreover, (Γ,ι) *is a coreflection of* $\mathbf{D}$, *then* $\psi(E,F)$ *and* $\psi'(E,F)$ *are unique.*

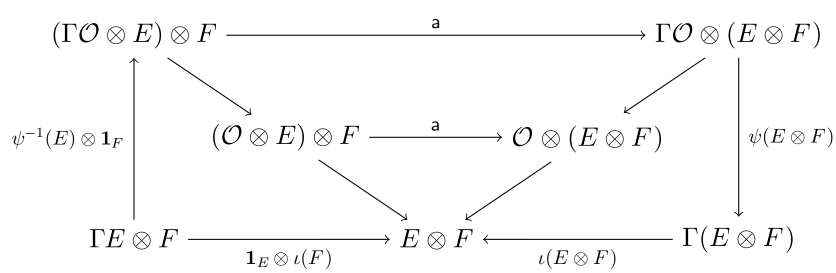
Proof. The easily-checked (via (1.4.1.5) and (1.5.2)) commutativity of the following natural diagram shows that the composite isomorphism

$$\psi(E,F)\colon \Gamma E\otimes F \underset{(1.5.1)}{\xrightarrow{\sim}} (\Gamma\mathcal{O}\otimes E)\otimes F \underset{\mathsf{a}}{\xrightarrow{\sim}} \Gamma\mathcal{O}\otimes(E\otimes F) \underset{(1.5.1)}{\xrightarrow{\sim}} \Gamma(E\otimes F)$$

makes subdiagram ① commute. It follows that the natural composite isomorphism

$$\psi'(E,F)\colon E\otimes\Gamma F \xrightarrow{\sim} \Gamma F\otimes E \xrightarrow{\psi(F,E)} \Gamma(F\otimes E) \xrightarrow{\sim} \Gamma(E\otimes F)$$

makes ①′ commute.



When (Γ,ι) is a coreflection, the unicity of ψ and ψ' follow from 1.3.2 (with both F and G replaced by $E\otimes F$). □

Remark 1.5.5. One checks that a map $\psi(F)$ makes (1.5.2) commute if and only if $\psi(F) = \Gamma\mathsf{l}_F\circ\psi(\mathcal{O},F)$ for some $\psi(\mathcal{O},F)$ as in 1.5.4. Hence when such a $\psi(\mathcal{O},F)$ is unique then so is such a $\psi(F)$.

* * * * *

Definition 1.5.6. A pair (Γ,ι) with $\Gamma\colon \mathbf{D}\to\mathbf{D}$ a functor and $\iota\colon\Gamma\to\mathbf{1}$ a map of functors is a $\otimes$*-coreflection of* $\mathbf{D}$ (or $\otimes$*-coreflecting in* $\mathbf{D}$) if it is a coreflection of $\mathbf{D}$ that lies in $\mathbf{E}^{\otimes}$. The functor Γ is a $\otimes$*-coreflector* if there exists an ι such that (Γ,ι) is a $\otimes$-coreflection.

Proposition 1.5.7. *The above equivalence between* $\mathbf{D}/\mathcal{O}$ *and* $\mathbf{E}^{\otimes}$ (see (1.5.3)) *induces an equivalence between the category* $\mathbf{I_D}$ *of* $\mathbf{D}$*-idempotent pairs and the category of* $\otimes$*-coreflections of* $\mathbf{D}$.

Proof. To be shown is that (A,α) is idempotent if and only if $(\Gamma,\iota):=(\Gamma_A,\iota_\alpha)$ is a $\otimes$-coreflection.

Suppose first that (A,α) is idempotent. As before, $\Gamma:=\Gamma_A$ is a $\otimes$-endofunctor of $\mathbf{D}$. That (Γ,ι) is coreflecting means that for any $E\in\mathbf{D}$, the following diagram commutes and moreover, the maps $\mathbf{1}\otimes(\alpha\otimes\mathbf{1})$ and $(\alpha\otimes\mathbf{1})\otimes\mathbf{1}$ are isomorphisms:

$$\begin{array}{ccc}
A\otimes(A\otimes E) & \xrightarrow{\alpha\otimes(\mathbf{1}\otimes\mathbf{1})} & \mathcal{O}\otimes(A\otimes E)\\
{\scriptstyle \mathbf{1}\otimes(\alpha\otimes\mathbf{1})}\downarrow & & \downarrow{\scriptstyle \mathsf{l}_{A\otimes E}}\\
A\otimes(\mathcal{O}\otimes E) & \xrightarrow[\mathbf{1}\otimes\mathsf{l}_E]{} & A\otimes E
\end{array}$$

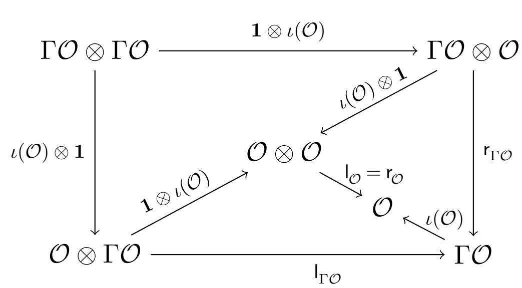
Using (1.4.1.1) and the functoriality of a, one expands this diagram as

$$\begin{array}{ccccccc}
A\otimes(A\otimes E) & \xrightarrow{\mathsf{a}^{-1}} & (A\otimes A)\otimes E & \xrightarrow{(\alpha\otimes\mathbf{1})\otimes\mathbf{1}} & (\mathcal{O}\otimes A)\otimes E & \xrightarrow{\mathsf{a}} & \mathcal{O}\otimes(A\otimes E)\\
{\scriptstyle \mathbf{1}\otimes(\alpha\otimes\mathbf{1})}\downarrow & \square_1 & {\scriptstyle (\mathbf{1}\otimes\alpha)}\downarrow{\scriptstyle \otimes\mathbf{1}} & \square_2 & {\scriptstyle \mathsf{l}_A\otimes}\downarrow{\scriptstyle \mathbf{1}} & \square_3 & \downarrow{\scriptstyle \mathsf{l}_{A\otimes E}}\\
A\otimes(\mathcal{O}\otimes E) & \xrightarrow[\mathsf{a}^{-1}]{} & (A\otimes\mathcal{O})\otimes E & \xrightarrow[\mathsf{r}_A\otimes\mathbf{1}]{} & A\otimes E & = & A\otimes E
\end{array}$$

The top row consists entirely of isomorphisms, so $(\alpha\otimes\mathbf{1})\otimes\mathbf{1}$ is an isomorphism. The commutativity of square $\square_1$ holds because a is functorial, and since $1\otimes\alpha$ is an isomorphism, therefore so is $\mathbf{1}\otimes(\alpha\otimes\mathbf{1})$. The commutativity of $\square_2$ holds by idempotence of (A,α), and of $\square_3$ by (1.4.1.5). So (Γ,ι) is indeed $\otimes$-coreflecting.

Suppose, conversely, that (Γ,ι) is a $\otimes$-coreflection. What's needed is that the maps $p:=\mathsf{l}_{\Gamma\mathcal{O}}\circ(\iota(\mathcal{O})\otimes\mathbf{1})$ and $q:=\mathsf{r}_{\Gamma\mathcal{O}}\circ(\mathbf{1}\otimes\iota(\mathcal{O}))$ from $\Gamma\mathcal{O}\otimes\Gamma\mathcal{O}$ to $\Gamma\mathcal{O}$ are equal.

As in the proof of 1.4.6, $\mathsf{l}_{\mathcal{O}}=\mathsf{r}_{\mathcal{O}}$, and so commutativity of the subdiagrams of the following diagram is clear, whence $\iota(\mathcal{O})\circ p=\iota(\mathcal{O})\circ q$. As there is an isomorphism $\psi(\Gamma\mathcal{O})\colon\Gamma\mathcal{O}\otimes\Gamma\mathcal{O}\xrightarrow{\sim}\Gamma\Gamma\mathcal{O}$ (see (1.5.1)), 1.3.2 implies that, indeed, $p=q$. $\square$



Corollary 1.5.8. *The natural functors taking A to Γ_A (respectively, Γ to $\Gamma\mathcal{O}$) are quasi-inverse equivalences between the category of idempotent* $\mathbf{D}$*-objects and that of* $\otimes$*-coreflectors of* $\mathbf{D}$. $\square$

Proposition 1.5.9. *For an object (A,α) in $\mathbf{D}/\mathcal{O}$, the following are equivalent.*

(i) *(A,α) is a $\mathbf{D}$-idempotent pair.*

(ii) *For all $F,\ G\in\mathbf{D}$ the composite map*

$$j_{F,G}\colon \mathrm{Hom}_{\mathbf{D}}(A\otimes F, A\otimes G)\xrightarrow[\text{via }\alpha]{}\mathrm{Hom}_{\mathbf{D}}(A\otimes F,\mathcal{O}\otimes G)\xrightarrow[\text{via }\mathsf{l}_G]{\sim}\mathrm{Hom}_{\mathbf{D}}(A\otimes F,G)$$

is an isomorphism.

(iii) *The maps $j_{A,A}$ and $j_{A,\mathcal{O}}$ in* (ii) *are injective, and $j_{\mathcal{O},A}$ is surjective.*

Proof. (i) $\Leftrightarrow$ (ii). By Lemma 1.3.2, (ii) says that (Γ_A,ι_α) (see (1.5.3)) is coreflecting, which, by 1.5.7, just means that (A,α) is idempotent.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. In the (obviously) commutative diagram

$$\begin{array}{ccccc}
A\otimes A & \xrightarrow{\alpha\otimes\mathbf{1}_A} & \mathcal{O}\otimes A & \xrightarrow{\mathsf{s}_{\mathcal{O},A}} & A\otimes\mathcal{O}\\
{\scriptstyle\mathbf{1}_A\otimes\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\mathbf{1}_{\mathcal{O}}\otimes\alpha} & & \big\downarrow{\scriptstyle\alpha\otimes\mathbf{1}_{\mathcal{O}}}\\
A\otimes\mathcal{O} & \xrightarrow[\alpha\otimes\mathbf{1}_{\mathcal{O}}]{} & \mathcal{O}\otimes\mathcal{O} & \xrightarrow[\mathsf{s}_{\mathcal{O},\mathcal{O}}]{} & \mathcal{O}\otimes\mathcal{O}
\end{array}$$

the map $\mathsf{s}_{\mathcal{O},\mathcal{O}}$ is the identity of $\mathcal{O}\otimes\mathcal{O}$ (see proof of Lemma 1.4.6), so by injectivity of $j_{A,\mathcal{O}}$, $\mathbf{1}_A\otimes\alpha$ factors as $A\otimes A\xrightarrow{\alpha\otimes\mathbf{1}_A}\mathcal{O}\otimes A\xrightarrow{\mathsf{s}_{\mathcal{O},A}}A\otimes\mathcal{O}$, whence

$$\mathsf{r}_A\circ(\mathbf{1}_A\otimes\alpha)=\mathsf{r}_A\circ\mathsf{s}_{\mathcal{O},A}\circ(\alpha\otimes\mathbf{1}_A)\underset{(1.4.1.4)}{=}\mathsf{l}_A\circ(\alpha\otimes\mathbf{1}_A).$$

It will suffice, therefore, to show that $\alpha\otimes\mathbf{1}_A$ is an isomorphism.

Surjectivity of $j_{\mathcal{O},A}$ entails the existence of a map $\chi\colon A\otimes\mathcal{O}\to A\otimes A$ such that

$$(\alpha\otimes\mathbf{1}_A)\circ\chi=\mathsf{s}_{A,\mathcal{O}}\colon A\otimes\mathcal{O}\xrightarrow{\sim}\mathcal{O}\otimes A,$$

whence $(\alpha\otimes\mathbf{1}_A)\circ\chi\circ\mathsf{s}_{\mathcal{O},A}=\mathbf{1}_{\mathcal{O}\otimes A}$. Moreover,

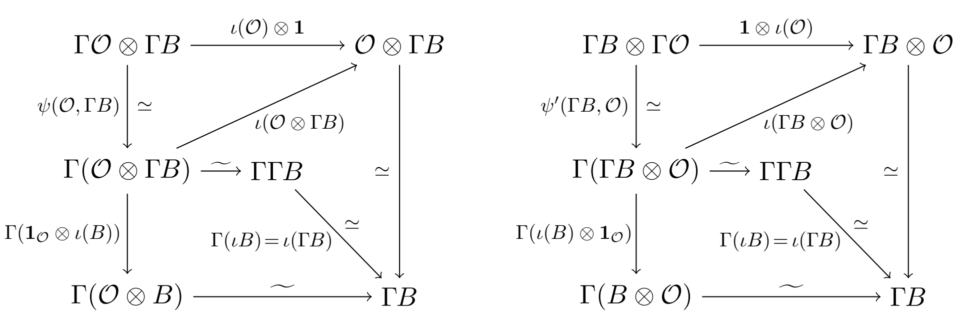
$$(\alpha\otimes\mathbf{1}_A)\circ\chi\circ\mathsf{s}_{\mathcal{O},A}\circ(\alpha\otimes\mathbf{1}_A)=\alpha\otimes\mathbf{1}_A=(\alpha\otimes\mathbf{1}_A)\circ\mathbf{1}_{A\otimes A},$$

and since $j_{A,A}$ is injective, therefore $\chi\circ\mathsf{s}_{\mathcal{O},A}\circ(\alpha\otimes\mathbf{1}_A)=\mathbf{1}_{A\otimes A}$.

Thus $\alpha\otimes\mathbf{1}_A$ is indeed an isomorphism, with inverse $\chi\circ\mathsf{s}_{\mathcal{O},A}$. □

Proposition 1.5.10. *Let (Γ,ι) be a $\otimes$-coreflection of $\mathbf{D}$. If the pair (B,β) is $\mathbf{D}$-idempotent then so is $(\Gamma B,\iota(\mathcal{O})\circ\Gamma\beta)$.*

Proof. One checks, via 1.5.4, that the following natural diagrams commute.



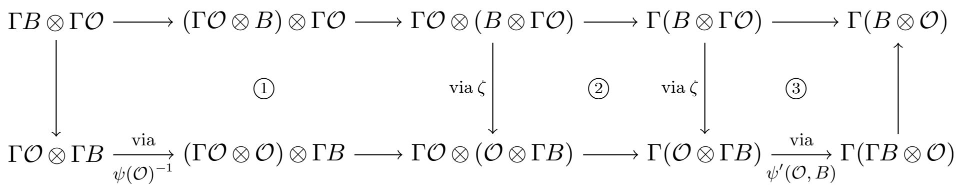
Then by 1.4.8, with $\xi:=\Gamma$, $u:=\iota(\mathcal{O})$ and $v(E,F):=\Gamma(\mathbf{1}_E\otimes\iota(F))\circ\psi(E,\Gamma F)$ (see 1.5.4), Proposition 1.5.10 follows from the fact—to be shown—that

$$\Gamma(\iota(B)\otimes\mathbf{1}_{\mathcal{O}})\circ\psi'(\Gamma B,\mathcal{O})=v(B,\mathcal{O}):=\Gamma(\mathbf{1}_B\otimes\iota(\mathcal{O}))\circ\psi(B,\Gamma\mathcal{O}),$$

that is, the border of the following natural diagram, with $\zeta = \zeta(B)$ the composite isomorphism

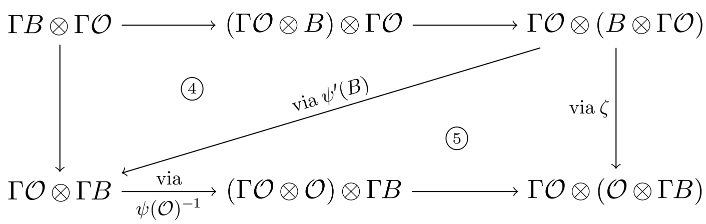
$$B \otimes \Gamma\mathcal{O} \xrightarrow{\psi'(B,\mathcal{O})} \Gamma(B\otimes\mathcal{O}) \xrightarrow{\Gamma(\mathsf{r}_B)} \Gamma B \xrightarrow{\mathsf{l}^{-1}_{\Gamma B}} \mathcal{O}\otimes\Gamma B,$$

commutes.



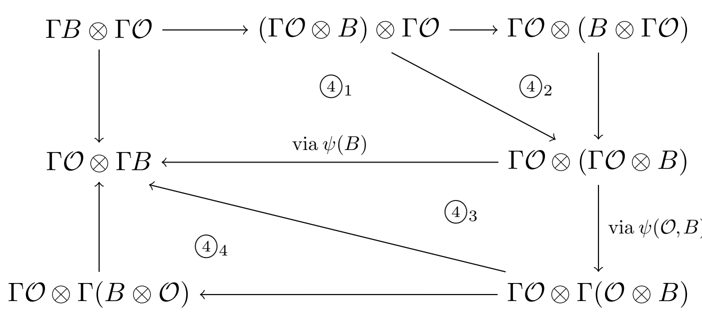
Subdiagram ② clearly commutes.

Subdiagram ① expands as follows, with $\psi'(B) := \Gamma(\mathsf{r}_B)\circ\psi'(B,\mathcal{O})$:



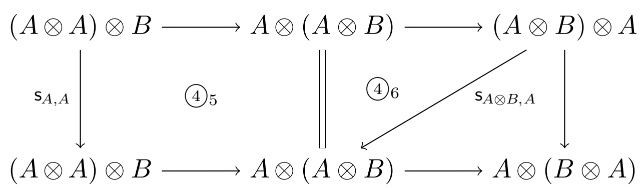
Since $\mathsf{l}_{\mathcal{O}} = \mathsf{r}_{\mathcal{O}}$ (see proof of 1.4.6), it follows from 1.5.5 that $\psi(\mathcal{O}) = \mathsf{r}_{\Gamma\mathcal{O}}$, so by (1.4.1.4), the bottom row composes to the map $\mathbf{1}_{\Gamma\mathcal{O}}\otimes \mathsf{l}^{-1}_{\Gamma B}$. The commutativity of ⑤ results then from the definition of ζ.

Subdiagram ④ expands naturally as



The commutativity of subdiagram ④$_1$ is obvious. That of ④$_3$ is given by 1.5.5, and of ④$_4$ by (1.4.1.4).

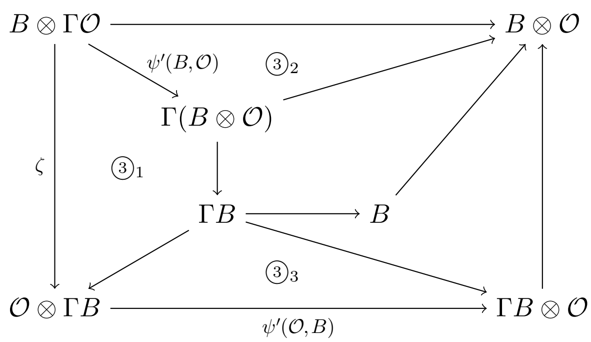
Finally, (1.4.1.3) gives that for any $A \in \mathbf{D}$, the border of the following natural diagram of isomorphisms commutes:



Clearly ④$_6$ commutes. If $A = \Gamma\mathcal{O}$—which, by 1.5.7, is idempotent—then $\mathsf{s}_{A,A}$ is the identity (see 1.4.7), so ④$_5$ commutes, whence so does the unlabeled subdiagram, which is just ④$_2$.

Thus ④, and hence ①, commutes.

Subdiagram ③, with the leading "Γ" omitted, expands naturally as



Subdiagram ③$_1$ commutes by the definition of ζ, and ③$_2$, ③$_3$ by 1.5.4. The commutativity of the unlabeled subdiagrams is obvious. Thus ③ commutes.

This completes the proof of 1.5.10. $\square$

Via 1.5.7, two alternate formulations of 1.5.10 are:

Proposition 1.5.11. (i) *Let* $\mu\colon \mathcal{O}\otimes\mathcal{O} \xrightarrow{\sim} \mathcal{O}$ *be the map* $\mathsf{l}_{\mathcal{O}} = \mathsf{r}_{\mathcal{O}}$ *(see 1.4.6). If* (A,α) *and* (B,β) *are* $\mathbf{D}$*-idempotent pairs, then so is* $\big(A\otimes B, \mu\circ(\alpha\otimes\beta)\big)$.

(ii) *If* (Γ_1, ι_1) *and* (Γ_2, ι_2) *are* $\otimes$*-coreflections of* $\mathbf{D}$ *then so is* $(\Gamma_2\circ\Gamma_1, \iota_2\circ\Gamma_2(\iota_1))$. $\square$

1.5.12. *To illustrate,* let $(X, \mathcal{O}_X)$ be a ringed space. The category $\mathbf{D}(X)$ carries a well-known monoidal structure with $\otimes := \underline{\underline{\otimes}}$, $\mathcal{O} := \mathcal{O}_X$, and $(\mathsf{a},\mathsf{l},\mathsf{r},\mathsf{s})$ the standard isomorphisms. (It suffices to check the axioms on the full subcategory spanned by the K-flat complexes.)[6] If X is a scheme, then $\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}(X)$ contains $\mathcal{O}_X$ and is closed under $\underline{\underline{\otimes}}$, (see [Lp1, 2.5.8.1]); thus it is a *monoidal subcategory* of $\mathbf{D}(X)$.

Recall that if Φ is a finitary s.o.s. in a scheme X, then $\mathbf{R}\varGamma_{\Phi}\mathbf{D}_{\mathrm{qc}}(X) \subset \mathbf{D}_{\mathrm{qc}}(X)$ (Proposition 1.2.2).

Proposition 1.5.13. (i) *Let* X *be a locally noetherian scheme, and* $\mathcal{I}$ *an* $\mathcal{O}_X$*-base. The pair* $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ *is a* $\otimes$*-coreflection of* $\mathbf{D}_{\mathrm{qc}}(X)$ *and of* $\mathbf{D}(X)$.

(ii) *Let* X *be a scheme, and* Φ *a finitary s.o.s. in* X. *The pair* $(\mathbf{R}\varGamma_{\Phi}, \iota_{\Phi})$ *is a* $\otimes$*-coreflection of* $\mathbf{D}_{\mathrm{qc}}(X)$.

Proof. (i). As in 1.3.3(c), $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ is a coreflection of $\mathbf{D}_{\mathrm{qc}}(X)$ and of $\mathbf{D}(X)$.

Moreover, there is a natural functorial map

(1.5.13.1) $$\psi_{\mathcal{I}}(E,F)\colon \mathbf{R}\varGamma_{\mathcal{I}}E \underline{\underline{\otimes}}_X F \to \mathbf{R}\varGamma_{\mathcal{I}}(E \underline{\underline{\otimes}}_X F) \qquad (E, F \in \mathbf{D}(X)),$$

defined as follows.

