ANALYTIC CELL DECOMPOSITION AND ANALYTIC MOTIVIC INTEGRATION

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To Paul J. Cohen and Simon Kochen on their seventieth birthdays. Their work on this subject in the 1960s has cast a long shadow.

Abstract. The main results of this paper are a Cell Decomposition Theorem for Henselian valued fields with analytic structure in an analytic Denef-Pas language, and its application to analytic motivic integrals and analytic integrals over $\mathbf{F}_q((t))$ of big enough characteristic. To accomplish this, we introduce a general framework for Henselian valued fields K with analytic structure, and we investigate the structure of analytic functions in one variable, defined on annuli over K. We also prove that, after parameterization, definable analytic functions are given by terms. The results in this paper pave the way for a theory of *analytic* motivic integration and *analytic* motivic constructible functions in the line of R. Cluckers and F. Loeser [*Fonctions constructible et intégration motivique I*, Comptes rendus de l'Académie des Sciences, **339** (2004) 411 - 416].

0.1. **Résumé. Décomposition cellulaire analytique et intégration motivique analytique.** Dans cet article nous établissons une décomposition cellulaire pour des corps valués Henseliens munis d'une structure analytique induite par un langage de Denef-Pas analytique. En particulier, nous appliquons cet énoncé à l' étude des intégrales analytiques motiviques et des intégrales analytiques sur $\mathbf{F}_q((t))$ de charactéristic assez grand. Pour cela, il est nécessaire d'introduire une définition général des corps valués Henselien K avec structure analytique. On examine alors la structure de fonctions analytiques dans une variable définies sur des annuli sur K et l'on établit que, dans ce contexte, les fonctions définissables sont exactement données par des termes après paramétrisation. Plus généralement, les résultats de cet article préparent le chemin pour définir une théorie d'intégration analytique motivique et des fonctions analytiques motiviques constructibles dans l'esprit de R. Cluckers et F. Loeser [*Fonctions constructible et intégration motivique I*, Comptes rendus de l'Académie des Sciences, **339** (2004) 411 - 416].

1. Introduction

The main results of this paper are a Denef-Pas Cell Decomposition Theorem for Henselian valued fields with analytic structure, Theorem 7.4, and its application

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to analytic motivic integrals[1] and analytic integrals over $\mathbf{F}_q((t))$ of big enough characteristic. In section 2 we introduce a framework for Henselian valued fields K with analytic structure, both strictly convergent and separated[2], that generalizes [2], [19], [21], [32], [34], and that works in all characteristics. An analytic structure is induced by rings of power series over a Noetherian ring A that is complete and separated with respect to the I-adic topology for some ideal I of A. This framework facilitates the use of standard techniques from the theory of analytic rings in a more general model theoretic setting.

Another necessary ingredient for cell decomposition is an analysis of analytic functions in one variable, defined on annuli over K. This is carried out in Section 3. Theorem 3.9 relates such functions piecewise to a (strong) unit times a quotient of polynomials. This result is extended to functions in one variable given by terms in Theorem 5.1. The results of this section extend the analysis of the ring of analytic functions on an affinoid subdomain of K, carried out in [24, Sections 2.1 and 2.2], in the case that K is algebraically closed and complete, in three directions: (i) K not necessarily algebraically closed, (ii) the case of quasi-affinoid subdomains of K and (iii) the case that K is not complete but carries an analytic structure. This analysis will be pursued further in the forthcoming paper [5].

We also prove a fundamental structure result on definable analytic functions, namely, that any definable function is given, after parameterization using auxiliary sorts, by terms in a somewhat bigger language, cf. Theorem 7.5. This structure result is new also in the algebraic case, and is used in [6] to prove a change of variables formula for motivic integrals.