[6] An $\mathcal{O}_X$-complex P is *K-flat* if for every $\mathcal{O}_X$-quasi-isomorphism $Q_1 \to Q_2$ the resulting map $P\otimes Q_1 \to P\otimes Q_2$ is also a quasi-isomorphism; or equivalently, if for every exact $\mathcal{O}_X$-complex Q, the complex $P\otimes Q$ is also exact. Every $\mathcal{O}_X$-complex Q admits a *K-flat resolution,* i.e., there exists a quasi-isomorphism $P \to Q$ with P K-flat [Spn, p. 139, 5.6]. If P is K-flat then for any $\mathcal{O}_X$-complex Q, the natural maps, with $\underline{\underline{\otimes}}$ denoting *left-derived tensor product,* are isomorphisms $P \underline{\underline{\otimes}} Q \xrightarrow{\sim} P\otimes Q$, $\quad Q \underline{\underline{\otimes}} P \xrightarrow{\sim} Q\otimes P$, see [Spn, p. 147, 6.5], [Lp1, §2.5].

Note first that for any $\mathcal{O}_X$-complexes E, F, one has $\varGamma_{\mathcal{J}} E \otimes_X F = \varGamma_{\mathcal{J}}(\varGamma_{\mathcal{J}} E \otimes_X F)$: by 1.1.19, it's enough to show this when E and F are $\mathcal{O}_X$-modules, a simple task left to the reader. Hence if E is K-injective and F is K-flat, and $E \otimes_X F \to G$ is a K-injective resolution, then the image of the natural composite map

$$\varGamma_{\mathcal{J}} E \otimes_X F \to E \otimes_X F \to G$$

lies in $\varGamma_{\mathcal{J}} G$. Via standard considerations (e.g., [Lp1, p. 69, 2.6.5]), the map $\psi(E, F)$ for *arbitrary* $E, F \in \mathbf{D}(X)$ results.

From this description of $\psi_{\mathcal{J}}(E, F)$ one gets a commutative diagram

$$\begin{array}{ccc} \mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X \underline{\otimes}_X F & \xrightarrow{\psi_{\mathcal{J}}(\mathcal{O}_X, F)} & \mathbf{R}\varGamma_{\mathcal{J}} F \\ {\scriptstyle \iota_{\mathcal{J}}(\mathcal{O}_X)\otimes \mathbf{1}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{\mathcal{J}}(F)} \\ \mathcal{O}_X \underline{\otimes}_X F & \underset{\mathsf{l}_F}{\xrightarrow{\ \sim\ }} & F \end{array}$$

It remains to be shown that $\psi_{\mathcal{J}}(\mathcal{O}_X, F)$ is an isomorphism (see (1.5.2)).

Actually, $\psi_{\mathcal{J}}(E, F)$ is an isomorphism for all E. This assertion is local, so assume $X = \operatorname{Spec}(R)$ (R a noetherian ring). If $\mathcal{J} = \mathcal{J}_J$ for some quasi-coherent $\mathcal{O}_X$-ideal J, then by [AJL1, (3.1.2)], $\psi_{\mathcal{J}}(E, F)$ is indeed an isomorphism. Thus for arbitrary $\mathcal{J}$, the natural composite maps

$$\varGamma_I E \otimes_X F \to \varGamma_I(E \otimes_X F) \to \varGamma_I G \qquad (I \in \mathcal{J})$$

are all quasi-isomorphisms, and one can apply $\varinjlim_{I\in\mathcal{J}}$ to get a quasi-isomorphism $\varGamma_{\mathcal{J}} E \otimes_X F \to \varGamma_{\mathcal{J}} G$, whose $\mathbf{D}(X)$-image $\psi_{\mathcal{J}}(E, F)$ is an isomorphism, as desired.

(ii). Proceed as in the proof of (i), with Φ in place of $\mathcal{J}$ and [AJL1, p. 25, (3.2.5)(i)] in place of [AJL1, p. 20, (3.1.2)].

Alternatively, assuming—as one may—that X is affine, check, using Proposition 1.2.16, that the $E \in \mathbf{D}_{\mathrm{qc}}(X)$ for which $\psi_\Phi(E)$ is an isomorphism span a localizing subcategory $\mathbf{D}_\otimes \subset \mathbf{D}_{\mathrm{qc}}(X)$. Since $\mathcal{O}_X \in \mathbf{D}_\otimes$, [Nm2, p. 222, Lemma 3.2] gives $\mathbf{D}_\otimes = \mathbf{D}_{\mathrm{qc}}(X)$. □

From 1.5.13 and 1.5.7 one gets:

Corollary 1.5.14. *Let X be a locally noetherian scheme, and $\mathcal{J}$ an $\mathcal{O}_X$-base. The pair $(\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X, \iota_{\mathcal{J}}(\mathcal{O}_X))$ is $\mathbf{D}_{\mathrm{qc}}(X)$-idempotent and $\mathbf{D}(X)$-idempotent.*

More generally (see 1.2.3), if Φ is a finitary s.o.s. in a scheme X, then the pair $(\mathbf{R}\varGamma_\Phi\mathcal{O}_X, \iota_\Phi(\mathcal{O}_X))$ is $\mathbf{D}_{\mathrm{qc}}(X)$-idempotent, hence $\mathbf{D}(X)$-idempotent. □

1.6. **Morphisms of idempotent pairs.** Notation remains as in Section 1.4.

Proposition 1.6.1. *Let (A, α) and (B, β) be $\mathbf{D}$-idempotent pairs.*

There is at most one *morphism $\lambda\colon (B, \beta) \to (A, \alpha)$. Such a λ exists if and only if $\mathsf{l}_B \circ (\alpha \otimes \mathbf{1}_B)\colon A \otimes B \to B$ is an isomorphism.*

Proof. We'll need:

Lemma 1.6.2. *Let (C, γ) be a $\mathbf{D}$-idempotent pair, and $(B, \beta) \in \mathbf{D}/\mathcal{O}$. Suppose the $\mathbf{D}$-maps $B \xrightarrow{q} C \xrightarrow{p} B$ satisfy $\beta p = \gamma$ and $pq = \mathbf{1}_B$. Then p is an isomorphism.*

Proof. The composition $\mathbf{1}_C \otimes \gamma\colon C \otimes C \xrightarrow{\mathbf{1}_C \otimes p} C \otimes B \xrightarrow{\mathbf{1}_C \otimes \beta} C \otimes \mathcal{O}$ is, by 1.4.3, an isomorphism. So $\mathbf{1}_C \otimes p$ has both a left inverse and a right inverse, and thus must be an isomorphism.

In the following commutative diagram, the isomorphisms $j_{\bullet,\bullet}$ are as in 1.5.9 (with (A,α) replaced by (C,γ)):

$$\begin{array}{ccc} \mathrm{Hom}(C, C\otimes C) & \xrightarrow[\text{via } \mathbf{1}_C\otimes p]{\sim} & \mathrm{Hom}(C, C\otimes B) \\ j_{\mathcal{O},C}\big\downarrow\simeq & & \simeq\big\downarrow j_{\mathcal{O},B} \\ \mathrm{Hom}(C, \mathcal{O}\otimes C) & \xrightarrow[\text{via } \mathbf{1}_{\mathcal{O}}\otimes p]{} & \mathrm{Hom}(C, \mathcal{O}\otimes B) \end{array}$$

Hence $\phi \mapsto p\phi$ is an isomorphism from $\mathrm{Hom}(C,C)$ to $\mathrm{Hom}(C,B)$; and since

$$pqp = \mathbf{1}_B p = p = p\mathbf{1}_C,$$

therefore $qp = \mathbf{1}_C$, so p is indeed an isomorphism. □

Assuming that λ exists, and having in mind Remark 1.4.7 and Proposition 1.5.11, one finds that Lemma 1.6.2, with $(C,\gamma) := (A\otimes B, \mathsf{l}_{\mathcal{O}}\circ(\alpha\otimes\beta))$, applies to

$$B \xrightarrow{\mathsf{l}_B^{-1}} \mathcal{O}\otimes B \xrightarrow{(\beta\otimes\mathbf{1}_B)^{-1}} B\otimes B \xrightarrow{\lambda\otimes\mathbf{1}_B} A\otimes B \xrightarrow{\alpha\otimes\mathbf{1}_B} \mathcal{O}\otimes B \xrightarrow{\mathsf{l}_B} B,$$

giving that $\alpha\otimes\mathbf{1}_B$ is an isomorphism, whence so is $\mathsf{l}_B\circ(\alpha\otimes\mathbf{1}_B)$; and conversely, the composites

$$B \xleftarrow{\mathsf{l}_B} \mathcal{O}\otimes B \xleftarrow{\alpha\otimes\mathbf{1}_B} A\otimes B \qquad \text{and} \qquad A\otimes B \xrightarrow{\mathbf{1}_A\otimes\beta} A\otimes\mathcal{O} \xrightarrow{\mathsf{r}_A} A$$

are $\mathbf{D}/\mathcal{O}$-morphisms, so if $\alpha\otimes\mathbf{1}_B$ is an isomorphism then λ exists.

Uniqueness of λ results from the following isomorphisms (the first and third induced by $\mathsf{l}_B\circ(\alpha\otimes\mathbf{1}_B)\colon A\otimes B \xrightarrow{\sim} B$), whose composition takes λ to $\alpha\lambda = \beta$:

$$\mathrm{Hom}(B,A) \xrightarrow{\sim} \mathrm{Hom}(A\otimes B, A\otimes\mathcal{O}) \xrightarrow[j_{B,\mathcal{O}}]{\sim} \mathrm{Hom}(A\otimes B,\mathcal{O}) \xrightarrow{\sim} \mathrm{Hom}(B,\mathcal{O}).$$ □

Remark 1.6.3. Recall from Remark 1.4.5 that the $\mathbf{D}$-idempotent pairs span a strictly full subcategory $\mathbf{I_D}$ of the slice category $\mathbf{D}/\mathcal{O}$.

It follows from Proposition 1.6.1 that $\mathbf{I_D}$ *is a preordered monoidal category* (see Example 1.4.4(d)). Indeed, $(\mathcal{O}, \text{identity})$ is clearly a largest object; and, maps of idempotent pairs $(C,\gamma)\to(A,\alpha)$ and $(C,\gamma)\to(B,\beta)$ give rise naturally to a composite $\mathbf{D}/\mathcal{O}$-map $(C,\gamma) \xrightarrow{\sim} (C\otimes C, \mu\circ(\gamma\otimes\gamma)) \to (A\otimes B, \mu\circ(\alpha\otimes\beta))$, whence $(A\otimes B, \mu\circ(\alpha\otimes\beta))$ is, via the maps $\mathsf{r}\circ(\mathbf{1}_A\otimes\beta)$ and $\mathsf{l}\circ(\alpha\otimes\mathbf{1}_B)$, a greatest lower bound for (A,α) and (B,β). So the unit object and the product functor in $\mathbf{I_D}$ are the same as those in $\mathbf{D}/\mathcal{O}$, and the associated functorial maps a, l, r and s are necessarily the same as those inherited from $\mathbf{D}/\mathcal{O}$.

Remark 1.6.4. If (A,β) and (A,α) are idempotent pairs then there is a unique $\lambda\colon A\to A$ such that $\beta = \alpha\lambda$. This λ is an *automorphism*, with inverse the unique $\lambda'\colon A\to A$ such that $\alpha = \beta\lambda'$. (By 1.6.1, $\alpha\lambda\lambda' = \alpha 1_A \implies \lambda\lambda' = 1_A$; and similarly, $\lambda'\lambda = 1_A$.) Explicitly, using 1.4.7 and the functoriality of s, one finds that

$$\alpha\otimes\beta = \beta\otimes\alpha\colon A\otimes A \to \mathcal{O}\otimes\mathcal{O},$$

whence the following diagram commutes:

$$\begin{array}{ccccccc}
\mathcal{O}\otimes A & \xleftarrow{\alpha\otimes 1} & A\otimes A & \xrightarrow{1\otimes\beta} & A\otimes\mathcal{O} & = & A\otimes\mathcal{O}\\
{\scriptstyle \mathsf{l}}\downarrow & & {\scriptstyle 1\otimes\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\otimes 1} & & \\
A & \xleftarrow[\mathsf{r}]{} & A\otimes\mathcal{O} & \xrightarrow[\beta\otimes 1]{} & \mathcal{O}\otimes\mathcal{O} & & \downarrow{\scriptstyle\mathsf{r}}\\
\| & & & & \downarrow{\scriptstyle\mathsf{r}} & & \\
A & & \xrightarrow[\beta]{\qquad\qquad} & & \mathcal{O} & \xleftarrow[\alpha]{} & A,
\end{array}$$

so that λ is the composite isomorphism

$$A \xrightarrow[\mathsf{l}^{-1}]{\sim} \mathcal{O}\otimes A \xrightarrow[(\alpha\otimes\mathbf{1})^{-1}]{\sim} A\otimes A \xrightarrow[\mathbf{1}\otimes\beta]{\sim} A\otimes\mathcal{O} \xrightarrow[\mathsf{r}]{\sim} A.$$

Conversely, it follows, e.g., from (i) $\Leftrightarrow$ (ii) in 1.5.8, that if (A,α) is idempotent and $\lambda\colon B \xrightarrow{\sim} A$ is a $\mathbf{D}$-isomorphism then $(B,\alpha\lambda)$ is idempotent.

Thus, *the automorphism group of any idempotent $A\in\mathbf{D}$ acts faithfully and transitively on the set of $\alpha\colon A\to\mathcal{O}$ such that (A,α) is an idempotent pair.*

Remark 1.6.5. *Idempotent pairs (B,β) and (A,α) are isomorphic $\Leftrightarrow$ there exists a $\mathbf{D}$-isomorphism $\lambda\colon B\xrightarrow{\sim} A$.* The implication $\Rightarrow$ is trivial. Conversely, if such a λ exists then (B,β) and $(B,\alpha\lambda)$ are both idempotent whence, as in 1.6.4, there is an automorphism $\kappa\colon B\to B$ such that $\beta=\alpha\lambda\kappa$; so $\lambda\kappa\colon(B,\beta)\to(A,\alpha)$ is an isomorphism of idempotent pairs.

Definition 1.6.6. For idempotent B and A, $B\preccurlyeq A$ means there exist β and α and a map—unique, by 1.6.1—of idempotent pairs $(B,\beta)\to(A,\alpha)$, a condition which is independent of the choice of β and α.

By Remark 1.6.5, B is $\mathbf{D}$-isomorphic to $A \iff B\preccurlyeq A$ and $A\preccurlyeq B$.

Of course $A\preccurlyeq A$, and $C\preccurlyeq B$ together with $B\preccurlyeq A$ implies $C\preccurlyeq A$. So we have a preordering on the idempotent $\mathbf{D}$-objects, such that $\mathcal{O}$ is a largest object and, as in Remark 1.6.3, $A\otimes B$ is a greatest lower bound for A and B.

* * * * *

Definition 1.6.7. For $A\in\mathbf{D}$, the category $\mathbf{D}_A := \mathbf{D}_{\Gamma_A}\subset\mathbf{D}$ is the essential image of the functor $\Gamma_A(-) := A\otimes -$.

Lemma 1.6.8. *If $\alpha\colon A\to\mathcal{O}$ is a $\mathbf{D}$-morphism such that $\alpha\otimes\mathbf{1}\colon A\otimes A\to\mathcal{O}\otimes A$ is an isomorphism, then $E\in\mathbf{D}_A$ if and only if*

$$\iota_\alpha(E) := \mathsf{l}_E\circ(\alpha\otimes\mathbf{1})\colon A\otimes E\to E$$

is an isomorphism.

Proof. "If" is trivial, and "only if" follows from the commutativity of the following diagram, with $F\in\mathbf{D}$ such that $A\otimes F\cong E$ (see (1.4.1.5)):

$$\begin{array}{ccccc}
A\otimes(A\otimes F) & \xrightarrow{\alpha\otimes(\mathbf{1}\otimes\mathbf{1})} & \mathcal{O}\otimes(A\otimes F) & \xrightarrow{\mathsf{l}_{A\otimes F}} & A\otimes F\\
{\scriptstyle\mathsf{a}}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\mathsf{a}} & & \|\\
(A\otimes A)\otimes F & \xrightarrow[(\alpha\otimes\mathbf{1})\otimes\mathbf{1}]{\sim} & (\mathcal{O}\otimes A)\otimes F & \xrightarrow[\mathsf{l}_A\otimes\mathbf{1}]{\sim} & A\otimes F
\end{array}$$

$\square$

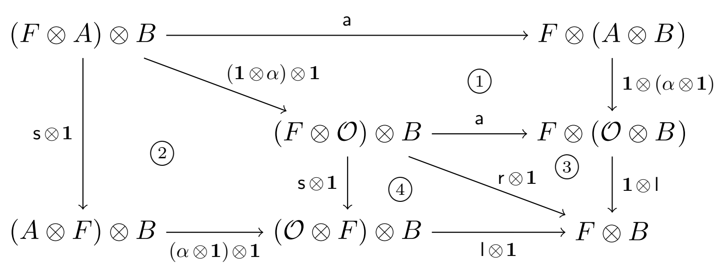
If $\alpha\colon A \to \mathcal{O}$ is as in 1.6.8 then for $E \in \mathbf{D}_A$, one has the functorial isomorphism $\mathsf{l}_\alpha(E) := \iota_\alpha(E)$; and likewise, the functorial isomorphism

$$\mathsf{r}_\alpha(E) := \mathsf{r}_E \circ (\mathbf{1} \otimes \alpha)\colon E \otimes A \xrightarrow{\sim} E.$$

Clearly, $A \cong A \otimes \mathcal{O} \in \mathbf{D}_A$. For any $E \cong A \otimes G \in \mathbf{D}_A$ and $F \in \mathbf{D}$, one has $E \otimes F \in \mathbf{D}_A$ and $F \otimes E \in \mathbf{D}_A$. In particular, $\mathbf{D}_A$ is closed under $\otimes$.

Lemma 1.6.9. *Let $\alpha\colon A \to \mathcal{O}$ be as in 1.6.8, and $\mathbf{D}_* \subset \mathbf{D}_A$ a full subcategory such that $A \in \mathbf{D}_*$ and such that if $E, F \in \mathbf{D}_*$ then $E \otimes F \in \mathbf{D}_*$. Then $(\otimes, A, \mathsf{a}, \mathsf{l}_\alpha, \mathsf{r}_\alpha, \mathsf{s})$ is a monoidal structure on $\mathbf{D}_*$.*

Proof. For any F and B in $\mathbf{D}$, one has the diagram



The subdiagrams commute: ① and ② clearly, ③ by (1.4.1.1), and ④ by (1.4.1.4). Therefore ③ plus ① give that (1.4.1.1) with $(A, \mathcal{O}, \mathsf{r}, \mathsf{l})$ replaced by $(F, A, \mathsf{r}_\alpha, \mathsf{l}_\alpha)$ commutes; and with $B := \mathcal{O}$, ④ plus ② give that (1.4.1.4) with $(A, \mathcal{O}, \mathsf{r}, \mathsf{l})$ replaced by $(F, A, \mathsf{r}_\alpha, \mathsf{l}_\alpha)$ commutes. The rest is obvious. □

In 1.6.9, l_α and r_α depend on α. However, for $\mathbf{D}/\mathcal{O}$-isomorphic objects $\alpha\colon A \to \mathcal{O}$ and $\alpha'\colon A' \to \mathcal{O}$ as in 1.6.8, it holds that $\mathbf{D}_A = \mathbf{D}_{A'}$, and the monoidal structures on $\mathbf{D}_*$ induced by α and α' are *equivalent,* where equivalence of monoidal structures $(\otimes, \mathcal{O}, \mathsf{a}, \mathsf{l}, \mathsf{r}, \mathsf{s})$ and $(\otimes, \mathcal{O}', \mathsf{a}, \mathsf{l}', \mathsf{r}', \mathsf{s})$ means, with $\lambda\colon \mathcal{O}' \to \mathcal{O}$ the isomorphism $\mathsf{r}'_{\mathcal{O}} \circ (\mathsf{l}^{-1}_{\mathcal{O}'}) = \mathsf{l}'_{\mathcal{O}} \circ (\mathsf{r}^{-1}_{\mathcal{O}'})$ (see (1.4.1.4)), that for all $E \in \mathbf{D_0}$ one has

$$\mathsf{l}'_E = \mathsf{l}_E \circ (\lambda \otimes \mathbf{1}_E)\colon \mathcal{O}' \otimes E \to E \quad \text{and} \quad \mathsf{r}'_E = \mathsf{r}_E \circ (\mathbf{1}_E \otimes \lambda)\colon E \otimes \mathcal{O}' \to E;$$

in other words, the identity functor of $\mathbf{D}_*$ along with the identity map of $E \otimes F$ $(E, F \in \mathbf{D}_*)$ and the isomorphism λ form an isomorphism of monoidal categories. (Details are left to the reader.)

Proposition 1.6.10. (i) *For $\mathbf{D}$-idempotents B and A,*

$$B \precsim A \iff B \in \mathbf{D}_A \iff \mathbf{D}_B \subset \mathbf{D}_A.$$

In particular, $\mathbf{D}_B = \mathbf{D}_A \iff B \cong A$.

(ii) *Let (A, α) be a $\mathbf{D}$-idempotent pair, and let $\mathbf{D}_A$ have the monoidal structure given in 1.6.9. The map Θ_α that sends $(B, \lambda) \in \mathbf{D}/A$ to $(B, \alpha\lambda) \in \mathbf{D}/\mathcal{O}$ restricts to a bijection from the set of $\mathbf{D}_A$-idempotent pairs to the set of $\mathbf{D}$-idempotent pairs (B, β) such that $B \precsim A$.*

Thus the $\mathbf{D}_A$-idempotents are just the $\mathbf{D}$-idempotents B such that $B \precsim A$.

Proof. (i) Left to the reader. (See Proposition 1.6.1.)

(ii) Let (B,λ) be a $\mathbf{D}_A$-idempotent pair. Since $\mathsf{s}_{B,B}\colon B\otimes B \xrightarrow{\,\sim\,} B\otimes B$ is the identity map (see 1.4.7), (1.4.1.4) ensures that the following diagram commutes:

$$\begin{array}{ccccccc}
B\otimes B & \xrightarrow[\mathbf{1}\otimes\lambda]{\sim} & B\otimes A & \xrightarrow[\mathbf{1}\otimes\alpha]{\sim} & B\otimes\mathcal{O} & \xrightarrow[\mathsf{r}_B]{\sim} & B\\
\Big\| & & {\scriptstyle \mathsf{s}_{B,A}}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle \mathsf{s}_{B,\mathcal{O}}} & & \Big\|\\
B\otimes B & \xrightarrow[\lambda\otimes\mathbf{1}]{\sim} & A\otimes B & \xrightarrow[\alpha\otimes\mathbf{1}]{\sim} & \mathcal{O}\otimes B & \xrightarrow[\mathsf{l}_B]{\sim} & B
\end{array}$$

Proposition 1.6.1 gives that $\mathsf{l}_B\circ(\alpha\otimes\mathbf{1})$ is an isomorphism, whence so is $\mathsf{r}_B\circ(\mathbf{1}\otimes\alpha)$, as are $\alpha\otimes\mathbf{1}$ and $\mathbf{1}\otimes\alpha$. Hence $(B,\alpha\lambda)$ is $\mathbf{D}$-idempotent; and λ is a map of $\mathbf{D}$-idempotent pairs $(B,\alpha\lambda)\to(A,\alpha)$, so that $B\preccurlyeq A$. Moreover, if (B,λ') is a $\mathbf{D}_A$-idempotent pair such that $\alpha\lambda'=\alpha\lambda$, then λ and λ' are maps from $(B,\alpha\lambda)$ to (A,α), so by Proposition 1.6.1, $\lambda=\lambda'$. Thus Θ_α acts injectively on $\mathbf{D}_A$-idempotent pairs.

Suppose (B,β) is a $\mathbf{D}$-idempotent pair such that $B\preccurlyeq A$, so that there exists a map of idempotent pairs $\lambda\colon (B,\beta)\to(A,\alpha)$. Then $\mathbf{1}\otimes\lambda\colon B\otimes B\to B\otimes A$ is an isomorphism, because its composition with the isomorphism $1\otimes\alpha\colon B\otimes A\to B\otimes\mathcal{O}$ is the isomorphism $\mathbf{1}\otimes\beta$. Similarly, $\lambda\otimes\mathbf{1}$ is an isomorphism.

Again, $\mathsf{s}_{B,B}$ is the identity map, so the preceding commutative diagram gives

$$\mathsf{r}_\alpha(B)\circ(\mathbf{1}\otimes\lambda)=\mathsf{l}_\alpha(B)\circ(\lambda\otimes\mathbf{1}),$$

so that (B,λ) is a $\mathbf{D}_A$-idempotent pair; and $\Theta_\alpha(B,\lambda)=(B,\alpha\lambda)=(B,\beta)$. Thus Θ_α is surjective, as well as injective.

Verifying the last assertion is now straightforward. □

Corollary 1.6.11. *Let (A,α) and (B,β) be $\mathbf{D}$-idempotent pairs.*

(i) $\big(A\otimes B,\mathsf{r}_A\circ(\mathbf{1}_A\otimes\beta)\big)$ *is $\mathbf{D}_A$-idempotent.*

(ii) $\big(A\otimes B,\mathsf{l}_B\circ(\alpha\otimes\mathbf{1}_B)\big)$ *is $\mathbf{D}_B$-idempotent.*

Proof. By 1.5.11, it holds that

$$\big(A\otimes B,\alpha\circ\mathsf{r}_A\circ(\mathbf{1}_A\otimes\beta)\big)=\big(A\otimes B,\mathsf{r}_\mathcal{O}\circ(\alpha\otimes\beta)\big)$$

is $\mathbf{D}$-idempotent, and so (i) results as in the latter part of the proof of 1.6.10(ii). The proof of (ii) is similar. □

* * * * *

A *closed category* is a monoidal category $\mathbf{D}$ (with product functor $\otimes$) together with an *internal hom* functor

$$[-,-]\colon \mathbf{D}^{\mathrm{op}}\times\mathbf{D}\to\mathbf{D}$$

and a trifunctorial isomorphism

$$\text{(1.6.12)}\qquad \boldsymbol{h}\colon \mathrm{Hom}_{\mathbf{D}}(E\otimes F,G)\xrightarrow{\,\sim\,}\mathrm{Hom}_{\mathbf{D}}(E,[F,G])\qquad (E,F,G\in\mathbf{D}).$$

(See, e.g., [Lp1, Definition (3.5.1)] and the references following it.)

Elementary considerations show that the existence and functoriality of $\boldsymbol{h}$ are equivalent to the existence for all F and G of an *evaluation map,* functorial in G,

$$\text{(1.6.12)}'\qquad \mathrm{ev}=\mathrm{ev}_{F,G}\colon [F,G]\otimes F\to G$$

such that for all E, the map taking $\phi\colon E\to[F,G]$ to the map $\mathrm{ev}\circ(\phi\otimes\mathbf{1})\colon E\otimes F\to G$ is an isomorphism $\mathrm{Hom}_{\mathbf{D}}(E,[F,G])\xrightarrow{\,\sim\,}\mathrm{Hom}_{\mathbf{D}}(E\otimes F,G)$, and also such that for any map $F \to F'$ the following naturally induced diagram commutes:

$$\begin{array}{ccc} [F',G]\otimes F & \longrightarrow & [F,G]\otimes F \\ \big\downarrow & & \big\downarrow \scriptstyle{\text{ev}} \\ [F',G]\otimes F' & \xrightarrow[\text{ev}]{} & G \end{array} \tag{1.6.12$''$}$$

By definition, $\mathbf{D}$ has a unit object $\mathcal{O}$ equipped with a functorial isomorphism

$$\mathsf{r}_G\colon G\otimes\mathcal{O} \xrightarrow{\ \sim\ } G \qquad (G\in\mathbf{D}). \tag{1.6.13}$$

The natural composite isomorphism

$$\operatorname{Hom}(F,G)\cong\operatorname{Hom}(F\otimes\mathcal{O},G)\cong\operatorname{Hom}(F,[\mathcal{O},G]) \qquad (F,G\in\mathbf{D})$$

takes the identity map $\mathbf{1}_G$ to a functorial isomorphism

$$G \xrightarrow{\ \sim\ } [\mathcal{O},G] \tag{1.6.13$'$}$$

corresponding under 1.6.12 to r_G.

For $(A,\alpha)\in\mathbf{D}/\mathcal{O}$, one shows, via (1.6.12)$''$ with $F\to F'$ the map $\alpha\colon A\to\mathcal{O}$, that the map $G\otimes A\to G\otimes\mathcal{O}\cong G$ induced by α factors as

$$G\otimes A \underset{(1.6.13)'}{\xrightarrow{\ \sim\ }} [\mathcal{O},G]\otimes A \underset{\text{via }\alpha}{\xrightarrow{\ \sim\ }} [A,G]\otimes A \underset{\text{ev}}{\longrightarrow} G. \tag{1.6.14}$$

In particular, the evaluation map is an isomorphism $[\mathcal{O},G]\otimes\mathcal{O} \xrightarrow{\ \sim\ } G$.

In [AJL2, pp. 69–70], there are a number of formal relations which hold for any $\mathbf{D}$-coreflector $\boldsymbol{\varGamma}$ that has a right adjoint $\mathbf{\Lambda}$—for example, if (A,α) is $\mathbf{D}$-idempotent, the natural adjoint functors specified objectwise by

$$\boldsymbol{\varGamma}G := G\otimes A, \qquad \mathbf{\Lambda}G := [A,G] \qquad (G\in\mathbf{D}).$$

One such relation is the existence of an isomorphism $\boldsymbol{\varGamma} \xrightarrow{\ \sim\ } \boldsymbol{\varGamma}\mathbf{\Lambda}$, which, for the preceding example, is just the composition of the first two maps in (1.6.14).

Remarks 1.6.15. (a) Internal hom is related to $\operatorname{Hom}_{\mathbf{D}}$ thus: for $G\in\mathbf{D}$ set

$$\mathrm{H}^0G := \operatorname{Hom}_{\mathbf{D}}(\mathcal{O},G); \tag{1.6.15.1}$$

then there are natural isomorphisms

$$\mathrm{H}^0[E,F] \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}}(\mathcal{O}\otimes E,F) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}}(E,F) \qquad (E,F\in\mathbf{D}). \tag{1.6.15.2}$$

(b) Let $\mathbf{D}$ be a closed category having an initial object A, and $\alpha\colon A\to\mathcal{O}$ the unique morphism. Then (A,α) is idempotent. Indeed, 1.6.12 shows, for any $F\in\mathbf{D}$, that $A\otimes F$ is also an initial object, so that there are unique maps $A\otimes F\to A$ and $A\to A\otimes F$, both isomorphisms. In particular, $\mathsf{r}\circ(\mathbf{1}\otimes\alpha)$ and $\mathsf{l}\circ(\alpha\otimes\mathbf{1})$ are equal isomorphisms from $A\otimes A$ to A.

Clearly, $(A,\alpha)\preccurlyeq(B,\beta)$ for any idempotent (B,β), i.e., (A,α) is initial in $\mathbf{I}_{\mathbf{D}}$.

Examples 1.6.16. (a) For a ringed space $(X,\mathcal{O}_X)$, the derived category $\mathbf{D}(X)$ is closed, with product $\underline{\underline{\otimes}}_S$ (derived tensor product, see footnote in section 1.5.12), unit $\mathcal{O}_X$, and $[E,F] := \mathbf{R}\mathrm{Hom}^{\bullet}_X(E,F)$. (For (1.6.12) see e.g. [Spn, p. 147, 6.6], or in more detail, [Lp1, §2.6].) The maps $(\mathsf{a},\mathsf{l},\mathsf{r},\mathsf{s})$ are the obvious ones.

In particular, if $S$is a ring (i.e., a ringed space $(X,\mathcal{O}_X)$ with X a single point), then $\mathbf{D}(S)$ is closed.

(b) Suppose $\mathbf{D}$ is closed, let (A,α) be a $\mathbf{D}$-idempotent pair, and let $\mathbf{D}_A \subset \mathbf{D}$ be the corresponding monoidal category (see Definition 1.6.7 and Lemma 1.6.9). The natural isomorphisms, with $E, F, G \in \mathbf{D}_A$,

$$\operatorname{Hom}_{\mathbf{D}}(G\otimes E, F) \underset{(1.6.12)}{\xrightarrow{\sim}} \operatorname{Hom}_{\mathbf{D}}(G, [E,F]) \underset{(1.5.9)}{\xrightarrow{\sim}} \operatorname{Hom}_{\mathbf{D}}(G, [E,F]\otimes A)$$

show that $\mathbf{D}_A$ is a closed category, whose internal hom is $[E,F]_A := [E,F]\otimes A$.

When $\mathbf{D}$ is closed one can expand on 1.5.9 and 1.3.2 in terms of $[-,-]$:

Corollary 1.6.17. *For a closed category* $\mathbf{D}$, *and* $(A,\alpha) \in \mathbf{D}/\mathcal{O}$, 1.5.9(ii) *holds if and only if for all* $F, G \in \mathbf{D}$ *the following composite map is an isomorphism*:

(1.6.17.1) $$[A\otimes F, A\otimes G] \xrightarrow[\text{via } \alpha]{} [A\otimes F, \mathcal{O}\otimes G] \xrightarrow[\text{via } \mathsf{l}]{\sim} [A\otimes F, G].$$

Consequently (*see* (1.5.7)), *for any* $\otimes$*-coreflection* (Γ,ι) *the map induced by* $\iota(G)$ *is an isomorphism*

(1.6.17.2) $$[\Gamma F, \Gamma G] \xrightarrow{\sim} [\Gamma F, G].$$

Proof. That (1.6.17.1) is an isomorphism follows, upon application of the functor $\operatorname{Hom}(E,-)$ with $E \in \mathbf{D}$ arbitrary, from the same for the map $j_{E\otimes A,F}$ in 1.5.9(ii). The converse is given by application of the functor H^0—see (1.6.15.2). □

1.7. **Cohomology with supports: topological rings.** Prior considerations are rehearsed here in the context of topological rings. This provides, among other things, a formulation, encapsulated in 1.7.10 (appearing also in [Lp2, §3.5]), suited to subsequent developments, of some basic facts about cohomology with supports. The underlying idea, which will emerge fully only in the next section (see 1.8.4) is to establish a categorical equivalence between "decently topologized" noetherian rings and noetherian rings S furnished with idempotent $\mathbf{D}(S)$-pairs.

This approach owes much to communications with Amnon Neeman.

1.7.1. (Topologies on a commutative noetherian ring.) A *topological ring* $(S,\mathfrak{U})$ is understood to be a *noetherian* ring S with topology $\mathfrak{U}$ such that addition and multiplication are continuous and such that there is a basis $\mathfrak{B}$ of neighborhoods of 0 consisting of ideals whose squares are open. (Any member of $\mathfrak{B}$ must itself be open, since an ideal J that contains an open neighborhood U of 0 also contains the open neighborhood $a+U$ of any $a\in J$.) Such a topology on S will be called *decent*.

For example, the *preadic* $(S,\mathfrak{U})$ are those having a $\mathfrak{B}$ consisting of all the powers of a single ideal [GD, p. 172, (7.1.9)].

In a topological ring, any product $I_1I_2\dots I_n$ of open ideals is open: induction reduces the proof to where $n=2$, and since $I_1\cap I_2$ contains some $J\in\mathfrak{B}$, therefore I_1I_2 contains the open ideal $U := J^2$, and so, as above, I_1I_2 is open.

Since every open ideal contains a finite product of open prime ideals, therefore such products constitute a basis of neighborhoods of 0.

Thus, for fixed S, there is a bijection between decent $\mathfrak{U}$ and sets Y of prime ideals such that for any prime ideals $p\subset p'$, $p\in Y \Rightarrow p'\in Y$, i.e., specialization-stable subsets of $X:=\operatorname{Spec}(S)$, or equivalently (see §1.1.1), between decent $\mathfrak{U}$ and systems of supports (necessarily finitary) in X, or equivalently (see 1.1.8), between decent $\mathfrak{U}$ and $\mathcal{O}_X$-bases.

The specialization-stable subset of X corresponding to $\mathfrak{U}$ consists of all $\mathfrak{U}$-open prime ideals. The corresponding system of supports $\Phi_{\mathfrak{U}}$ consists of those closed subsets of X all of whose members are $\mathfrak{U}$-open prime ideals, i.e., with "$\tilde{}$" denoting sheafification, $\Phi_{\mathfrak{U}} = \{\, Z(\tilde{I}) \mid I \text{ is } \mathfrak{U}\text{-open}\}$. The corresponding $\mathcal{O}_X$-base $\mathcal{I}_{\mathfrak{U}}$ is the set of sheafifications of $\mathfrak{U}$-open S-ideals; and vice versa, for any $\mathcal{O}_X$-base $\mathcal{J}$, the global section functor takes the members of $\mathcal{J}$ to the set of open ideals for a decent topology $\mathfrak{U} = \mathfrak{U}_{\mathcal{J}}$ such that $\mathcal{J} = \mathcal{I}_{\mathfrak{U}}$.

For a topological ring $(S, \mathfrak{U})$, let $\Gamma' = \Gamma'_{\mathfrak{U}}$ be the *left-exact subfunctor of the identity functor on the category $\mathcal{A}(S)$ of S-modules* such that for any S-module M,

$$\Gamma' M = \{\, x \in M \mid \text{for some open ideal } J,\ Jx = 0\,\}.$$

If p is a prime S-ideal and I_p is an injective hull of S/p—or of its fraction field, so that I_p is an S_p-module—then $\Gamma' I_p = 0$ if p is not open, and since every element of I_p is annihilated by a power of p, $\Gamma' I_p = I_p$ if p is open. Thus Γ' determines the set of open primes, and hence determines the topology $\mathfrak{U}$.

With $\mathfrak{s}M$ the sheafification of M, one has

$$\Gamma' M = \Gamma_{\mathcal{I}_{\mathfrak{U}}}(X, \mathfrak{s}M) = \Gamma_{\Phi_{\mathfrak{U}}}(X, \mathfrak{s}M) = \Gamma(X, \mathit{\Gamma}_{\Phi_{\mathfrak{U}}}\mathfrak{s}M), \tag{1.7.1.1}$$

see 1.1.17 and (1.1.13.3). Consequently, by 1.1.20, *the functor Γ' commutes with small filtered colimits, hence with small direct sums.*

More directly, if $x \in \varinjlim M_\alpha$ is annihilated by an open ideal $J = (a_1, a_2, \dots, a_n)S$, then for some α, x is the natural image of an $x_\alpha \in M_\alpha$, and $a_i x_\alpha = 0$ for all i, i.e., $Jx_\alpha = 0$.

Moreover, Γ' *preserves injectivity of S-modules,* since every injective S-module is a direct sum of ones of the form I_p, and any such direct sum is injective.

In fact, $\mathfrak{U} \mapsto \Gamma'_{\mathfrak{U}}$ is a bijection from decent topologies on S t left-exact subfunctors $\mathit{\Gamma}$ of the identity functor on $\mathcal{A}(S)$ that commute with direct sums and preserve injectivity. For, since I_p is indecomposable, its injective submodule $\mathit{\Gamma} I_p$ is I_p or 0; and if $p \subset p'$ then by left-exactness, $\mathit{\Gamma} I_p \subset \mathit{\Gamma} I_{p'}$; so the set of p such that $\mathit{\Gamma} I_p = I_p$ is the set of open primes for a decent topology $\mathfrak{U}$. One checks then that $\mathit{\Gamma} = \Gamma'_{\mathfrak{U}}$ by applying both functors to representations of S-modules as kernels of maps between injectives.

During the rest of this section, $(S, \mathfrak{U})$ will be a topological ring. By and large, the presented properties of $\Gamma' := \Gamma'_{\mathfrak{U}}$ and its derived functor $\mathbf{R}\Gamma'$ correspond, via sheafification, to previously discussed properties, over $\operatorname{Spec}(S)$, of $\mathit{\Gamma}_{\mathcal{I}_{\mathfrak{U}}}$ and $\mathbf{R}\mathit{\Gamma}_{\mathcal{I}_{\mathfrak{U}}}$.

Lemma 1.7.2. *Any injective S-complex is Γ'-acyclic.*

Proof. The proof, via that of [AJL1, (3.1.1)(2)′ ⇒ (3.1.1)(2)]), *mutatis mutandis*, is like that of 1.2.6(ii).

For another proof—Koszul-free—see [Lp2, Lemma 3.5.1]. □

Proposition 1.7.3. *Set $\mathrm{H}^n_{\mathfrak{U}} := \mathrm{H}^n \circ \mathbf{R}\Gamma'_{\mathfrak{U}}$. Let A be a small filtered category, $\mathcal{M}$ a functor from A to the category of S-complexes, and $n \in \mathbb{Z}$. Then the natural map is an isomorphism*

$$\varinjlim_{A} (\mathrm{H}^n_{\mathfrak{U}} \circ \mathcal{M}) \xrightarrow{\ \sim\ } \mathrm{H}^n_{\mathfrak{U}} \varinjlim_{A} \mathcal{M}$$

In particular, $\mathbf{R}\Gamma'_{\mathfrak{U}}$ commutes with small direct sums in $\mathbf{D}(R)$.

Proof. As in the proof of 1.2.16, reduce to where $\mathfrak{U}$ has an open base consisting of powers of a single ideal $\mathbf{t}S$, in which case the functor $\mathbf{R}\mathit{\Gamma}_{\mathfrak{U}}$ is given by tensoring with the bounded flat complex $\Gamma(\operatorname{Spec}(S), \mathcal{K}^\bullet_\infty(\mathbf{t}))$, rendering 1.7.3 obvious.

Or, make use of the existence of functorial K-injective resolutions [St, Tag 079P], commutativity of $\Gamma'_{\mathfrak{U}}$ with small filtered colimits, preservation of quasi-isomorphisms by such colimits, and (S being noetherian) injectivity of filtered colimits of injective S-modules. □

One has natural functorial maps

$$\iota'_{\mathfrak U}\colon \mathbf R\Gamma'_{\mathfrak U} \to \mathbf 1$$

and, for decent topologies $\mathfrak U$ and $\mathfrak V$, with $\mathfrak U \cap \mathfrak V$ the topology whose open sets are those sets which are open for both $\mathfrak U$ and $\mathfrak V$ (the decent topology whose corresponding specialization-stable subset of X is the intersection of those of $\mathfrak U$ and $\mathfrak V$),

$$\gamma'_{\mathfrak U,\mathfrak V}\colon \mathbf R\Gamma'_{\mathfrak U\cap\mathfrak V} \xrightarrow{\ \sim\ } \mathbf R\Gamma'_{\mathfrak U}\mathbf R\Gamma'_{\mathfrak V},$$

an isomorphism because $\Gamma'_{\mathfrak U\cap\mathfrak V} = \Gamma'_{\mathfrak U}\Gamma'_{\mathfrak V}$, and, as above, $\Gamma'_{\mathfrak V}$ preserves injectivity, so Lemma 1.7.2 can be applied.

Proposition 1.7.4. *Let $\mathfrak U$, $\mathfrak V$ be decent topologies on S. The subtriangles in the following natural functorial diagram commute.*

$$\begin{array}{ccc}
\mathbf R\Gamma'_{\mathfrak U\cap\mathfrak V} & \longrightarrow & \mathbf R\Gamma'_{\mathfrak V} \\
\downarrow & \overset{\simeq}{\underset{\gamma'_{\mathfrak U,\mathfrak V}}{\searrow}} & \uparrow \\
\mathbf R\Gamma'_{\mathfrak U} & \longleftarrow & \mathbf R\Gamma'_{\mathfrak U}\mathbf R\Gamma'_{\mathfrak V}
\end{array}$$

In particular, $(\mathbf R\Gamma'_{\mathfrak U}, \iota'_{\mathfrak U})$ is coreflecting in $\mathbf D(S)$.

Proof. Imitate the proof of 1.3.4. (For the last assertion, set $\mathfrak V := \mathfrak U$.) □

1.7.5. Let $\mathcal A(S)$ be the category of small S-modules, and let $\mathcal A_{\mathfrak U}(S) \subset \mathcal A(S)$ be the essential image of $\Gamma' := \Gamma'_{\mathfrak U}$—the Serre subcategory (cf. 1.3.8) whose objects are the $\mathfrak U$-*torsion S-modules,* that is, those S-modules M such that $\Gamma' M = M$, or equivalently, such that the localization M_p vanishes for every non-open prime S-ideal p. One can regard Γ' as being right-adjoint to the inclusion $\mathcal A_{\mathfrak U}(S) \hookrightarrow \mathcal A(S)$.

At the derived level, let $\mathbf D_{\mathfrak U}(S) \subset \mathbf D(S)$ be the full subcategory whose objects are those complexes E whose homology modules are all in $\mathcal A_{\mathfrak U}(S)$, that is, whose localization E_p is exact for every non-open prime S-ideal p. Any complex in $\mathcal A_{\mathfrak U}(S)$ is in $\mathbf D_{\mathfrak U}(S)$. As in the remarks after 1.3.8, $\mathbf D_{\mathfrak U}(S)$ is a *localizing subcategory* of $\mathbf D(S)$.

Proposition 1.7.6. *An S-complex E is in $\mathbf D_{\mathfrak U}(S)$ if and only if the natural map $\iota'(E) := \iota'_{\mathfrak U}(E)\colon \mathbf R\Gamma' E \to E$ is an isomorphism. So $\mathbf D_{\mathfrak U}(S)$ is the essential image of the functor $\mathbf R\Gamma'\colon \mathbf D(S) \to \mathbf D(S)$.*

Proof. Set $X := \operatorname{Spec}(S)$. Let $\mathfrak s$ be the sheafification functor, an equivalence of categories from $\mathcal A(S)$ to $\mathcal A_{\mathrm{qc}}(X)$.

One can assume that E is injective. Since S is noetherian, the $\mathcal O_X$-module $\mathfrak s E$ is injective, and the first assertion is given by the following logical equivalences:

$$\begin{aligned}
E \in \mathbf D_{\mathfrak U}(S) &\iff \forall n \in \mathbb Z,\ \Gamma' \mathrm H^n E = \mathrm H^n E \\
&\underset{(1.7.1.1)}{\iff} \forall n \in \mathbb Z,\ \Gamma(X, \varGamma_{\Phi_{\mathfrak U}} H^n \mathfrak s E) = \mathrm H^n E \\
&\underset{1.1.18}{\iff} \forall n \in \mathbb Z,\ \varGamma_{\Phi_{\mathfrak U}} H^n \mathfrak s E = \mathfrak s \mathrm H^n E \\
&\iff \mathfrak s E \in \mathbf D_{\Phi_{\mathfrak U}}(X) \\
&\underset{1.3.10}{\iff} \varGamma_{\Phi_{\mathfrak U}} \mathfrak s E = \mathfrak s E \\
&\underset{1.1.18}{\iff} \mathfrak s \Gamma(X, \varGamma_{\Phi_{\mathfrak U}} \mathfrak s E) = \mathfrak s E \\
&\underset{(1.7.1.1)}{\iff} \mathfrak s \Gamma' E = \mathfrak s E \iff \Gamma' E = E.
\end{aligned}$$

The last assertion results then from the last assertion in 1.7.4. □

Here is another argument for the first assertion in 1.7.6.

If $\sigma_E\colon E \to I_E$ is a K-injective resolution then $\mathbf{R}\Gamma' E \cong \Gamma' I_E \in \mathcal{A}_{\mathfrak{U}}(S) \subset \mathbf{D}_{\mathfrak{U}}(S)$, and so $\mathbf{R}\Gamma'\mathbf{D}(S) \subset \mathbf{D}_{\mathfrak{U}}(S)$. Thus if $\iota'(E)$ is an isomorphism then $E \in \mathbf{D}_{\mathfrak{U}}(S)$.

Conversely, note via 1.7.3 that the $E \in \mathbf{D}_{\mathfrak{U}}(S)$ for which $\iota'(E)$ is an isomorphism span a localizing subcategory $\mathbf{L} \subset \mathbf{D}_{\mathfrak{U}}(S)$. Now [Nm1, p. 526, Theorem 2.8] says that any localizing subcategory $\mathbf{L}' \subset \mathbf{D}(S)$ is determined by the set of prime ideals p such that $\mathbf{L}'$ contains the fraction field $k(p)$ of S/p. Since $k(p)$ is in $\mathbf{D}_{\mathfrak{U}}(S) \iff k(p)$ is $\mathfrak{U}$-torsion $\iff p$ is open, therefore $\mathbf{L} = \mathbf{D}_{\mathfrak{U}}(S)$ if $\iota'(k(p))$ is an isomorphism for any open p, which indeed it is, because $k(p)$ admits a quasi-isomorphism into a bounded-below complex of $\mathfrak{U}$-torsion S-injective modules (which follows easily from the fact that if an $\mathfrak{U}$-torsion module M is contained in an injective S-module J then M is contained in the $\mathfrak{U}$-torsion injective module $\Gamma' J$).

Once again, set $\mathbf{R}\Gamma' := \mathbf{R}\Gamma'_{\mathfrak{U}}$ and $\iota' := \iota'_{\mathfrak{U}}$.

Corollary 1.7.7. *For $F \in \mathbf{D}_{\mathfrak{U}}(S)$ and $G \in \mathbf{D}(S)$, $\iota'(G)\colon \mathbf{R}\Gamma' G \to G$ induces an isomorphism*

$$\operatorname{Hom}_{\mathbf{D}_{\mathfrak{U}}(S)}(F, \mathbf{R}\Gamma' G) = \operatorname{Hom}_{\mathbf{D}(S)}(F, \mathbf{R}\Gamma' G) \xrightarrow{\ \sim\ } \operatorname{Hom}_{\mathbf{D}(S)}(F, G).$$

Proof. In view of the last assertion in 1.7.4, this results from 1.7.6 and 1.3.2. $\square$

Proposition 1.7.8. *The natural functor is an equivalence of categories*

$$\mathbf{D}(\mathcal{A}_{\mathfrak{U}}(S)) \xrightarrow{\ \approx\ } \mathbf{D}_{\mathfrak{U}}(S),$$

with quasi inverse $\mathbf{R}\Gamma'|_{\mathbf{D}_{\mathfrak{U}}(S)}$.

Proof. Apply [AJL2, p. 49, 5.2.2] (where the second "let" should be "let $\boldsymbol{j}$ be the"). $\square$

Let $\underset{=}{\otimes}$ denote derived tensor product in $\mathbf{D}(S)$—defined via K-flat resolutions, see footnote in section 1.5.12.

Proposition 1.7.9. *There is a unique bifunctorial $\mathbf{D}(S)$-isomorphism*

$$\psi(E,F)\colon \mathbf{R}\Gamma' E \underset{=}{\otimes} F \xrightarrow{\ \sim\ } \mathbf{R}\Gamma'(E \underset{=}{\otimes} F) \qquad \big(E,\ F \in \mathbf{D}(S)\big)$$

making the following diagram commute:

$$\begin{array}{ccc}
\mathbf{R}\Gamma' E \underset{=}{\otimes} F & \xrightarrow{\ \psi(E,F)\ } & \mathbf{R}\Gamma'(E \underset{=}{\otimes} F) \\
& {\scriptstyle \iota'(E) \underset{=}{\otimes} \mathbf{1}_F} \searrow & \downarrow {\scriptstyle \iota'(E \underset{=}{\otimes} F)} \\
& & E \underset{=}{\otimes} F
\end{array}$$

Thus the coreflecting pair $(\mathbf{R}\Gamma', \iota')$ (see 1.7.4) is $\otimes$-coreflecting in $\mathbf{D}(S)$.

Proof. First, $\mathbf{R}\Gamma' E \underset{=}{\otimes} F \in \mathbf{D}_{\mathfrak{U}}(S)$—just note that if E is K-injective, F is K-flat, and p is a non-open prime S-ideal, then $(\Gamma' E \otimes_S F)_p \cong (\Gamma' E)_p \otimes_{S_p} F_p = 0$. Hence the existence and uniqueness of the map $\psi(E,F)$ is given by 1.7.7.

To show that $\psi(E,F)$ is an isomorphism one reduces, as in the proof of 1.5.13(i), to the preadic case (i.e., a basis of neighborhoods of 0 is given by the powers of a single ideal), and then applies [AJL1, (3.1.2)].

Alternatively, it's enough, by 1.5.4, to show that $\psi(S,F)$ is an isomorphism. For variable F, $\psi(S,F)$ is compatible with triangles and direct sums; hence, and by 1.7.3, the F for which $\psi(S,F)$ is an isomorphism span a localizing subcategory $\mathbf{F} \subset \mathbf{D}(S)$. As $S \in \mathbf{F}$, [Nm2, p. 222, Lemma 3.2] gives $\mathbf{F} = \mathbf{D}(\mathbf{S})$. $\square$

Corollary 1.7.10. *The pair* $(\mathbf{R}\Gamma' S, \iota'(S))$ *is* $\mathbf{D}(S)$*-idempotent, and there is a unique functorial isomorphism*

$$\psi_F\colon \mathbf{R}\Gamma' S \underset{=}{\otimes} F \xrightarrow{\sim} \mathbf{R}\Gamma' F \qquad \big(F \in \mathbf{D}(S)\big)$$

making the following diagram commute:

$$\begin{array}{ccc} \mathbf{R}\Gamma' S \underset{=}{\otimes} F & \xrightarrow[\psi_F]{\sim} & \mathbf{R}\Gamma' F \\ {\scriptstyle \iota'_S \underset{=}{\otimes} \mathbf{1}_F}\big\downarrow & & \big\downarrow{\scriptstyle \iota'(F)} \\ S \underset{=}{\otimes} F & \xrightarrow[\mathsf{l}_F]{\sim} & F \end{array} \tag{1.7.10.1}$$

Proof. The first assertion follows from 1.5.7, and the rest from 1.7.9 with $E = S$. □

Remark. By 1.6.7, 1.7.6 and 1.7.10, $\mathbf{D}_{\mathfrak{U}} = \mathbf{D}_{\mathbf{R}\Gamma'_{\mathfrak{U}} S}$.

1.8. **Idempotent pairs and topological rings.** In a later chapter, the discussion of Duality will involve, in particular, the behavior of functors vis-à-vis compositions

$$(R, \mathfrak{T}) \xrightarrow{\varphi} (S, \mathfrak{U}) \xrightarrow{\psi} (T, \mathfrak{V}) \xrightarrow{\chi} (U, \mathfrak{W})$$

of continuous topological-ring homomorphisms and vis-à-vis certain commutative "base-change" diagrams. For that discussion, the formal basics can be set up more efficiently, and more generally, in an expanded category obtained by substituting idempotent pairs for topologies and dropping noetherian hypotheses. This section explicates the expansion.

1.8.1. For a noetherian ring S, a decent topology $\mathfrak{U}$ determines the isomorphism class of the idempotent pair $(A, \alpha) := (\mathbf{R}\Gamma'_{\mathfrak{U}} S, \iota'_{\mathfrak{U}}(S))$ (see 1.7.10); and conversely, this (A, α) determines $\mathfrak{U}$, since an S-ideal J is $\mathfrak{U}$-open if and only if $S/J \in \mathbf{D}_{\mathfrak{U}}(S)$, that is, by 1.7.6 and (1.7.10.1), if and only if $\alpha \underset{=}{\otimes}_S \mathbf{1}\colon A \underset{=}{\otimes}_S S/J \to S \underset{=}{\otimes}_S S/J = S/J$ is an isomorphism.

Alternatively, it holds that an S-prime ideal p is $\mathfrak{U}$-open if and only if, with $k(p)$ the fraction field of S/p, $\alpha \underset{=}{\otimes}_S \mathbf{1}\colon A \underset{=}{\otimes}_S k(p) \to S \underset{=}{\otimes}_S k(p) = k(p)$ is an isomorphism.

So the map

$$\{\,\text{decent topologies}\,\} \longrightarrow \{\,\text{isomorphism classes of } \mathbf{D}(S)\text{-idempotent pairs}\,\}$$

that takes $\mathfrak{U}$ to the class of $(\mathbf{R}\Gamma'_{\mathfrak{U}} S, \iota'_{\mathfrak{U}}(S))$ has a left inverse. In fact it is *bijective* [Lp2, p. 65, 3.5.7], a result generalized to formal schemes below, in 1.9.20. It is also order-preserving: for decent topologies $\mathfrak{U}$, $\mathfrak{V}$ with $\mathfrak{U} \subset \mathfrak{V}$ (as collections of open sets), and any S-module M, one has $\Gamma'_{\mathfrak{U}} M \subset \Gamma'_{\mathfrak{V}} M$, and hence $\mathbf{R}\Gamma'_{\mathfrak{U}} S \preccurlyeq \mathbf{R}\Gamma'_{\mathfrak{V}} S$. More generally, from 1.7.4 and 1.7.9 one gets that for any decent topologies $\mathfrak{U}$, $\mathfrak{V}$,

$$(\mathbf{R}\Gamma'_{\mathfrak{U}} S \underset{=}{\otimes}_S \mathbf{R}\Gamma'_{\mathfrak{V}} S,\ \iota'_{\mathfrak{U}}(S) \underset{=}{\otimes}_S \iota'_{\mathfrak{V}}(S)) \cong (\mathbf{R}\Gamma'_{\mathfrak{U}\cap\mathfrak{V}} S,\ \iota'_{\mathfrak{U}\cap\mathfrak{V}}(S)).$$

1.8.2. Next, a reformulation of continuity of maps of topological rings, in terms of idempotent pairs.

For a ring homomorphism $\psi\colon S \to T$, let ψ_* be the restriction-of-scalars functor from the category $\mathcal{A}(T)$ of T-modules to the category $\mathcal{A}(S)$. This functor is exact, so its derived functor, from $\mathbf{D}(T)$ to $\mathbf{D}(S)$, will also be denoted by "ψ_*".

The extension-of-scalars functor $-\otimes_S T$ from $\mathcal{A}(S)$ to $\mathcal{A}(T)$, together with the counit map $\psi_* M \otimes_S T \to M$ $(M \in \mathcal{A}(T))$ given by scalar multiplication, is left-adjoint to ψ_*. Standard arguments (cf. e.g., [Lp1, §(2.5.7)]) show that this functor has a left-derived functor $\psi^*\colon \mathbf{D}(S) \to \mathbf{D}(T)$, constructed objectwise by choosing for each S-complex E a K-flat resolution $\varsigma_E\colon P_E \to E$, and setting $\psi^* E := P_E \otimes_S T$, furnished with the $\mathbf{D}(T)$-map $\psi^* E = P_E \otimes_S T \xrightarrow{\varsigma_E \otimes \mathbf{1}} E \otimes_S T$.

There is a natural identification $\psi^*S = T$.

The functor ψ^* is left-adjoint to ψ_*, with counit map at any T-complex F being the natural composite

$$\psi^*\psi_*F = P_{\psi_*F} \otimes_S T \xrightarrow{\varsigma\otimes\mathbf{1}} \psi_*F \otimes_S T \longrightarrow F,$$

cf. [Lp1, 3.1–3.2.2], or see [St, Tag 09T5]).

There is a unique bifunctorial isomorphism $\tau = \tau(E,E')$ $(E, E' \in \mathbf{D}(S))$ such that the following otherwise natural diagram commutes.

$$\begin{array}{ccc}
\psi^*(E \underset{=}{\otimes}_S E') & \xrightarrow[\tau]{\;\;\sim\;\;} & \psi^*E \underset{=}{\otimes}_T \psi^*E' \\
\big\downarrow & & \big\downarrow \\
(E \otimes_S E') \otimes_S T & \xrightarrow{\;\;\sim\;\;} & (E \otimes_S T) \otimes_T (E' \otimes_S T)
\end{array}$$

This follows from [Lp1, (2.6.5)], cf. proof of [Lp1, (3.2.4(i)].

One checks that the following natural diagram commutes:

$$(1.8.2.1)\qquad \begin{array}{ccc}
\psi^*(E \underset{=}{\otimes}_S S) & \xrightarrow[\tau]{\;\sim\;} & \psi^*E \underset{=}{\otimes}_T \psi^*S \\
\simeq\big\downarrow & & \big\| \\
\psi^*E & \xrightarrow{\;\sim\;} & \psi^*E \underset{=}{\otimes}_T T
\end{array}$$

For an S-complex E and a T-complex F, let $E \otimes_\psi F$ be the T-complex

$$E \otimes_\psi F := (E \otimes_S T) \otimes_T F = E \otimes_S F,$$

and set

$$E \underset{=}{\otimes}_\psi F := \psi^*E \underset{=}{\otimes}_T F.$$

As $P_E \otimes_S T$ is K-flat over T, there is a canonical $\mathbf{D}(T)$-map

$$E \underset{=}{\otimes}_\psi F = (P_E \otimes_S T) \otimes_T F \to E \otimes_\psi F,$$

making $\underset{=}{\otimes}_\psi$ a two-variable derived functor of $\otimes_\psi$.

In particular, since S is K-flat as an S-complex vanishing in all nonzero degrees, there is a canonical functorial $\mathbf{D}(T)$-isomorphism

$$(1.8.2.2)\qquad S \underset{=}{\otimes}_\psi F \xrightarrow{\;\sim\;} F \qquad \big(F \in \mathbf{D}(T)\big).$$

There is a unique bifunctorial "projection" isomorphism

(1.8.2.3)

$$\rho\colon E \underset{=}{\otimes}_S \psi_*F \xrightarrow{\;\sim\;} \psi_*(\psi^*E \underset{=}{\otimes}_T F) = \psi_*(E \underset{=}{\otimes}_\psi F) \qquad \big(E \in \mathbf{D}(S),\ F \in \mathbf{D}(T)\big)$$

whose composition with the natural $\mathbf{D}(S)$-map $\zeta\colon \psi_*(E \underset{=}{\otimes}_\psi F) \to E \otimes_S F$ is the natural map $\beta\colon E \underset{=}{\otimes}_S \psi_*F \to E \otimes_S F$—an isomorphism when E is K-flat. This ρ can be identified with the natural $\mathbf{D}(S)$ isomorphism $P_E \otimes_S F \xrightarrow{\;\sim\;} (P_E \otimes_S T) \otimes_T F$.

One checks that ρ is an instance of the map p_2 in [Lp1, p. 107, 3.4.6].

Recall the definition of $\mathbf{D}_A$ (see 1.6.7), and the discussion of $\mathbf{D}_{\mathfrak{U}}$ preceding 1.7.6. Recall further Lemma 1.4.8, which for $\xi := \psi^*$ (as in 1.8.2) and $u := \mathbf{1}$ gives that for any $\mathbf{D}(S)$-idempotent pair (A, α), the pair $(\psi^*A, \psi^*\alpha)$ is $\mathbf{D}(T)$-idempotent.

Proposition 1.8.3. *Let $\psi\colon S\to T$ be a ring homomorphism. Let (A,α) be $\mathbf{D}(S)$-idempotent and let (B,β) be $\mathbf{D}(T)$-idempotent. The following are equivalent.*

(i) *$B\preccurlyeq\psi^*A$, that is (see 1.6.6), there exists a $\mathbf{D}(T)$-map $\lambda: B\to\psi^*A$, necessarily unique, such that the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\ \lambda\ } & \psi^*A \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\psi^*\alpha} \\ T & = & \psi^*S \end{array}$$

(ii) *The map $\alpha\underset{=}{\otimes}_\psi\mathbf{1}_B$ is an isomorphism $A\underset{=}{\otimes}_\psi B\xrightarrow{\ \sim\ }S\underset{=}{\otimes}_\psi B\underset{(1.8.2.2)}{=}B$.*

(ii)′ *The map $\alpha\underset{=}{\otimes}_S\mathbf{1}_{\psi_*B}$ is an isomorphism $A\underset{=}{\otimes}_S\psi_*B\xrightarrow{\ \sim\ }S\underset{=}{\otimes}_S\psi_*B=\psi_*B$.*

(iii) *For $E\in\mathbf{D}(S)$ the map $\alpha\underset{=}{\otimes}_\psi\mathbf{1}_E$ induces a $\mathbf{D}(T)$-isomorphism*

$$(A\underset{=}{\otimes}_S E)\underset{=}{\otimes}_\psi B\xrightarrow{\ \sim\ }(S\underset{=}{\otimes}_S E)\underset{=}{\otimes}_\psi B=E\underset{=}{\otimes}_\psi B.$$

(iii)′ *$\psi_*\mathbf{D}_B(T)\subset\mathbf{D}_A(S)$.*

If $(S,\mathfrak{U})$ and $(T,\mathfrak{V})$ are topological rings, and (A,α) (respectively (B,β)) is the idempotent pair $(\mathbf{R}\Gamma'_{\mathfrak{U}}S,\iota_{\mathfrak{U}}(S))$ (respectively $(\mathbf{R}\Gamma'_{\mathfrak{V}}T,\iota_{\mathfrak{V}}(T))$), then each of the preceding conditions is equivalent to each of the following ones.

(iv) *The map ψ is continuous.*

(v) *For $G\in\mathbf{D}_{\mathfrak{V}}(T)$ the map $\iota_{\mathfrak{U}}(S)\underset{=}{\otimes}_\psi\mathbf{1}_G$ is a $\mathbf{D}(T)$-isomorphism*

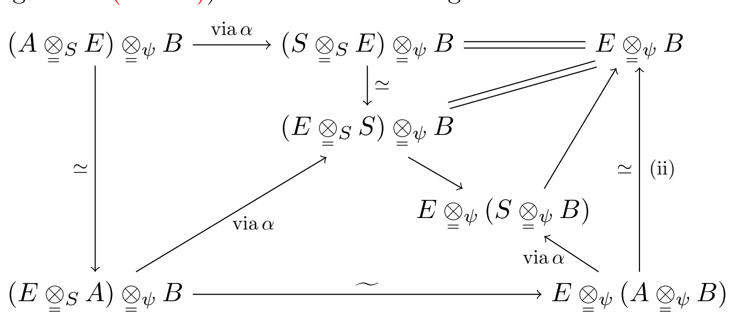
$$\mathbf{R}\Gamma'_{\mathfrak{U}}S\underset{=}{\otimes}_\psi G\xrightarrow{\ \sim\ }S\underset{=}{\otimes}_\psi G\underset{(1.8.2.2)}{=}G.$$

(v)′ *$\psi_*\mathbf{D}_{\mathfrak{V}}(T)\subset\mathbf{D}_{\mathfrak{U}}(S)$.*

Proof. (i) $\Leftrightarrow$ (ii). This results from 1.6.1.

(ii) $\Leftrightarrow$ (ii)′. The map $\alpha\underset{=}{\otimes}_\psi\mathbf{1}_B$ is an isomorphism, that is, it induces homology isomorphisms, if and only if its image under the exact functor ψ_* does so; and by (1.8.2.3), that image is (up to isomorphism) the map $\alpha\underset{=}{\otimes}_S\mathbf{1}_{\psi_*B}$ in (ii)′.

(iii) $\Leftrightarrow$ (ii). Condition (ii) is the case $E=S$ of (iii). That (ii) $\Rightarrow$ (iii) results from the commutativity—elementary to check, e.g., by unwinding the relevant definitions and making use of (1.8.2.1))—of the natural diagram



(ii) $\Rightarrow$ (iii)′ $\Rightarrow$ (ii)′. By 1.6.8, (iii)′ means that for all $G\in\mathbf{D}_B(T)$, the map $\alpha\underset{=}{\otimes}_S\mathbf{1}\colon A\underset{=}{\otimes}_S\psi_*G\to S\underset{=}{\otimes}_S\psi_*G$ is an isomorphism, to prove which it suffices to consider those G of the form $B\underset{=}{\otimes}_T F$ ($F\in\mathbf{D}(T)$). For such G, assuming (ii), apply the functor $\psi_*\circ(-\underset{=}{\otimes}_T F)$ to the map $\alpha\underset{=}{\otimes}_\psi\mathbf{1}_B$, and then use (1.8.2.3) to get (iii)′. Conversely, (ii)′ is the case $G=B$ of (iii)′.

(iii)$'$ $\Rightarrow$ (v)$'$. By the Remark after 1.7.10, $\mathbf{D}_{\mathfrak{V}}(T)=\mathbf{D}_B(T)$ and $\mathbf{D}_{\mathfrak{U}}(S)=\mathbf{D}_A(S)$.

(iv) $\Rightarrow$ (v)$'$. For $G\in\mathbf{D}_{\mathfrak{V}}(T)$, $n\in\mathbb{Z}$, and $a\in \mathrm{H}^nG$, the annihilator $\mathrm{ann}_T(a)$ is a $\mathfrak{V}$-open T-ideal, so that if ψ is continuous then $\mathrm{ann}_S(a)=\psi^{-1}\mathrm{ann}_T(a)$ is a $\mathfrak{U}$-open S-ideal. It follows that $\psi_*G\in\mathbf{D}_{\mathfrak{U}}(S)$.

(v)$'$ $\Rightarrow$ (iv). Let q be a $\mathfrak{V}$-open prime T-ideal. Every x in the T-injective hull I_q of T/q is annihilated by a power of q, so $\mathbf{R}\Gamma'_{\mathfrak{V}}I_q\cong\Gamma'_{\mathfrak{V}}I_q=I_q$. Hence, 1.7.6 and (v)$'$ give $\mathbf{R}\Gamma'_{\mathfrak{U}}\psi_*I_q\cong\psi_*I_q$.

Set $p:=\psi^{-1}q$. Then ψ_*I_q has a natural S_p-module structure. So tensoring an S-injective resolution of ψ_*I_q by S_p produces an injective resolution J of ψ_*I_q such that multiplication by any element in $S\setminus p$ is an isomorphism of J, so that if p is not $\mathfrak{U}$-open, then $0=\Gamma'_{\mathfrak{U}}J\cong\mathbf{R}\Gamma'_{\mathfrak{U}}\psi_*I_q\cong\psi_*I_q$, which is absurd; thus p must be $\mathfrak{U}$-open. Since an ideal in a topological ring is open if and only if it contains an intersection of open prime ideals, it follows that ψ^{-1} takes open T-ideals to open S-ideals, whence ψ is continuous.

(v) $\Leftrightarrow$ (v)$'$. By (1.8.2.3), application of ψ_* to $\iota_{\mathfrak{U}}(S)\underline{\underline{\otimes}}_{\psi}\mathbf{1}_G$ produces the map

$$\iota_{\mathfrak{U}}(S)\underline{\underline{\otimes}}_S\mathbf{1}_{\psi_*G}\colon \mathbf{R}\Gamma'_{\mathfrak{U}}S\underline{\underline{\otimes}}_S\psi_*G\longrightarrow S\underline{\underline{\otimes}}_S\psi_*G=\psi_*G,$$

so that the map $\iota_{\mathfrak{U}}(S)\underline{\underline{\otimes}}_{\psi}\mathbf{1}_G$ is an isomorphism iff so is $\iota_{\mathfrak{U}}(S)\underline{\underline{\otimes}}_S\mathbf{1}_{\psi_*G}$ (see the above proof that (ii) $\Leftrightarrow$ (ii)$'$), that is, by 1.7.6 and 1.7.10, iff $\psi_*G\in\mathbf{D}_{\mathfrak{U}}(S)$.

(v) $\Leftrightarrow$ (ii). Using 1.7.10 one gets (v) for $G\cong B\underline{\underline{\otimes}}_T F$ from (ii) by applying the functor $-\underline{\underline{\otimes}}_T F$. Conversely, (ii) is the case $G=B$ of (v). $\square$

Scholium 1.8.4. Consider the category $\mathcal{T}$ of triples (S,A,α) with S a commutative ring and (A,α) a $\mathbf{D}(S)$-idempotent pair, morphisms $(S,A,\alpha)\to(T,B,\beta)$ being ring homomorphisms $\psi\colon S\to T$ satisfying the equivalent conditions (i), (ii), (ii)$'$, (iii) and (iii)$'$ in 1.8.3. The functor that takes $(S,\mathfrak{U})$ to $(S,\mathbf{R}\Gamma'_{\mathfrak{U}}S,\iota'_{\mathfrak{U}}(S))$, and homomorphisms to themselves, embeds the category of continuous homomorphisms of topological rings *fully faithfully* into $\mathcal{T}$, with essential image the full subcategory spanned by all (S,A,α) with S noetherian (see 1.8.1, 1.8.3).

1.8.5. Let $(S,\mathfrak{U})$ be a topological ring, and $\psi\colon S\to T$ a homomorphism of noetherian rings. Let $\mathfrak{U}T$ be the (decent) topology on T for which a basis of neighborhoods of 0 is the family of ideals $\{JT\mid J$ an $\mathfrak{U}$-open S-ideal$\}$. Then

$$(\psi^*\mathbf{R}\Gamma'_{\mathfrak{U}}S,\psi^*\iota'_{\mathfrak{U}}(S))\cong(\mathbf{R}\Gamma'_{\mathfrak{U}T}T,\iota'_{\mathfrak{U}T}(T)).$$

In view of the bijective order-preserving map from T-topologies to isomorphism classes of $\mathbf{D}(T)$-idempotent pairs (see 1.8.1), this results from the equivalence of (i) and (iv) in Proposition 1.8.3 and the fact that $\mathfrak{U}T$ is the strongest among the T-topologies $\mathfrak{V}$ that make ψ continuous.

Alternatively, if for any S-ideal J, $\mathfrak{U}_J$ is the topology with the powers of J as a basis of neighborhoods of 0, then $\Gamma'_{\mathfrak{U}}=\varinjlim_{J\,\text{open}}\Gamma'_{\mathfrak{U}_J}$, allowing one to assume $\mathfrak{U}=\mathfrak{U}_J$, in which case one can use the representation of $\mathbf{R}\Gamma'_{\mathfrak{U}}S$ by a Koszul complex… (see proof of 1.2.6).

It follows, under the assumptions preceding 1.8.3(iv), that *the map λ in* 1.8.3(i) *is an isomorphism if and only if the topology $\mathfrak{V}$ equals $\mathfrak{U}T$*. For, 1.8.3(i)$\Leftrightarrow$1.8.3(iv) shows that the (continuous) identity map $(T,\mathfrak{U}T)\to(T,\mathfrak{V})$ has a continuous inverse if and only if $\mathbf{R}\Gamma'_{\mathfrak{V}}T\preccurlyeq\mathbf{R}\Gamma'_{\mathfrak{U}T}T\preccurlyeq\mathbf{R}\Gamma'_{\mathfrak{V}}T$, that is, if and only if λ is an isomorphism (see line following Definition 1.6.6).

In terms of prime ideals, $\mathfrak{V}=\mathfrak{U}T$ signifies that a prime T-ideal p is $\mathfrak{V}$-open if and only if $\psi^{-1}(p)$ is $\mathfrak{U}$-open.

1.8.6. Given morphisms $\varphi\colon (R,D,\delta)\to(S,A,\alpha)$ and $\mu\colon (R,D,\delta)\to(U,B,\beta)$ (as in 1.8.4), one checks, with $V:=S\otimes_R U$, and $\nu\colon S\to V$, $\xi\colon U\to V$ the canonical maps, that $(V,\nu^*A\,\underline{\underline{\otimes}}_V\,\xi^*B,\nu^*\alpha\,\underline{\underline{\otimes}}_V\,\xi^*\beta)$ is, together with ν and ξ, a fibered direct sum of φ and μ. It follows (or can be shown directly) that if S, U and V are noetherian, and (A,α), (B,β) correspond to the S-topology $\mathfrak{U}$ and the T-topology $\mathfrak{T}$ respectively, then $(\nu^*A\,\underline{\underline{\otimes}}_V\,\xi^*B,\nu^*\alpha\,\underline{\underline{\otimes}}_V\,\xi^*\beta)$ corresponds to the tensor-product topology $\mathfrak{U}V\cap\mathfrak{T}V$ on V.

1.9. **Cohomology with supports: formal schemes.** In this section, X will be a *noetherian formal scheme* [GD, p. 407, (10.4.2)], equipped with a specialization-stable subset Z—or equivalently, with an s.o.s, see Section 1.1.1. An $\mathcal{O}_X$-ideal will be called *open* if it contains an ideal of definition of X. A noetherian formal scheme X has an ideal of definition all of whose powers are ideals of definition, whence any power of an open $\mathcal{O}_X$-ideal is open.

The main results extend those in the preceding two sections, where X is just an ordinary noetherian affine scheme. For noetherian formal schemes, some basics on cohomology with supports are gone over in 1.9.1–1.9.16; the close relation (given in [AJS2]) between specialization-stable subsets of X and idempotent pairs in the derived torsion category is reviewed in 1.9.17–1.9.24; and the interaction between derived torsion functors with maps of formal schemes is addressed in 1.9.25–1.9.27.

The foundations of the theory of formal schemes, as presented in [GD, §10], are largely taken for granted. The notation and terminology to be used here can be chased down via the index in [AJL2, p. 125]. Full justification of statements to be made requires, as indicated by references, numerous results which can be found in chapters 1–3 of [Lp1] (an exposition of standard material about unbounded derived categories and the derived direct- and inverse-image functors associated to maps of ringed spaces) and in [AJL2] (a study of duality on formal schemes).

1.9.1. An $\mathcal{O}_X$-base $\mathfrak{I}$ is as in 1.1.6, with the constraint that members of $\mathfrak{I}$ be *coherent* and *open.*

Since $\mathcal{O}_X$ is coherent [GD, p. 428, (10.10.2.7)]), therefore $\mathcal{O}_X\in\mathfrak{I}$.

An $\mathcal{O}_X$-ideal belongs to such an $\mathfrak{I}$ if and only if so does its radical, so if J is an ideal of definition (necessarily coherent, see [GD, p. 429, (10.10.2.9)]) and $\overline{X}$ is the noetherian scheme $(X,\mathcal{O}_X/J)$, and if $\pi\colon\mathcal{O}_X\to\mathcal{O}_{\overline{X}}=\mathcal{O}_X/J$ is the canonical surjection, then there is a natural bijection

$$\mathfrak{I}\mapsto\overline{\mathfrak{I}}:=\{\,\mathcal{O}_{\overline{X}}\text{-ideals } I\mid \pi^{-1}I\in\mathfrak{I}\,\}\tag{1.9.1.1}$$

from the set of $\mathcal{O}_X$-bases onto the set of $\mathcal{O}_{\overline{X}}\,(=\mathcal{O}_X/J)$-bases. Proposition 1.1.8 holds for open coherent I, giving an inclusion-preserving bijection from $\mathcal{O}_X$-bases to specialization-stable subsets of X (see section 1.1.1).

1.9.2. Let $\mathcal{A}:=\mathcal{A}(X)$ be the abelian category of $\mathcal{O}_X$-modules. For any $\mathcal{O}_X$-base $\mathfrak{I}$, one has the left-exact subfunctor $\varGamma_{\mathfrak{I}}\colon\mathcal{A}\to\mathcal{A}$ of the identity functor, see (1.1.14.1).

If $\mathfrak{I}$ and $\mathfrak{J}$ are $\mathcal{O}_X$-bases then $\varGamma_{\mathfrak{I}}\varGamma_{\mathfrak{J}}=\varGamma_{\mathfrak{I}\cap\mathfrak{J}}$, and so the functor $\varGamma_{\mathfrak{I}}$ is idempotent, see (1.1.14.5). Hence the essential image $\mathcal{A}_{\mathfrak{I}}$ of $\varGamma_{\mathfrak{I}}$ is the full subcategory of $\mathcal{A}$ spanned by the $\mathcal{O}_X$-modules M such that $\varGamma_{\mathfrak{I}}M=M$.

If $\mathcal{J}\subsetneq\mathcal{I}$ then $\mathcal{A}_{\mathcal{J}}\subsetneq\mathcal{A}_{\mathcal{I}}$. For,

$M\in\mathcal{A}_{\mathcal{J}}\implies\{M=\mathit{\Gamma}_{\mathcal{J}}M\}\implies\{\mathit{\Gamma}_{\mathcal{I}}M=\mathit{\Gamma}_{\mathcal{I}}\mathit{\Gamma}_{\mathcal{J}}M=\mathit{\Gamma}_{\mathcal{I}\cap\mathcal{J}}M=\mathit{\Gamma}_{\mathcal{J}}M=M\}\implies M\in\mathcal{A}_{\mathcal{I}}.$

Furthermore, if $I\in\mathcal{I}$ then

$$\begin{aligned}&\mathcal{O}_X/I\in\mathcal{A}_{\mathcal{J}}\\ &\iff \mathcal{O}_X/I=\mathit{\Gamma}_{\mathcal{J}}(\mathcal{O}_X/I)=\varinjlim_{J\in\mathcal{J}}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/J,\mathcal{O}_X/I)=\varinjlim_{J\in\mathcal{J}}(I:J)/I\\ &\iff \text{for some } J,\ 1\in\Gamma(X,I:J)\ (\text{because } \Gamma(X,-) \text{ respects } \varinjlim\,)\iff I\in\mathcal{J},\end{aligned}$$

so if $I\notin\mathcal{J}$ then $\mathcal{O}_X/I\in\mathcal{A}_{\mathcal{I}}\setminus\mathcal{A}_{\mathcal{J}}$. Thus $\mathcal{J}\neq\mathcal{I}\implies\mathcal{A}_{\mathcal{J}}\neq\mathcal{A}_{\mathcal{I}}$.

Also, for any $\mathcal{I}$, $\mathcal{J}$, it holds that $\mathit{\Gamma}_{\mathcal{I}}\mathcal{A}_{\mathcal{J}}=\mathcal{A}_{\mathcal{I}\cap\mathcal{J}}=\mathcal{A}_{\mathcal{I}}\cap\mathcal{A}_{\mathcal{J}}$. For, as above, $M\in\mathcal{A}_{\mathcal{J}}\Rightarrow\{\mathit{\Gamma}_{\mathcal{I}}M=\mathit{\Gamma}_{\mathcal{I}\cap\mathcal{J}}M\}$, so that $\mathit{\Gamma}_{\mathcal{I}}\mathcal{A}_{\mathcal{J}}\subset\mathcal{A}_{\mathcal{I}\cap\mathcal{J}}\subset\mathcal{A}_{\mathcal{I}}\cap\mathcal{A}_{\mathcal{J}}$; and if $M\in\mathcal{A}_{\mathcal{I}}\cap\mathcal{A}_{\mathcal{J}}$ then $M=\mathit{\Gamma}_{\mathcal{I}}M\in\mathit{\Gamma}_{\mathcal{I}}\mathcal{A}_{\mathcal{J}}$.[7]

Reasoning as in the proof of 1.3.8(ii), one sees that $\mathcal{A}_{\mathcal{I}}$ is a Serre—hence plump—subcategory of $\mathcal{A}$. Also, as in the proof of 1.1.19(ii) one sees that $\mathit{\Gamma}_{\mathcal{I}}$ preserves small filtered colimits, so that $\mathcal{A}_{\mathcal{I}}$ is closed under such colimits.

1.9.3. Let $\mathcal{A}_{\mathrm{qc}}\subset\mathcal{A}$ (respectively $\mathcal{A}_{\vec{c}}\subset\mathcal{A}$) be the full subcategory spanned by the quasi-coherent $\mathcal{O}_X$-modules (respectively the $\mathcal{O}_X$-modules which are small filtered colimits of coherent ones—or equivalently, by [AJL2, p. 33, 3.1.7], unions of coherent submodules). With $\mathcal{I}'$ the $\mathcal{O}_X$-base comprising *all* open coherent $\mathcal{O}_X$-ideals, and $\mathcal{A}_{\mathrm{qct}}:=\mathcal{A}_{\mathrm{qc}}\cap\mathcal{A}_{\mathcal{I}'}$, one has $\mathcal{A}_{\mathrm{qct}}\subset\mathcal{A}_{\vec{c}}\subset\mathcal{A}_{\mathrm{qc}}$ [AJL2, p. 32, 3.1.5 and p. 48, 5.1.4]. If X is affine, then $\mathcal{A}_{\vec{c}}=\mathcal{A}_{\mathrm{qc}}$ [AJL2, p. 32, 3.1.4]. These are all *plump* subcategories of $\mathcal{A}$, see [AJL2, p. 34, 3.2.2 and p. 48, 5.1.3]. It is clear that $\mathcal{A}_{\vec{c}}$ is closed under small filtered colimits; and so is $\mathcal{A}_{\mathrm{qct}}$ [AJL2, p. 48, 5.1.3]. As in the proof of *loc. cit.,* 5.1.4, *mutatis mutandis,* one finds that for any $\mathcal{O}_X$-base $\mathcal{I}$, $\mathit{\Gamma}_{\mathcal{I}}\mathcal{A}_{\mathrm{qct}}\subset\mathcal{A}_{\mathrm{qct}}$. Also, $\mathit{\Gamma}_{\mathcal{I}}\mathcal{A}_{\vec{c}}\subset\mathcal{A}_{\vec{c}}$: for if $(M_\alpha)_{\alpha\in A}$ is a directed system of coherent $\mathcal{O}_X$-modules, then for all $\alpha\in\mathcal{A}$ and $I\in\mathcal{I}$, $\mathcal{H}om_{\mathcal{O}_X}(A/I,M_\alpha)$ is coherent [AJL2, p. 33, 3.1.6(d)], and so

$$\mathit{\Gamma}_{\mathcal{I}}\varinjlim_{\alpha}M_\alpha\cong\varinjlim_{\alpha}\mathit{\Gamma}_{\mathcal{I}}M_\alpha=\varinjlim_{\alpha}\varinjlim_{I\in\mathcal{I}}\mathcal{H}om_{\mathcal{O}_X}(A/I,M_\alpha)\in\mathcal{A}_{\vec{c}}.$$

If X is an ordinary scheme then $\mathcal{A}_{\mathrm{qct}}=\mathcal{A}_{\vec{c}}=\mathcal{A}_{\mathrm{qc}}$.

Let $\mathbf{D}_{\mathrm{qct}}\subset\mathbf{D}_{\vec{c}}\subset\mathbf{D}_{\mathrm{qc}}$ be the full subcategories of $\mathbf{D}$ spanned by the complexes whose homology modules are all in $\mathcal{A}_{\mathrm{qct}}$ (resp. in $\mathcal{A}_{\vec{c}}$, resp. in $\mathcal{A}_{\mathrm{qc}}$). If X is affine then $\mathbf{D}_{\vec{c}}=\mathbf{D}_{\mathrm{qc}}$. If X is an ordinary scheme then $\mathbf{D}_{\mathrm{qct}}=\mathbf{D}_{\vec{c}}=\mathbf{D}_{\mathrm{qc}}$.

Since $\mathcal{A}_{\dots}$ is plump in $\mathcal{A}$, therefore $\mathbf{D}_{\dots}$ is a triangulated subcategory of $\mathbf{D}$.

* * * * *

Let $Z\subset X$ be specialization-stable, and let Φ_Z be the set consisting of all subsets of Z that are closed in X. As noted in section 1.1.1, the map sending any such Z to Φ_Z is a bijection from the set of specialization-stable subsets of X to the set of systems of supports in X.

Let $\mathit{\Gamma}_Z:=\mathit{\Gamma}_{\Phi_Z}:\mathcal{A}\to\mathcal{A}$ be the functor of sections supported in Z: for all $M\in\mathcal{A}$ and open $U\subset X$,

$$\big(\mathit{\Gamma}_Z M\big)(U):=\{\,\xi\in M(U)\mid \xi_x=0\text{ for all }x\in U\setminus Z\,\}.$$

The pair with components $\mathit{\Gamma}_Z$ and its inclusion map into the identity functor is coreflecting in $\mathcal{A}$.

[7] More generally, for endofunctors Γ_1, Γ_2 of a category $\mathbf{D}$, $\Gamma_1\mathbf{D}_{\Gamma_2}=\mathbf{D}_{\Gamma_1\circ\Gamma_2}=\mathbf{D}_{\Gamma_1}\cap\mathbf{D}_{\Gamma_2}$.

Proposition 1.9.4. *With preceding notation, it holds that* $\mathbf{R}\varGamma_{\!Z}\mathbf{D}_{\mathrm{qct}} \subset \mathbf{D}_{\mathrm{qct}}$.

Proof. As $\mathcal{A}_{\mathrm{qct}}$ is closed under $\varinjlim$, one can reduce via (1.2.1.1) to where Z is closed in X, and then refer to the first sentence in [AJS2, §2.1]). □

Lemma 1.9.5. *Let* $\mathcal{I}$, $\mathcal{J}$ *be* $\mathcal{O}_X$*-bases and* E *an injective* $\mathcal{O}_X$*-complex. Then* $\varGamma_{\mathcal{J}}E$ *is* $\varGamma_{\mathcal{I}}$*-acyclic.*

Proof. To be proved is that the natural map $\varGamma_{\mathcal{I}}\varGamma_{\mathcal{J}}E \to \mathbf{R}\varGamma_{\mathcal{I}}\varGamma_{\mathcal{J}}E$ is an isomorphism, a local property, so that one can restrict to where $X = \mathrm{Spf}(S)$ for some adic noetherian ring S with ideal of definition, say, L.

The completion map $\kappa\colon X = \mathrm{Spf}(S) \to \mathrm{Spec}(S) =: X_0$ corresponds to the identity map of S, considered as a continuous map of topological rings, with discrete source and L-adically topologized target (see [GD, p. 403, (10.2.1)]). Topologically, κ is the closed immersion $\mathrm{Spec}(S/L) \hookrightarrow \mathrm{Spec}(S)$.

Accordingly, $\Phi_{\mathcal{I}} := \{\, Z(I) \mid I \in \mathcal{I}\,\}$ can be regarded as an s.o.s. in X_0, with corresponding $\mathcal{O}_{X_0}$-base $\mathcal{I}_0$. Similarly, $\mathcal{J}$ determines an $\mathcal{O}_{X_0}$-base $\mathcal{J}_0$.

To proceed, we'll need:

Lemma 1.9.6. *With preceding notation,*

$$\mathcal{I} = \{\, I_0\mathcal{O}_X \mid I_0 \in \mathcal{I}_0\,\}.$$

Proof. For any $I_0 \in \mathcal{I}_0$, $Z(I_0) \subset \mathrm{Spec}(S/L)$, so $\sqrt{I_0}$ contains the sheafification $\widetilde{L}$. The locally-ringed-space map κ being *flat* [GD, p. 185, (7.6.13), p. 187, (7.6.18) and p. 403, (10.1.5)], one has

$$\kappa^*\sqrt{I_0} \cong \sqrt{I_0}\mathcal{O}_X \supset \widetilde{L}\mathcal{O}_X \cong \kappa^*\widetilde{L}.$$

Since $\widetilde{L}\mathcal{O}_X \cong \kappa^*\widetilde{L}$ is an ideal of definition of X (see [GD, p. 420, (10.8.5), p. 421, (10.8.8)(ii) and p. 427, (10.10.1), second paragraph]), therefore the $\mathcal{O}_X$-ideal $I_0\mathcal{O}_X$ is *open*. Also, $I_0\mathcal{O}_X \cong \kappa^*I_0$ is *coherent* (see [GD, p. 115, (5.3.14)]). Moreover, $Z(I_0\mathcal{O}_X) = \kappa^{-1}Z(I_0) \in \Phi_{\mathcal{I}}$, so $I_0\mathcal{O}_X \in \mathcal{I}$. Thus $\{\, I_0\mathcal{O}_X \mid I_0 \in \mathcal{I}_0\,\} \subset \mathcal{I}$.

Let $I \in \mathcal{I}$, and let $I_0 \subset \mathcal{O}_{X_0}$ be the sheafication of $\Gamma(X, I) \subset \Gamma(X, \mathcal{O}_X) = S$ (see [GD, p. 402, (10.1.3)]). Then $I_0\mathcal{O}_X \cong \kappa^*I_0 \cong I$, see [GD, p. 420, (10.8.5), p. 421, (10.8.8)(ii) with $i = \kappa$ and $\mathcal{F} = I_0$, and p. 429, (10.10.2.9)(ii) with $M = \Gamma(X, I)$]. Since κ^*I_0 is generated by its global sections, therefore so is I, whence $I = I_0\mathcal{O}_X$.

Since $\widetilde{L}\mathcal{O}_X$ is an ideal of definition of X, therefore $I \supset \widetilde{L}^n\mathcal{O}_X$ for some $n > 0$, so $\Gamma(X, I) \supset \Gamma(X, \widetilde{L}^n\mathcal{O}_X) \supset L^n$, whence $Z(I_0) \subset \mathrm{Spec}(S/L)$. And since

$$Z(I) = Z(I_0\mathcal{O}_X) = \kappa^{-1}Z(I_0) \in \Phi_{\mathcal{I}},$$

therefore $I_0 \in \mathcal{I}_0$. Thus $\mathcal{I} \subset \{\, I_0\mathcal{O}_X \mid I_0 \in \mathcal{I}_0\,\}$. □

Using the commutativity of $\varinjlim$ with global sections over noetherian spaces [Kf, p. 641, Prop. 6], plus Lemma 1.9.6, plus the natural isomorphism $\kappa^*I_0 \xrightarrow{\sim} I_0\mathcal{O}_X$ one gets, for any $\mathcal{O}_X$-complex F, the natural composite isomorphism

$$\begin{aligned}
\xi_{\kappa,\mathcal{I},F}\colon \kappa_*\varGamma_{\mathcal{I}}F &= \kappa_*\varinjlim_{I\in\mathcal{I}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I, F) \\
&\xrightarrow{\sim} \varinjlim_{I\in\mathcal{I}} \kappa_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I, F) \\
&\xrightarrow{\sim} \varinjlim_{I_0\in\mathcal{I}_0} \kappa_*\mathcal{H}om_{\mathcal{O}_X}(\kappa^*(\mathcal{O}_{X_0}/I_0), F) \\
&\xrightarrow{\sim} \varinjlim_{I_0\in\mathcal{I}_0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X_0}/I_0, \kappa_*F) = \varGamma_{\mathcal{I}_0}\kappa_*F.
\end{aligned} \tag{1.9.5.1}$$

Replacing F by a K-injective resolution, one derives a functorial isomorphism $\bar\xi_{\kappa,\mathcal{I},F}\colon \kappa_*\mathbf{R}\varGamma_{\mathcal{I}}F \xrightarrow{\,\sim\,} \mathbf{R}\varGamma_{\mathcal{I}_0}\kappa_*F$.

The map κ being flat, the left adjoint κ^* of κ_* is exact, and therefore κ_*E is an injective $\mathcal{O}_{X_0}$-complex, as is $\varGamma_{\mathcal{J}_0}\kappa_*E$ (see 1.2.5). The obvious commutativity of the natural diagram

$$\begin{array}{ccccc}
\kappa_*\varGamma_{\mathcal{I}}\varGamma_{\mathcal{J}}E & \xrightarrow[\text{via}\,\xi]{\sim} & \varGamma_{\mathcal{I}_0}\kappa_*\varGamma_{\mathcal{J}}E & \xrightarrow[\text{via}\,\xi]{\sim} & \varGamma_{\mathcal{I}_0}\varGamma_{\mathcal{J}_0}\kappa_*E \\
{\scriptstyle \kappa_*\zeta}\big\downarrow & & \big\downarrow & & {\scriptstyle 1.2.6\text{(ii)}}\big\downarrow{\scriptstyle \simeq} \\
\kappa_*\mathbf{R}\varGamma_{\mathcal{I}}\varGamma_{\mathcal{J}}E & \xrightarrow[\text{via}\,\bar\xi]{\sim} & \mathbf{R}\varGamma_{\mathcal{I}_0}\kappa_*\varGamma_{\mathcal{J}}E & \xrightarrow[\text{via}\,\bar\xi]{\sim} & \mathbf{R}\varGamma_{\mathcal{I}_0}\varGamma_{\mathcal{J}_0}\kappa_*E
\end{array}$$

shows then that $\kappa_*\zeta$ is an isomorphism, whence so is ζ, because κ is, topologically, a closed immersion. Thus Lemma 1.9.5 holds. $\square$

Proposition 1.9.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be $\mathcal{O}_X$-bases and E an $\mathcal{O}_X$-complex. The natural map is an isomorphism*

$$\gamma_{\mathcal{I},\mathcal{J}}\colon \mathbf{R}\varGamma_{\mathcal{I}\cap\mathcal{J}}E \xrightarrow{\,\sim\,} \mathbf{R}\varGamma_{\mathcal{I}}\mathbf{R}\varGamma_{\mathcal{J}}E.$$

such that the following natural diagram commutes:

$$\begin{array}{ccc}
\mathbf{R}\varGamma_{\mathcal{I}\cap\mathcal{J}}E & \longrightarrow & \mathbf{R}\varGamma_{\mathcal{J}}E \\
\big\downarrow & \searrow^{\gamma_{\mathcal{I},\mathcal{J}}} & \big\uparrow{\scriptstyle \iota_{\mathcal{I}}(\mathbf{R}\varGamma_{\mathcal{J}}E)} \\
\mathbf{R}\varGamma_{\mathcal{I}}E & \xleftarrow[\mathbf{R}\varGamma_{\mathcal{I}}(\iota_{\mathcal{J}}E)]{} & \mathbf{R}\varGamma_{\mathcal{I}}\mathbf{R}\varGamma_{\mathcal{J}}E
\end{array}$$

Proof. By 1.9.5, for $\gamma_{\mathcal{I},\mathcal{J}}$ to be an isomorphism it suffices that $\varGamma_{\mathcal{I}}\varGamma_{\mathcal{J}} = \varGamma_{\mathcal{I}\cap\mathcal{J}}$, which one can show by imitating the argument used to establish (1.1.14.5).

The commutativity can be shown by arguing just as in the proof of 1.3.4. $\square$

As in 1.4.2, derived tensor product makes $\mathbf{D} := \mathbf{D}(X)$ into a symmetric monoidal category, with unit object $\mathcal{O}_X$.

Deriving the inclusion $\varGamma_{\mathcal{I}} \hookrightarrow \mathbf{1}_{\mathcal{A}}$ (the identity functor of $\mathcal{A}$) produces a functorial map $\iota_{\mathcal{I}}\colon \mathbf{R}\varGamma_{\mathcal{I}} \to \mathbf{1}_{\mathbf{D}}$.

Corollary 1.9.8. *Set*

(1.9.8.1) $\qquad (\mathbf{R}\varGamma', \iota') := (\mathbf{R}\varGamma_{\mathcal{I}'}, \iota_{\mathcal{I}'}) \qquad (\mathcal{I}'$ as in 1.9.3).

Then:

(i) $\mathbf{R}\varGamma'\mathbf{D}_{\mathrm{qc}} \subset \mathbf{D}_{\mathrm{qct}}$.

(ii) $\mathbf{D}_{\mathrm{qct}}$ *is the essential image of* $\mathbf{R}\varGamma'\colon \mathbf{D}_{\mathrm{qct}} \to \mathbf{D}$.

Proof. By [AJL2, p. 49, 5.2.1(a)], a complex $E \in \mathbf{D}_{\mathrm{qc}}$ lies in $\mathbf{D}_{\mathrm{qct}}$ if and only if $\iota'(E)\colon \mathbf{R}\varGamma'E \to E$ is an isomorphism. In particular, $\mathbf{D}_{\mathrm{qct}}$ is contained in the essential image of $\mathbf{R}\varGamma'\colon \mathbf{D}_{\mathrm{qct}} \to \mathbf{D}$. Moreover, if $E \cong \mathbf{R}\varGamma'F$ ($F \in \mathbf{D}_{\mathrm{qc}}$) then $E \in \mathbf{D}_{\mathrm{qct}}$, since by 1.9.7,

$$\mathbf{R}\varGamma'E \cong \mathbf{R}\varGamma'\mathbf{R}\varGamma'F \underset{\iota'(\mathbf{R}\varGamma_{\mathcal{I}'}F)}{\cong} \mathbf{R}\varGamma'F \cong E.$$

Thus $\mathbf{D}_{\mathrm{qct}}$ contains the essential image of $\mathbf{R}\varGamma'\colon \mathbf{D}_{\mathrm{qc}} \to \mathbf{D}$; and 1.9.8 follows. $\square$

Proposition 1.9.9. *For any $\mathcal{O}_X$-base $\mathcal{I}$, the pair $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ is a $\otimes$-coreflection of $\mathbf{D}$; and so $(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_X, \iota_{\mathcal{I}}(\mathcal{O}_X))$ is $\mathbf{D}(X)$-idempotent.*

Proof. That $(\mathbf{R}\varGamma_{\mathcal{J}}, \iota_{\mathcal{J}})$ is coreflecting in $\mathbf{D}$ results from 1.9.7. For compatibility with $\underline{\underline{\otimes}}$, argue as in the proof of 1.5.13(i), except for replacing Spec(R) there by an affine formal scheme $X := \mathrm{Spf}(R)$ with R an admissible noetherian ring, and taking J to be a coherent open ideal. The last assertion results from 1.5.7. $\square$

1.9.10. By 1.6.9, the essential image $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}}}$ of $\mathbf{R}\varGamma_{\mathcal{J}}$ is a monoidal category, with product $\underline{\underline{\otimes}}_X$ and unit $\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X$. By 1.9.9 and 1.6.8, an $\mathcal{O}_X$-complex E lies in $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}}}$ if and only if the natural map is an isomorphism $\mathbf{R}\varGamma_{\mathcal{J}}E \xrightarrow{\sim} E$.

The plumpness of $\mathcal{A}_{\mathcal{J}}$ in $\mathcal{A}$ implies that $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}}} \subset \mathbf{D}_{\mathcal{J}}$, the full subcategory of $\mathbf{D}$ spanned by the complexes whose homology sheaves are all in $\mathcal{A}_{\mathcal{J}}$. The converse holds if for some open coherent $\mathcal{O}_X$-ideal I,

$$\mathcal{J} := \{\, \text{open coherent } \mathcal{O}_X\text{-ideals } G \mid \sqrt{G} \supset I \,\},$$

as can be seen, via Koszul complexes, just as in the proof of [AJL2, p. 49, 5.2.1(a)] (with the ideal $\mathscr{J}$ there replaced by I).

Proposition 1.9.11. *For any $\mathcal{O}_X$-base $\mathcal{J}$, it holds that $\mathbf{R}\varGamma_{\mathcal{J}}\mathbf{D}_{\mathrm{qc}} \subset \mathbf{D}_{\mathrm{qct}}$.*

Proof. For any $E \in \mathbf{D}_{\mathrm{qc}}$, one has

$$\mathbf{R}\varGamma_{\mathcal{J}}E \underset{1.9.7}{\cong} \mathbf{R}\varGamma_{\mathcal{J}}\mathbf{R}\varGamma' E \underset{1.9.8}{\in} \mathbf{R}\varGamma_{\mathcal{J}}\mathbf{D}_{\mathrm{qct}},$$

so one need only see that $\mathbf{R}\varGamma_{\mathcal{J}}\mathbf{D}_{\mathrm{qct}} \subset \mathbf{D}_{\mathrm{qct}}$. Hence, one may assume that $E \in \mathbf{D}_{\mathrm{qct}}$ and E is K-injective. Then for any open immersion $u\colon U \hookrightarrow X$, u^*E is K-injective and $u^*\varGamma_{\mathcal{J}}E \cong \varGamma_{\mathcal{J}|_U}u^*E$, so one can assume $X = \mathrm{Spf}(S)$ for an adic noetherian ring S. Since $\mathcal{A}_{\mathrm{qct}}$ is closed under $\varinjlim$, (1.2.1.2) allows one to assume that $\varGamma_{\mathcal{J}} = \varGamma_I$ with $\mathcal{J}$ an open coherent $\mathcal{O}_X$-ideal.

Let $\kappa\colon X \to X_0 := \mathrm{Spec}(S)$ be the (flat) completion map (see proof of 1.9.5). By [AJL2, p. 47, 5.1.2], $I_0 := \kappa_*I$ is a coherent $\mathcal{O}_{X_0}$-ideal, and $I = \kappa^*I_0 = I_0\mathcal{O}_X$. By [AJL2, p. 50, 5.2.4], $E_0 := \kappa_*E \in \mathbf{D}_{\mathrm{qc}Z}(X_0)$ and $E \cong \kappa^*E_0$. Hence by [AJL2, p. 53, 5.2.8(b)], $\mathbf{R}\varGamma_{\mathcal{J}}E \cong \kappa^*\mathbf{R}\varGamma_{\mathcal{J}_0}E_0$, which, by [AJL2, p. 50, 5.2.4], lies in $\mathbf{D}_{\mathrm{qct}}$ since by 1.2.4, E_0 being exact outside Z, one has $\mathbf{R}\varGamma_{\mathcal{J}_0}E_0 \in \mathbf{D}_{\mathrm{qc}Z}(X_0)$. $\square$

Recalling that 1.1.8 holds for open coherent $\mathcal{O}_X$-ideals I, let $\mathcal{J}$ be the $\mathcal{O}_X$-base that corresponds to Φ_Z. Then for any $\mathcal{O}_X$-module M, $\varGamma_{\mathcal{J}}M \subset \varGamma_Z M$, *with equality if* $M \in \mathcal{A}_{\mathrm{qct}}$. The proof is the same as that of 1.1.17, modulo the observation that for any open $U \subset X$ and $s \in \Gamma(U, M)$, $\mathrm{ann}_U(s)$ is an open coherent $\mathcal{O}_U$-ideal. The following more general result comes from [AJS2, §§2.1–2.2].

Proposition 1.9.12. *Let $\mathcal{J}$ be the $\mathcal{O}_X$-base that corresponds to Φ_Z. The natural map $\theta_{Z,E}$ is an* isomorphism $\mathbf{R}\varGamma_{\mathcal{J}}E \xrightarrow{\sim} \mathbf{R}\varGamma_Z E$ *for all $E \in \mathbf{D}_{\mathrm{qct}}$.*

Proof. As in the proof of 1.9.11, one may assume $X = \mathrm{Spf}(S)$ for an adic noetherian ring S with ideal of definition, say, L. Let $\kappa\colon X = \mathrm{Spf}(S) \to \mathrm{Spec}(S) =: X_0$ and let $\mathcal{J}_0$ be as in the proof of 1.9.5.

By 1.9.4, $\mathbf{R}\varGamma_Z E \in \mathbf{D}_{\mathrm{qct}}$, and by 1.9.11, $\mathbf{R}\varGamma_{\mathcal{J}}E \in \mathbf{D}_{\mathrm{qct}}$. Since κ_* sends $\mathbf{D}_{\mathrm{qct}}$ fully faithfully into $\mathbf{D}_{\mathrm{qc}}(X_0)$ [AJL2, p. 50, 5.2.4(a)], it suffices for 1.9.12 to show that $\kappa_*\theta_{Z,E}$ is an isomorphism.

One may assume that the $\mathcal{O}_X$-complex E is K-injective, whence, κ being flat, the $\mathcal{O}_{X_0}$-complex κ_*E is K-injective. Thus one need only verify that the following $\mathbf{D}(X_0)$-diagram commutes:

$$\begin{array}{ccc}
\kappa_*\varGamma_{\mathcal{J}}E & \xrightarrow{\kappa_*\theta_{Z,E}} & \kappa_*\varGamma_{Z}E \\
{\scriptstyle (1.9.5.1)}\big\downarrow \simeq & & \big\| {\scriptstyle (1.1.13.5),\ 1.1.3} \\
\varGamma_{\mathcal{J}_0}\kappa_*E & \underset{1.2.3}{\xrightarrow{\ \sim\ }} & \varGamma_{Z}\kappa_*E
\end{array}$$

For this it's enough, by 1.3.2, to show that the natural map $\kappa_*\varGamma_{\mathcal{J}}E \to \kappa_*E$ factors naturally as $\kappa_*\varGamma_{\mathcal{J}}E \underset{(1.9.5.1)}{\xrightarrow{\sim}} \varGamma_{\mathcal{J}_0}\kappa_*E \longrightarrow \kappa_*E$, a task that comes down easily to verifying that the natural composite isomorphism

$$\kappa_*E \xrightarrow{\sim} \kappa_*\mathcal{H}om_{\mathcal{O}_X}(\kappa^*\mathcal{O}_{X_0}, E) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X_0}}(\mathcal{O}_{X_0}, \kappa_*E) \xrightarrow{\sim} \kappa_*E$$

is the identity map of κ_*E—which results from [Lp1, p. 117, 3.5.6(e)], or from an explicit description of the isomorphisms involved. Details are left to the reader. □

Another way to prove Proposition 1.9.12 is by upgrading the proof of Proposition 1.2.3. This means, ultimately, to adapt the proof of [AJL1, p. 25, Lemma (3.2.3)] to the formal-scheme context. For this, two points have to be addressed.

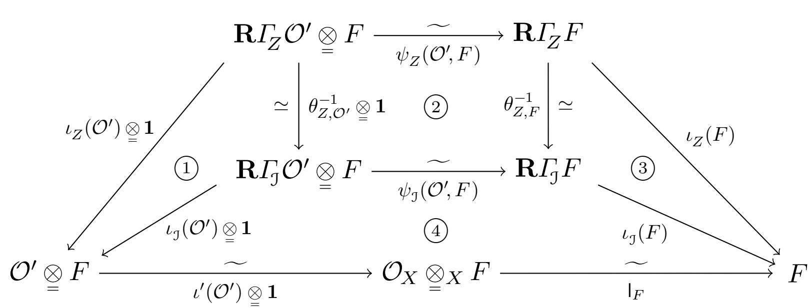
First, if V is an affine formal scheme and $g\colon V \to W$ is a separated—hence affine—map of formal schemes then the natural map is a $\mathbf{D}(V)$-isomorphism $g_*\mathcal{O}_V \xrightarrow{\sim} \mathbf{R}g_*\mathcal{O}_V$. In view of [GD3, p. 68, (13.3.1)], this follows from the well-known case where V and W are ordinary schemes. (For greater generality, see [AJL2, p. 39, 3.4.2].)

Second, one needs to extend the *projection isomorphism* to the formal-scheme context. This is done in Proposition 1.9.29 below.

$\mathbf{D}_{\text{qct}}$ has a monoidal structure with product $\underline{\underline{\otimes}}$ and unit object $\mathcal{O}' := \mathbf{R}\varGamma'\mathcal{O}_X$ (see 1.9.28). For any $\mathcal{O}_X$-base $\mathcal{I}$, $\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}' \cong \mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}_X$ (1.9.7 with $\mathcal{J} := \mathcal{I}'$).

Proposition 1.9.13. *Let $Z \subset X$ be specialization-stable, and let $\mathcal{I}$ be the $\mathcal{O}_X$-base corresponding to Φ_Z. Then $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ and $(\mathbf{R}\varGamma_{Z}, \iota_{Z})$ restrict to naturally isomorphic $\otimes$-coreflections of $\mathbf{D}_{\text{qct}}$, whence $(\mathbf{R}\varGamma_{\mathcal{I}}\mathcal{O}', \iota_{\mathcal{I}}(\mathcal{O}'))$ and $(\mathbf{R}\varGamma_{Z}\mathcal{O}', \iota_{Z}(\mathcal{O}'))$ are naturally isomorphic $\mathbf{D}_{\text{qct}}$-idempotent pairs.*

Proof. By 1.9.11, $\mathbf{R}\varGamma_{\mathcal{I}}\mathbf{D}_{\text{qct}} \subset \mathbf{D}_{\text{qct}}$. So by 1.9.9 and 1.3.3(a), $(\mathbf{R}\varGamma_{\mathcal{I}}, \iota_{\mathcal{I}})$ restricts to a coreflection of $\mathbf{D}_{\text{qct}}$, in fact a $\otimes$-coreflection because by 1.9.9, subdiagram ④ in the following diagram (with ψ as in 1.5.13, θ as in 1.9.12 and $F \in \mathbf{D}_{\text{qct}}(X)$) commutes:



It holds that $\iota_Z(F) \circ \theta_{Z,F} = \iota_{\mathcal{I}}(F)$—clearly for K-injective F, hence for all F. It follows easily that the restriction of $(\mathbf{R}\varGamma_Z, \iota_Z)$ to $\mathbf{D}_{\text{qct}}$ is coreflecting. Moreover, for $F \in \mathbf{D}_{\text{qct}}$, subdiagrams ① and ③ in the above diagram commute.

The commutativity of ② follows readily from the definitions of $\psi_{\mathcal{I}}$ and ψ_Z (details left to the reader). Thus the border of the diagram commutes, and so $(\mathbf{R}\varGamma_Z, \iota_Z)|_{\mathbf{D}_{\text{qct}}}$ is a $\otimes$-coreflection.

The last assertion results from 1.5.7. (For its converse, see 1.9.20 below.) □

Proposition 1.9.14. *If Z and Z' are specialization-stable subsets of X then for all $E \in \mathbf{D}_{\mathrm{qct}}$, the natural map is an isomorphism*

$$\mathbf{R}\varGamma_{Z\cap Z'}E \xrightarrow{\sim} \mathbf{R}\varGamma_Z\mathbf{R}\varGamma_{Z'}E.$$

Proof. This follows from 1.9.7, 1.9.12 and 1.9.4. □

Proposition 1.9.15. *Let $\mathcal{I}$ be an $\mathcal{O}_X$-base, A a small filtered category, $n \in \mathbb{Z}$, and $\mathcal{M}\colon A \to \{\mathcal{O}_X\text{-complexes}\}$ a functor. The natural map is an isomorphism*

$$\varinjlim_A H^n_{\mathcal{I}} \circ \mathcal{M} \xrightarrow{\sim} H^n_{\mathcal{I}}\big(\varinjlim_A \mathcal{M}\big).$$

In particular, $\mathbf{R}\varGamma_{\mathcal{I}}$ commutes with small direct sums in $\mathbf{D}(X)$.

Proof. Essentially the same as the (first) proof of 1.2.17. □

Proposition 1.9.16. *Let Z be a specialization-stable subset of X, A a small filtered category, $\mathcal{M}$ a functor from A to the category of $\mathcal{O}_X$-complexes all of whose homology modules are in $\mathcal{A}_{\mathrm{qct}}(X)$, and $n \in \mathbb{Z}$. The natural map is an isomorphism*

$$\varinjlim_A H^n_Z \circ \mathcal{M} \xrightarrow{\sim} H^n_Z\big(\varinjlim_A \mathcal{M}\big).$$

In particular, $\mathbf{R}\varGamma_Z$ commutes with small direct sums in $\mathbf{D}_{\mathrm{qct}}(X)$.

Proof. This follows from 1.9.15 and 1.9.12. □

* * * * *

Corollary 1.9.17. *The objects of the full subcategory $(\mathbf{D}_{\mathrm{qct}})_{\mathbf{R}\varGamma_Z\mathcal{O}'} \subset \mathbf{D}_{\mathrm{qct}}$ (1.6.7) are those $E \in \mathbf{D}_{\mathrm{qct}}$ with* $\mathrm{Supp}(E) \subset Z$.

Proof. For any $E \in \mathbf{D}_{\mathrm{qct}}$, 1.3.13 gives

$$\mathrm{Supp}(E) \subset Z \iff E \in \mathbf{D}_{\Phi_Z}(X) \cap \mathbf{D}_{\mathrm{qct}}.$$

In view of 1.9.13, 1.3.9(i) with $\mathbf{D} := \mathbf{D}_{\mathrm{qct}}$ (same proof) shows that $\mathbf{D}_{\Phi_Z}(X) \cap \mathbf{D}_{\mathrm{qct}}$ is the essential image $(\mathbf{D}_{\mathrm{qct}})_{\mathbf{R}\varGamma_Z\mathcal{O}'}$ of $\mathbf{R}\varGamma_Z\colon \mathbf{D}_{\mathrm{qct}} \to \mathbf{D}_{\mathrm{qct}}$. □

Proposition 1.9.18. $\mathrm{Supp}(\mathbf{R}\varGamma_Z\mathcal{O}') = Z$.

Proof. That $\mathrm{Supp}(\mathbf{R}\varGamma_Z\mathcal{O}') \subset Z$ is given by 1.3.14.

For the opposite inclusion, let $x \in Z$, let $\widehat{\mathcal{O}_{X,x}}$ be the maximal-ideal completion of $\mathcal{O}_{X,x}$, X_x the one-point formal scheme $\mathrm{Spf}(\widehat{\mathcal{O}_{X,x}})$ ($\widehat{\mathcal{O}_{X,x}}$ being topologized in the usual way), $\mathcal{K}_x$ the residue field of $\widehat{\mathcal{O}_{X,x}}$ (= residue field of $\mathcal{O}_{X,x}$) viewed as an object of $\mathcal{A}_{\mathrm{qct}}(X_x)$, $\iota_x\colon X_x \to X$ the canonical map, and $\mathcal{K}(x) := \iota_{x*}\mathcal{K}_x \in \mathcal{A}_{\mathrm{qct}}(X)$ (see [AJL2, p. 47, 5.1.1].)

Then $\mathcal{K}(x)$ has support $\overline{\{x\}} \subset Z$ and is flabby, hence K-flabby (section 1.2.8), hence $\varGamma_Z$-acyclic (section 1.2.7), so there are natural isomorphisms

$$\mathcal{K}(x) = \varGamma_Z\mathcal{K}(x) \xrightarrow{\sim} \mathbf{R}\varGamma_Z\mathcal{K}(x) \underset{1.9.13}{\xrightarrow{\sim}} \mathbf{R}\varGamma_Z\mathcal{O}' \underset{=}{\otimes}_X \mathcal{K}(x).$$

Since the stalk $\mathcal{K}(x)_x \neq 0$, therefore $(\mathbf{R}\varGamma_Z\mathcal{O}')_x \neq 0$, that is, $x \in \mathrm{Supp}(\mathbf{R}\varGamma_Z\mathcal{O}')$. Thus $Z \subset \mathrm{Supp}(\mathbf{R}\varGamma_Z\mathcal{O}')$. □

Corollary 1.9.19. $\mathbf{R}\varGamma_Z\mathcal{O}' \preccurlyeq \mathbf{R}\varGamma_{Z'}\mathcal{O}' \iff Z \subset Z'$.

Proof. One has

$$[\,\mathbf{R}\varGamma_{Z}\mathcal{O}' \preccurlyeq \mathbf{R}\varGamma_{Z'}\mathcal{O}'\,] \underset{1.6.10}{\iff} [\,\mathbf{R}\varGamma_{Z}\mathcal{O}' \in (\mathbf{D}_{\mathrm{qct}})_{\mathbf{R}\varGamma_{Z'}\mathcal{O}'}] \underset{1.9.17}{\iff} [\,\mathrm{Supp}(\mathbf{R}\varGamma_{Z}\mathcal{O}') \subset Z'\,] \underset{1.9.18}{\iff} Z \subset Z'. \qquad \square$$

As in 1.9.13, for any specialization-stable $Z \subset X$, the pair $\big(\mathbf{R}\varGamma_{Z}\mathcal{O}', \iota_Z(\mathcal{O}')\big)$ is $\mathbf{D}_{\mathrm{qct}}$-idempotent. The converse is, essentially, [AJS2, p. 604, Corollary 5.4]:

Proposition 1.9.20. *Every $\mathbf{D}_{\mathrm{qct}}$-idempotent pair (A, α) is isomorphic to a pair $\big(\mathbf{R}\varGamma_{Z}\mathcal{O}', \iota_Z(\mathcal{O}')\big)$ for some specialization-stable Z—necessarily* $\mathrm{Supp}(A)$ (*see* 1.9.18).

Proof. The idea is to reduce, via localization and completion, to the known case where $X = \mathrm{Spec}(S)$ for a noetherian ring S.

Since $H^iA \in \mathcal{A}_{\mathrm{qct}} \subset \mathcal{A}_{\vec{\mathrm{c}}}$ is the union of all its coherent submodules, each of whose support is closed (see §1.1.7), therefore $\mathrm{Supp}(A) = \cup_{i\in\mathbb{Z}}\,\mathrm{Supp}(H^iA)$ *is specialization-stable*. So by 1.9.18 and 1.6.5, it's enough to show that if (A, α) and (B, β) are $\mathbf{D}_{\mathrm{qct}}$-idempotent pairs with $\mathrm{Supp}(A) = \mathrm{Supp}(B)$, then $A \cong B$, i.e., by 1.6.1, the maps $A \underline{\underline{\otimes}}_X B \to B$ and $A \underline{\underline{\otimes}}_X B \to A$ induced by α and β, respectively, induce homology isomorphisms. In view of 1.4.9 with f an open immersion, the question is local, so one may assume that $X = \mathrm{Spf}(S)$ where S is an adic noetherian ring with ideal of definition, say, L.

Let $\kappa\colon X := \mathrm{Spf}(S) \to \mathrm{Spec}(S) := X_0$ be the completion map (as in the proof of 1.9.5). This map is flat, so the functor $\kappa^*\colon \mathcal{A}(X_0) \to \mathcal{A}(X)$ is exact, therefore extends to $\kappa^*\colon \mathbf{D}(X_0) \to \mathbf{D}(X)$.

Since X and $Y := \mathrm{Spec}(S/L)$ are homeomorphic, and κ is, topologically, the inclusion $Y \hookrightarrow X_0$, one can regard Y as a closed subset of X_0, whence the functor $\kappa_*\colon \mathcal{A}(X) \to \mathcal{A}(X_0)$ is exact, therefore extends to $\kappa_*\colon \mathbf{D}(X) \to \mathbf{D}(X_0)$. The category $\mathbf{D}_{\mathrm{qc}}(X_0)$ has a monoidal structure with product $\underline{\underline{\otimes}}_{X_0}$ and unit object $\mathcal{O}_{X_0}$ (see 1.4.2(a)). For any $W \subset X_0$, the pair $(\mathbf{R}\varGamma_W\mathcal{O}_{X_0}, \iota_W(\mathcal{O}_{X_0}))$ is $\mathbf{D}_{\mathrm{qc}}(X_0)$-idempotent (see 1.5.14). So by 1.6.9, $\mathbf{D}_{\mathrm{qc}W}(X_0) := (\mathbf{D}_{\mathrm{qc}}(X_0))_{\mathbf{R}\varGamma_W\mathcal{O}_{X_0}}$ has a monoidal structure with product $\underline{\underline{\otimes}}_{X_0}$ and unit object $\mathbf{R}\varGamma_W\mathcal{O}_{X_0}$. Also, if $W' \subset W$ then $\mathbf{R}\varGamma_{W'}\mathcal{O}_{X_0} \preccurlyeq \mathbf{R}\varGamma_W\mathcal{O}_{X_0}$, so by 1.6.10(ii), $\mathbf{R}\varGamma_{W'}\mathcal{O}_{X_0}$ is $\mathbf{D}_{\mathrm{qc}W}(X_0)$-idempotent.

By 1.9.17 for the discrete formal scheme X_0, $\mathbf{D}_{\mathrm{qc}Y}(X_0)$ is the full subcategory spanned by the $\mathcal{O}_{X_0}$-complexes whose homology modules are quasi-coherent and have support contained in Y. Thus the notation here agrees with that in [AJL2, p. 50, 5.2.4(a)], which shows that the functors κ^* and κ_* give inverse isomorphisms between $\mathbf{D}_{\mathrm{qc}Y}(X_0)$ and $\mathbf{D}_{\mathrm{qct}}(X)$. Also, one has the usual isomorphism

$$v(E, F)\colon \kappa^*E \underline{\underline{\otimes}}_X \kappa^*F \xrightarrow{\ \sim\ } \kappa^*(E \underline{\underline{\otimes}}_{X_0} F) \qquad (E, F \in \mathbf{D}(X_0)).$$

Taking $E := \mathbf{R}\varGamma_Y\mathcal{O}_{X_0}$, one deduces that $\kappa^*\mathbf{R}\varGamma_Y\mathcal{O}_{X_0}$ is a unit object in the monoidal category $\mathbf{D}_{\mathrm{qct}}$ (see 1.9.28); and one checks (directly, or via 1.4.8 with $\mathcal{O}_2 := \mathbf{R}\varGamma_Y\mathcal{O}_{X_0}$, $\mathcal{O}_1 := \kappa^*\mathcal{O}_2$, and $u := \mathbf{1}_{\mathcal{O}_1}$) that κ^* and κ_* induce inverse bijections between the sets of $\mathbf{D}_{\mathrm{qc}Y}(X_0)$-idempotents and $\mathbf{D}_{\mathrm{qct}}(X)$-idempotents.

Let $Z \subset Y$ be specialization-stable. Over X_0, set $\mathcal{I}_0 := \mathcal{I}_{\Phi_Z}$ (see 1.1.8.1, 1.1.1). The $\mathcal{O}_X$-base corresponding to the s.o.s. Φ_Z in X is

$$\mathcal{I} = \mathcal{I}_0\mathcal{O}_X := \{\, I_0\mathcal{O}_X \mid I_0 \in \mathcal{I}_0 \,\}$$

(see 1.9.6). For $F \in \mathbf{D}(X)$, one has, as in the lines following (1.9.5.1), the isomorphism $\bar\xi_{\kappa,\mathcal{J},F}\colon \kappa_*\mathbf{R}\varGamma_{\mathcal{J}}F \xrightarrow{\,\sim\,} \mathbf{R}\varGamma_{\mathcal{J}_0}\kappa_*F$, whence the natural composite map

$$(1.9.20.1)\qquad \kappa^*\mathbf{R}\varGamma_{\mathcal{J}_0}F \longrightarrow \kappa^*\mathbf{R}\varGamma_{\mathcal{J}_0}\kappa_*\kappa^*F \xrightarrow[\kappa^*\bar\xi^{-1}]{\,\sim\,} \kappa^*\kappa_*\mathbf{R}\varGamma_{\mathcal{J}}\kappa^*F \longrightarrow \mathbf{R}\varGamma_{\mathcal{J}}\kappa^*F.$$

This is an *isomorphism*: apply cohomology H^n $(n \in \mathbb{Z})$, then use (1.2.1.2) to reduce to where there is an open coherent $\mathcal{O}_{X_0}$-ideal I_0 such that

$$\mathcal{J}_0 := \{\,\text{open coherent } \mathcal{O}_X\text{-ideals } J \mid \sqrt{J} \supset I_0\,\};$$

then identify (1.9.20.1), via [AJL1, p. 18, 3.1.1], with the natural isomorphism $v(\mathcal{K}^\bullet_\infty(\mathbf{t}), F)$, where $\mathbf{t}$ is a finite sequence in S that generates I_0, and $\mathcal{K}^\bullet_\infty(\mathbf{t})$ is as in the proof of 1.2.6. (Details left to the reader.)[8]

One has then the composite isomorphism

$$\kappa^*\mathbf{R}\varGamma_Z\mathcal{O}_{X_0} \xrightarrow[1.2.3]{\,\sim\,} \kappa^*\mathbf{R}\varGamma_{\mathcal{J}_0}\mathcal{O}_{X_0} \xrightarrow[(1.9.20.1)]{\,\sim\,} \mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X \xrightarrow[1.9.7]{\,\sim\,} \mathbf{R}\varGamma_{\mathcal{J}}\mathbf{R}\varGamma'\mathcal{O}_X = R\varGamma_{\mathcal{J}}\mathcal{O}' \xrightarrow[1.9.13]{\,\sim\,} \mathbf{R}\varGamma_Z\mathcal{O}'.$$

Accordingly, it suffices that any $\mathbf{D}_{\mathrm{qc}Y}(X_0)$-idempotent be isomorphic to $\mathbf{R}\varGamma_Z\mathcal{O}_{X_0}$ for some specialization-stable $Z \subset Y$. But in view of the monoidal equivalence between $\mathbf{D}_{\mathrm{qc}}(X_0)$ and $\mathbf{D}(S)$ [BN, p. 225, Theorem 5.1], that results, essentially, from [Nm1, p. 526, Theorem 2.8] and its proof—cf. [Lp2, Proposition 3.5.7] and the remarks following it, keeping in mind the bijection between specialization-stable subsets of Spec(S) and decent topologies on S, as in 1.7.1 above. □

Corollary 1.9.21. *The mapping that takes the isomorphism class of A to* Supp(A) *induces an order-preserving bijection*

$$\{\text{isomorphism classes of } \mathbf{D}_{\mathrm{qct}}\text{-idempotents}\} \leftrightarrow \{\text{specialization-stable subsets of } X\}.$$

Proof. This follows from 1.9.13, 1.9.19 and 1.9.20. □

Corollary 1.9.22. *There is an order-reversing bijection*

$$\{\mathcal{O}_X\text{-bases}\} \leftrightarrow \{\text{isomorphism classes of } \mathbf{D}_{\mathrm{qct}}\text{-idempotents}\}$$

that sends an $\mathcal{O}_X$-base $\mathcal{J}$ to the isomorphism class of $\mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X$.

Proof. The order-reversing bijection arising from 1.1.8 takes each $\mathcal{O}_X$-base $\mathcal{J}$ to the specialization-stable set $Z = \cup_{I\in\mathcal{J}} Z(I)$; and the order-preserving bijection in 1.9.21 takes Z to the isomorphism class of $\mathbf{R}\varGamma_Z\mathcal{O}' \cong \mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}' \cong \mathbf{R}\varGamma_{\mathcal{J}}\mathcal{O}_X$ (see 1.9.12 and 1.9.7). □

Corollary 1.9.23. *If A is a $\mathbf{D}_{\mathrm{qct}}(X)$-idempotent and $E \in \mathbf{D}_{\mathrm{qct}}(X)$, then*

$$E \in (\mathbf{D}_{\mathrm{qct}})_A \iff \operatorname{Supp}(E) \subset \operatorname{Supp}(A).$$

Proof. With $Z := \operatorname{Supp}(A)$, Proposition 1.9.20 allows one to assume $A = \mathbf{R}\varGamma_Z\mathcal{O}'$, in which case the assertion is just Corollary 1.9.17. □

Corollary 1.9.24. *Let $x \in X$ and let $\mathcal{K}(x) \in \mathbf{D}_{\mathrm{qct}}(X)$ be as in the proof of* 1.9.18. *For any $\mathbf{D}_{\mathrm{qct}}(X)$-idempotent (A, α),*

$$\mathcal{K}(x) \in (\mathbf{D}_{\mathrm{qct}})_A \iff A \underset{=}{\otimes}_X \mathcal{K}(x) \underset{\text{via } \alpha}{\cong} \mathcal{K}(x) \iff A \underset{=}{\otimes}_X \mathcal{K}(x) \neq 0 \iff x \in \operatorname{Supp}(A).$$

---

[8] Alternatively, it is an instance of the isomorphism in 1.9.26.

Proof. Supp(A) is specialization-stable, so if $x \in \mathrm{Supp}(A)$ then $\overline{\{x\}} \subset \mathrm{Supp}(A)$, whence by 1.9.23, $\mathcal{K}(x) \in (\mathbf{D}_{\mathrm{qct}})_A$, that is, by 1.6.8, $A \underset{=}{\otimes}_X \mathcal{K}(x) \underset{\text{via}\,\alpha}{\cong} \mathcal{K}(x)$, whence $A \underset{=}{\otimes}_X \mathcal{K}(x) \neq 0$, whence, by 1.9.20, if $Z := \mathrm{Supp}(A)$ then $\mathbf{R}\varGamma_Z\mathcal{O}' \underset{=}{\otimes}_X \mathcal{K}(x) \neq 0$.

Conversely, by 1.9.13 and since $\mathcal{K}(x)$ is flabby, hence K-flabby, and since x lies in the support of any nonzero section of $\mathcal{K}(x)$ over any open set U, therefore

$$0 \neq \mathbf{R}\varGamma_Z\mathcal{O}' \underset{=}{\otimes}_X \mathcal{K}(x) \cong \mathbf{R}\varGamma_Z\mathcal{K}(x) \cong \varGamma_Z\mathcal{K}(x) \implies x \in Z = \mathrm{Supp}(A). \qquad \square$$

* * * * *

The next Proposition enhances Corollary 1.9.21.

Set $\mathcal{O}'_X := \mathbf{R}\varGamma'_X\mathcal{O}_X$ and $\mathcal{O}'_W := \mathbf{R}\varGamma'_W\mathcal{O}_W$.

Proposition 1.9.25. *Let $f\colon W \to X$ be a map of noetherian formal schemes. Set $\mathbf{L}'f^* := \mathbf{R}\varGamma'_W\mathbf{L}f^*$. Then the natural map is an isomorphism*

$$\mathbf{L}'f^*\mathcal{O}'_X = \mathbf{L}'f^*\mathbf{R}\varGamma'_X\mathcal{O}_X \xrightarrow{\ \sim\ } \mathbf{L}'f^*\mathcal{O}_X = \mathcal{O}'_W\,;$$

and if $\alpha\colon A \to \mathcal{O}'_X$ is a $\mathbf{D}_{\mathrm{qct}}(X)$-idempotent pair with Supp$(A) = Z$, *then*

$$\mathbf{L}'f^*\alpha\colon \mathbf{L}'f^*A \longrightarrow \mathbf{L}'f^*\mathcal{O}'_X = \mathcal{O}'_W$$

is a $\mathbf{D}_{\mathrm{qct}}(W)$-idempotent pair with Supp$(B) = f^{-1}Z$.

Proof. For the first assertion, see [AJL2, p. 53, 5.2.8(c)].

The pair (A,α) is clearly $\mathbf{D}(X)_{\mathcal{O}'}$-idempotent, whence $A \xrightarrow{\alpha} \mathcal{O}'_X \xrightarrow{\iota'(\mathcal{O}_X)} \mathcal{O}_X$ is $\mathbf{D}(X)$-idempotent, see 1.9.9, 1.6.10. By 1.4.9 and 1.5.10 with $(\Gamma,\iota) := (\mathbf{R}\varGamma'_W, \iota'_W)$ (see again 1.9.9), the composition

$$\mathbf{L}'f^*A \xrightarrow{\mathbf{L}'f^*\alpha} \mathbf{L}'f^*\mathcal{O}'_X \xrightarrow{\mathbf{L}'f^*\iota'_X(\mathcal{O}_X)} \mathbf{L}'f^*\mathcal{O}_X \xrightarrow{\iota'(\mathbf{L}f^*\mathcal{O}_X)} \mathbf{L}f^*\mathcal{O}_X = \mathcal{O}_W$$

is $\mathbf{D}(W)$-idempotent. In particular, this holds when $\alpha = \mathbf{1}_{\mathcal{O}'_X}$.

Note that $\mathbf{L}f^*A \in \mathbf{D}_{\mathrm{qc}}(W)$: the question being local on X, one can assume that $A \in \mathbf{D}_{\vec{\mathrm{c}}}(X)$ and apply [AJL2, p. 37, 3.3.5]. So by 1.9.11, $\mathbf{L}'f^*A$ in $\mathbf{D}_{\mathrm{qct}}(W)$. Hence, as in the proof of 1.6.10(ii), with $B := \mathbf{L}'f^*A$, $A := \mathbf{L}'f^*\mathcal{O}_X$ and $\lambda := \mathbf{L}'f^*\alpha$,

$$\mathbf{L}'f^*\alpha\colon \mathbf{L}'f^*A \to \mathbf{L}'f^*\mathcal{O}'_X = \mathcal{O}'_W$$

is $\mathbf{D}(W)_{\mathcal{O}'_W}$-idempotent, and thus $\mathbf{D}_{\mathrm{qct}}(W)$-idempotent.

It remains to be shown that $B := \mathbf{L}'f^*A$ has support $f^{-1}Z$.

Assuming, as one may, that A is K-flat, one has for $w \in W$ and $x := f(w)$ that

$$(\mathbf{L}f^*A)_w = (f^*A)_w = \mathcal{O}_{W,w} \otimes_{\mathcal{O}_{X,x}} A_x.$$

If $x \notin Z$ then A_x is exact and K-flat,[9] and therefore $(\mathbf{L}f^*A)_w$ is exact—as one sees upon replacing $\mathcal{O}_{W,w}$ by a quasi-isomorphic K-flat $\mathcal{O}_{X,x}$-complex; in other words, $w \notin \mathrm{Supp}(\mathbf{L}f^*A)$. Hence $\mathrm{Supp}(B) \subset f^{-1}Z$.

For the opposite inclusion, suppose $w \in f^{-1}Z \setminus \mathrm{Supp}(B)$. Let $\mathcal{K}(w) \in \mathcal{A}_{\mathrm{qct}}(W)$ be as in the proof of 1.9.18. This sheaf is K-flabby, hence f_*-acyclic (see section 1.2.7). One has $\mathbf{R}f_*\mathcal{K}(w) \cong f_*\mathcal{K}(w) \in \mathcal{A}_{\mathrm{qct}}(X)$ [AJL2, p. 47, 5.1.1], the stalk $(f_*\mathcal{K}(w))_x$ is the residue field of $\mathcal{O}_{W,w}$, and $f_*\mathcal{K}(w)$ vanishes outside $\overline{\{f(w)\}} \subset Z$, so that $\mathrm{Supp}(\mathbf{R}f_*\mathcal{K}(w)) \subset \mathrm{Supp}(A)$. Hence $0 \neq \mathbf{R}f_*\mathcal{K}(w) \cong A \underset{=}{\otimes}_X \mathbf{R}f_*\mathcal{K}(w)$, where the last isomorphism comes from 1.9.23 and 1.6.8.

[9] For any exact $\mathcal{O}_{X,x}$-complex C, the extension by 0 of the constant sheaf C on $\bar{x}$ is exact, as is its tensor product with the K-flat $\mathcal{O}_X$-complex A, whence $C \otimes_{\mathcal{O}_{X,x}} A_x$ is exact.

Since $A\in \mathbf{D}_{\mathrm{qct}}(X)\subset \mathbf{D}_{\vec{c}}(X)$ and $\mathcal{K}(w)\in \mathbf{D}_{\mathrm{qct}}(W)\subset \mathbf{D}_{\vec{c}}(W)$, Proposition 1.9.29 below gives a natural "projection" isomorphism

$$0\neq A \underline{\underline{\otimes}}_X \mathbf{R}f_*\mathcal{K}(w)\cong \mathbf{R}f_*\big(\mathbf{L}f^*A\underline{\underline{\otimes}}_W \mathcal{K}(w)\big);$$

so one has, via 1.5.4, 1.9.9 and (1.9.8.1), natural isomorphisms

$$\begin{aligned}0\neq \mathbf{L}f^*A\underline{\underline{\otimes}}_W \mathcal{K}(w)&\cong \mathbf{L}f^*A\underline{\underline{\otimes}}_W \mathbf{R}\varGamma'_W\mathcal{K}(w)\\ &\cong \mathbf{R}\varGamma'_W\big(\mathbf{L}f^*A\underline{\underline{\otimes}}_W \mathcal{K}(w)\big)\\ &\cong \mathbf{R}\varGamma'_W\mathbf{L}f^*A\underline{\underline{\otimes}}_W \mathcal{K}(w)=B\underline{\underline{\otimes}}_W \mathcal{K}(w)=0,\end{aligned}$$

where the last equality comes from 1.9.24. This contradiction shows w can't exist. Thus $\mathrm{Supp}(B)=f^{-1}Z$. □

Proposition 1.9.26. *Let $f\colon W\to X$ be a map of noetherian formal schemes, let $\mathcal{I}$ be an $\mathcal{O}_X$-basis, and let $\mathcal{I}_f$ be the $\mathcal{O}_W$-basis*

$$\mathcal{I}_f := \{\,\text{open coherent } \mathcal{O}_W\text{-ideals } J \mid \sqrt{J}\supset I\mathcal{O}_W \text{ for some } I\in\mathcal{I}\,\}.$$

There is a unique functorial isomorphism

$$\xi(\mathcal{I},E)\colon \mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{I}}E \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathcal{I}_f}\mathbf{L}f^*E \qquad (E\in\mathbf{D}(X))$$

whose composition with the natural map $s\colon \mathbf{R}\varGamma_{\mathcal{I}_f}\mathbf{L}f^*E\to \mathbf{L}f^*E$ is the natural map $q\colon \mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{I}}E\to\mathbf{L}f^*E$.

Proof. Set $\mathbf{D}:=\mathbf{D}(W)$. First of all, it holds that $\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{I}}E$ *is in the essential image* $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{I}_f}}$ *of* $\mathbf{R}\varGamma_{\mathcal{I}_f}$—which implies the existence and uniqueness of $\xi(\mathcal{I},E)$ as a $\mathbf{D}(W)$-*map* (see 1.9.9 and 1.3.2).

To see this, assume without loss of generality that E is K-injective. Regard the ordered set $\mathcal{I}$ as a category in the usual way (with containments as morphisms), and let P be a functor from $\mathcal{I}$ to the category of maps of $\mathcal{O}_X$-complexes such that for each $I\in\mathcal{I}$, $P(I):P_I\to\varGamma_I E$ is a K-flat resolution, and for each $\mathcal{I}$-morphism $I'\supset I$, the resulting map $\varGamma_{I'}E\to\varGamma_I E$ is the natural one. The existence of such a P is given, for instance, by [Lp1, p. 61, 2.5.5]. Then with $\varinjlim := \varinjlim_{I\in\mathcal{I}}$, $\varinjlim P_I$ is a K-flat resolution of $\varinjlim \varGamma_I E=\varGamma_{\mathcal{I}}E$; and so

$$\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{I}}E\cong f^*\varinjlim P_I\cong \varinjlim f^*P_I.$$

If for an open coherent $\mathcal{O}_X$-ideal I, $\mathcal{I}$ is the $\mathcal{O}_X$-basis

(1.9.26.1) $$\mathcal{I}_I := \{\,\text{open coherent } \mathcal{O}_X\text{-ideals } G \mid \sqrt{G}\supset I\,\},$$

then

$$\mathcal{I}_f=\mathcal{I}_{I\mathcal{O}_W}=\{\,\text{open coherent } \mathcal{O}_W\text{-ideals } J\mid \sqrt{J}\supset I\mathcal{O}_W\,\}.$$

So in this case, $\xi(\mathcal{I},E)$ is the isomorphism given by [AJL2, p. 53, 5.2.8(b)], whence

(1.9.26.2) $$f^*P_I\cong \mathbf{L}f^*\mathbf{R}\varGamma_I E\cong \mathbf{R}\varGamma_{I\mathcal{O}_W}\mathbf{L}f^*E=\mathbf{R}\varGamma_{\mathcal{I}_f}\mathbf{L}f^*E\in\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{I}_f}}.$$

The category $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{I}_f}}$ is a triangulated subcategory of $\mathbf{D}(X)$: it is clearly closed under translation, and if T is a $\mathbf{D}(X)$-triangle with two vertices in $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{I}_f}}$, then since $\mathbf{R}\varGamma_{\mathcal{I}_f}$ is coreflecting (see 1.9.9), the natural map is an isomorphism $\mathbf{R}\varGamma_{\mathcal{I}_f}T\xrightarrow{\ \sim\ }T$, so the third vertex is also in $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{I}_f}}$.

Moreover, 1.9.15 implies that $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}_f}}$ is closed under small direct sums, that is, $\mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}_f}}$ is a localizing subcategory of $\mathbf{D}$. So [AJS1, pp. 232–233, Theorems 2.2 and 3.1] give that

$$\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}E \cong \mathbf{L}f^*\varGamma_{\mathcal{J}}E \cong \varinjlim_{I\in\mathcal{J}} f^*P_I \in \mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}_f}},$$

as desired.

Note next that the natural map

$$\nu_I\colon \varinjlim_{I\in\mathcal{J}} H^n\mathbf{L}f^*\mathbf{R}\varGamma_I E \longrightarrow H^n\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}E$$

is the natural composite *isomorphism*

$$\varinjlim_{I\in\mathcal{J}} H^n\mathbf{L}f^*\mathbf{R}\varGamma_I E \xrightarrow{\ \sim\ } \varinjlim_{I\in\mathcal{J}} H^n f^*P_I \xrightarrow{\ \sim\ } H^n f^*\varinjlim_{I\in\mathcal{J}} P_I \xrightarrow{\ \sim\ } H^n\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}E.$$

So for $\xi(\mathcal{J},E)$to be an isomorphism it's enough that for all $n\in\mathbb{Z}$, subdiagram ① of the following natural diagram commutes, or equivalently, that the border of the whole diagram commutes.

$$\begin{array}{ccccc}
H^n\mathbf{L}f^*\mathbf{R}\varGamma_I E & \longrightarrow & \varinjlim_{I\in\mathcal{J}} H^n\mathbf{L}f^*\mathbf{R}\varGamma_I E & \xrightarrow[\text{via }\nu_I]{\sim} & H^n\mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}E \\
{\scriptstyle H^n\xi(\mathcal{J}_I,E)}\big\downarrow\simeq & & {\scriptstyle \varinjlim_{I\in\mathcal{J}} H^n\xi(\mathcal{J}_I,E)}\big\downarrow\simeq & ① & \big\downarrow{\scriptstyle H^n\xi(\mathcal{J},E)} \\
H^n\mathbf{R}\varGamma_{I\mathcal{O}_W}\mathbf{L}f^*E & \longrightarrow & \varinjlim_{I\in\mathcal{J}} H^n\mathbf{R}\varGamma_{I\mathcal{O}_W}\mathbf{L}f^*E & \xrightarrow[(1.2.1.2)]{\sim} & H^n\mathbf{R}\varGamma_{\mathcal{J}_f}\mathbf{L}f^*E
\end{array}$$

Thus one has to see that in the following natural diagram, $ca = db$.

$$\begin{array}{ccccc}
\mathbf{L}f^*\mathbf{R}\varGamma_I E & & \xrightarrow{\quad a\quad} & & \mathbf{L}f^*\mathbf{R}\varGamma_{\mathcal{J}}E \\
 & \searrow^{p} & & \swarrow^{q} & \\
{\scriptstyle b=\xi(\mathcal{J}_I,E)}\big\downarrow & & \mathbf{L}f^*E & & \big\downarrow{\scriptstyle c=\xi(\mathcal{J},E)} \\
 & \nearrow_{r} & & \nwarrow_{s} & \\
\mathbf{R}\varGamma_{I\mathcal{O}_W}\mathbf{L}f^*E & & \xrightarrow[\quad d\quad]{} & & \mathbf{R}\varGamma_{\mathcal{J}_f}\mathbf{L}f^*E
\end{array}$$

But $\mathbf{L}f^*\mathbf{R}\varGamma_I E \in \mathbf{D}_{\mathbf{R}\varGamma_{\mathcal{J}_f}}$ (see (1.9.26.2)) so by 1.3.2, it suffices to note that since (clearly) all the subdiagrams commute, therefore $sca = qa = p = rb = sdb$. $\square$

As an exercise, show that for $E=\mathcal{O}_X$ and $A\in\mathbf{D}_{\mathrm{qct}}(X)$, 1.9.26 implies 1.9.25.

Recall that $\mathcal{O}'_X := \mathcal{O}' = \mathbf{R}\varGamma'\mathcal{O}_X$, that $(\mathcal{O}'_X, \iota'_X(\mathcal{O}_X))$ is $\mathbf{D}(X)$-idempotent (see 1.9.9), and that for any ringed-space map $f\colon W\to X$, $(\mathbf{L}f^*\mathcal{O}'_X, \mathbf{L}f^*\iota'_X(\mathcal{O}_X))$ is $\mathbf{D}(W)$-idempotent (see 1.4.8 and the remarks following it). Recall also the meaning of $B\preccurlyeq A$ for $\mathcal{O}_W$-idempotents B and A (see 1.6.6).

Proposition 1.9.27. *Let $f\colon W\to X$ be a map of noetherian formal schemes. Then $\mathcal{O}'_W \preccurlyeq \mathbf{L}f^*\mathcal{O}'_X$; and $\mathcal{O}'_W \cong \mathbf{L}f^*\mathcal{O}'_X$ if and only if f is adic.*

Proof. Let I (respectively J) be an ideal of definition of X (respectively W). Then $I\mathcal{O}_W \subset \sqrt{J}$ (see [GD, p. 416, (10.6.10)(i)]), or equivalently, $\mathcal{J}_f = \mathcal{J}_{I\mathcal{O}_W} \supset \mathcal{J}_J$ (see (1.9.26.1) *ff.*), or equivalently (by 1.9.22),

$$\mathcal{O}'_W = \mathbf{R}\varGamma_J\mathcal{O}_W \preccurlyeq \mathbf{R}\varGamma_{I\mathcal{O}_W}\mathcal{O}_W \underset{1.9.26}{\cong} \mathbf{L}f^*\mathbf{R}\varGamma_I\mathcal{O}_X = \mathbf{L}f^*\mathcal{O}'_X. \qquad \square$$

If, in addition, $\mathbf{L}f^*\mathcal{O}'_X \preccurlyeq \mathcal{O}'_W$, i.e., $J\subset\sqrt{I\mathcal{O}_W}$, then $I\mathcal{O}_W$ is an ideal of definition of W, and so f is adic [GD, p. 436, (10.12.1)].

* * * * *

The following basic facts, 1.9.28 and 1.9.29, were referred to before.

Proposition 1.9.28. (i) *For an ordinary scheme X, the usual monoidal structure on $\mathbf{D}(X)$ restricts to one on $\mathbf{D}_{\mathrm{qc}}(X)$.*

(ii) *For a noetherian formal scheme X, the usual monoidal structure on $\mathbf{D}(X)$ restricts to one on $\mathbf{D}_{\vec{c}}(X)$.*

(iii) *For a noetherian formal scheme X, there is a monoidal structure on $\mathbf{D}_{\mathrm{qct}}(X)$ with product map and associativity and symmetry isomorphisms inherited from the usual monoidal structure on $\mathbf{D}(X)$, with unit element $\mathcal{O}' := \mathbf{R}\varGamma'\mathcal{O}_X$, and with unit isomorphisms $\mathsf{l}'_E := \mathsf{l}_E \circ (\iota'(\mathcal{O}_X) \underset{=}{\otimes} \mathbf{1}_E)$ and $\mathsf{r}'_E := \mathsf{r}_E \circ (\mathbf{1}_E \underset{=}{\otimes} \iota'(\mathcal{O}_X))$ $(E \in \mathbf{D}_{\mathrm{qct}})$.*

Proof. One needs to show that $\mathbf{D}_{\mathrm{qc}}(X)$ (resp. $\mathbf{D}_{\vec{c}}(X)$, $\mathbf{D}_{\mathrm{qct}}(X)$) is $\underset{=}{\otimes}_X$-closed, i.e., if all the cohomology sheaves of $\mathcal{O}_X$-complexes E and F lie in $\mathcal{A}_{\mathrm{qc}} := \mathcal{A}_{\mathrm{qc}}(X)$ (resp. $\mathcal{A}_{\vec{c}} := \mathcal{A}_{\vec{c}}(X)$, $\mathcal{A}_{\mathrm{qc}} := \mathcal{A}_{\mathrm{qct}}(X)$) then the same holds for $E \underset{=}{\otimes}_X F$. After that, one applies 1.6.9 with $\mathbf{D}_* := \mathbf{D}_{\mathrm{qc}}$ and $\alpha := \mathbf{1}_{\mathcal{O}_X}$ or $\mathbf{D}_{\vec{c}}$, or (keeping in mind 1.9.8) with $\mathbf{D}_* := \mathbf{D}_{\mathrm{qct}}$ and $\alpha := \iota'(\mathcal{O}_X) \colon \mathcal{O}' \to \mathcal{O}_X$.

Let $P \to E$, $P' \to F$ be K-flat $\mathcal{O}_X$-resolutions, so that, with $P^{\le u}$ the complex obtained from P by replacing P^n by 0 for all $n > u$ and P^u by the kernel of $P^u \to P^{u+1}$, one has, for all $i \in \mathbb{Z}$,

$$H^i(E \underset{=}{\otimes}_X F) \cong H^i(P \otimes_X P') = \varinjlim_u H^i(P^{\le u} \otimes_X P').$$

Since $\mathcal{A}_{\mathrm{qct}}$ is closed under $\varinjlim$ [AJL2, p. 48, 5.1.3], and the same clearly holds for $\mathcal{A}_{\vec{c}}$ and for $\mathcal{A}_{\mathrm{qc}}$, therefore P can be replaced by a bounded-above flat resolution of $P^{\le u}$. Then one can do likewise with P'. So one may assume E and F bounded-above. Since $\mathbf{D}_{\mathrm{qct}}(X)$ (resp. $\mathbf{D}_{\vec{c}}(X)$, $\mathbf{D}_{\mathrm{qc}}(X)$) is a triangulated subcategory of $\mathbf{D}(X)$ ([AJL2, p. 48, 5.1.3], [AJL2, p. 34, 3.2.2], [GD, p. 217, (2.2.2)(iii)]), [Hrt, p. 73, Proposition 7.3(ii)] (dualized, and for whose terminology see [Hrt, p. 38, Definition]) yields a further reduction to where E and F are single sheaves (complexes vanishing in nonzero degrees). To be shown then is that $\mathcal{T}or_i^X(E,F) \in \mathcal{A}_{\mathrm{qct}}$ (resp. $\mathcal{A}_{\vec{c}}$, $\mathcal{A}_{\mathrm{qc}}$).

For $\mathcal{A}_{\mathrm{qc}}$ the problem is local, say $X = \mathrm{Spec}(R)$, and is easily disposed of via the standard equivalence of categories between $\mathcal{A}_{\mathrm{qc}}$ and the category of R-modules (an equivalence which preserves free resolutions). For $\mathcal{A}_{\mathrm{qct}}$, one has more generally that if $E \in \mathcal{A}_{\mathrm{qc}}$ and $F \in \mathcal{A}_{\mathrm{qct}}$ then $\mathcal{T}or_i^X(E,F) \in \mathcal{A}_{\mathrm{qct}}$: one localizes to the case $X = \mathrm{Spf}(S)$ where S is a noetherian ring complete with respect to the topology defined by powers of an ideal I, such that E is a cokernel of a map of free $\mathcal{O}_X$-modules, so that $E \in \mathcal{A}_{\vec{c}}(X)$ [AJL2, p. 32, 3.1.4]; and one uses the equivalences of categories described in [AJL2, p. 31, 3.1.1] and [AJL2, p. 47, 5.1.2] to reduce to consideration of two S-modules E_0, F_0, such that $F_0 = \varinjlim_{n>0} \mathrm{Hom}_S(S/I^n, F_0)$. Since Tor_i^S commutes with $\varinjlim$ one may assume that F_0 is annihilated by some fixed power I^n, whence so is $\mathrm{Tor}_i^S(E_0, F_0)$, whence the conclusion.

As for $\mathcal{A}_{\vec{c}}$, some caution must be taken because being in $\mathcal{A}_{\vec{c}}$ is not a local property. But since $\mathcal{T}or_i$ commutes with $\varinjlim$ one may assume that E and F have coherent homology, and then the problem is to show that so does $\mathcal{T}or_i(E,F)$. This problem *is* local, and so one can use the equivalence of categories described in [AJL2, p. 31, Proposition 3.1.1] to reduce to the analogous—and easily handled—problem for finitely-generated modules over a noetherian ring. □

Proposition 1.9.29. *Let $\psi\colon X \to Y$ be a map of noetherian formal schemes. For all $F \in \mathbf{D}_{\vec{c}}(X)$, $G \in \mathbf{D}_{\vec{c}}(Y)$, the projection map is an* isomorphism

$$\mathbf{R}\psi_*F \underset{=}{\otimes}_X G \xrightarrow{\sim} \mathbf{R}\psi_*(F \underset{=}{\otimes}_X \mathbf{L}\psi^*G).$$

Proof. Once the necessary preliminaries are in place the proof is essentially that of [Lp1, Proposition 3.9.4]. These preliminaries are as follows.

1) The question is local on Y (cf. e.g., *loc. cit.*), so one can assume that Y is affine. Then [AJL2, p. 37, Prop. 3.3.5] gives $\mathbf{L}\psi^*G \in \mathbf{D}_{\vec{c}}(X)$, and so, by 1.9.28, $F \otimes \mathbf{L}\psi^*G \in \mathbf{D}_{\vec{c}}(X)$.

2) The functor $\mathbf{R}\psi_*$ is bounded-above on $\mathbf{D}_{\vec{c}}(X)$ [AJL2, p. 39, Prop. 3.4.3(b)].

3) The functors $\mathbf{R}\psi_*$, $\mathbf{L}\psi^*$ and $\underset{=}{\otimes}_X$ all commute with direct sums: for the first, see [AJL2, p. 41, Prop. 3.5.2], and for the last two see [Lp1, Prop. (3.8.2)].

4) For any noetherian formal scheme Z, $\mathcal{A}_{\vec{c}}(Z)$ is a *plump* subcategory of $\mathcal{A}(Z)$ [AJL2, p. 34, Prop. 3.2.2].

5) Over an affine noetherian formal scheme Z, every object in $\mathcal{A}_{\vec{c}}(Z)$ is a homomorphic image of a free $\mathcal{O}_Z$-module [AJL2, p. 32, Corollary 3.1.4].

These facts enable a "way-out" reduction of the proof of Proposition 1.9.29 for bounded-above $\mathbf{D}_{\vec{c}}$-complexes to the simple case where $G = \mathcal{O}_Y$ (cf. proof of [Lp1, Proposition 3.9.4]). Then for the unbounded case, one uses that $\mathcal{A}_{\vec{c}}(X)$ is stable under $\varinjlim$.

The rest is left to the reader. $\square$

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