In section 8 we apply our results to study analytic motivic integrals in the sense that we uniformly interpolate analytic p-adic and $\mathbf{F}_p((t))$ integrals for p big enough and for boundedly ramified p-adic field extensions for any fixed p. These results for $\mathbf{F}_p((t))$ and the uniformity are completely new. For fixed p-adic fields these integrals are calculated in [18], and in [4] the relative case over a parameter space is treated. The results for the fields $\mathbf{F}_p((t))$ with p big are new, even for fixed p, but they also follow in fact in a classical way from uniformity for p-adic fields, cf. the algebraic case in [9] (but this is already implicit in work of Denef, and Pas). More generally, the results in this paper pave the way for a theory of analytic motivic integration and analytic motivic constructible functions along the lines of [6], [7] and [8], in particular, for calculating relative motivic integrals over a parameter space. Another approach to analytic motivic integration, based on entire models of rigid varieties and the theory of Néron models instead of cell decomposition, is developed by Sebag and Loeser in [35], and by Sebag in [42]. This alternative approach is pursued in [41] for the study of generating power series and in [37] for the study of the monodromy conjecture. In [6], apart from cell decomposition, also a dimension theory is used; in the analytic case this can be developed along the lines of work by Çelikler [3].

1.1. Cell decomposition is a technique of breaking definable sets into finitely many definable pieces each of which is particularly simple in a chosen coordinate direction. For example, in the real case, Fubini's Theorem often reduces the computation of an

[1]Motivic here stands for the idea of giving a geometric meaning to p-adic integrals, uniform in p.

[2]The term "separated" usually means that the (intended) domains of the power series considered are Cartesian products of the valuation ring and the maximal ideal.

integral over a complicated set to an iterated integral over the region between two graphs, on which the integrand is of a simple form with respect to this coordinate, cf. the Preparation Theorem and its use for integration by Lion and Rolin in [28].

In [11], Cohen reproved Tarski's real quantifier elimination using his real cell decomposition for definable sets. In the same paper, he gave a cell decomposition for some Henselian fields, e.g. p-adic fields, extending results of Ax and Kochen, [1]. A cell over a real field is a set given by conditions of the form $f(x) < y < g(x)$ or $y = f(x)$, where f, g are definable. That quantifier elimination follows from cell decomposition is fairly clear; the other implication a bit more complicated. A cell over a Henselian field is specified by simple conditions on the order and angular component of $y - c(x)$, where c is definable (see below for definitions). This reflects the idea that for many Henselian fields, a statement about the field can be reduced to statements about the value group and the residue field.

Denef [14] refined Cohen's techniques to reprove Macintyre's quantifier elimination for p-adic fields and to obtain a p-adic integration technique which he used to prove the rationality of certain p-adic Poincaré series [12]. Pas [38], [39], and Macintyre [36] extended this method to study uniform properties of p-adic integrals. Denef and van den Dries [18] extended the Ax-Kochen-Cohen-Macintyre p-adic quantifier elimination to the analytic category based on strictly convergent power series. (See also [19].) These ideas were extended to the algebraically closed analytic category using separated power series by the second and third authors, see [29] and [33]. The first author [4], using work of Haskell, Macpherson, and van den Dries [21], obtained an analytic variant of the p-adic cell decomposition and an application to p-adic analytic integrals.

In this paper, we extend the ideas of quantifier elimination and cell decomposition to a wider class of Henselian fields with analytic structure, cf. Theorems 4.2 and 7.4.

1.2. Let us elaborate on the application to analytic motivic integrals. We repeat that the contribution here is the uniformity and the ability of working relatively over a parameter space (which is immediate from cell decomposition but not written out in this paper). Let $\mathcal{A}$ be the class of all fields $\mathbf{Q}_p$ for all primes p together with all their finite field extensions and let $\mathcal{B}$ be the class of all the fields $\mathbf{F}_q((t))$ with q running over all prime powers. For each fixed prime p and integer $n > 0$ let $\mathcal{A}_{p,n}$ be the subset of $\mathcal{A}$ consisting of all finite field extensions of $\mathbf{Q}_p$ with degree of ramification fixed by $\mathrm{ord}_p(p) = n$. For $K \in \mathcal{A} \cup \mathcal{B}$ write K° for the valuation ring, $\widetilde{K}$ for the residue field, π_K for a uniformizer of K°, and q_K for $\sharp\widetilde{K}$.

Denote by $\mathbf{Z}[[t]]\langle x_1, \ldots, x_n\rangle$ the ring of strictly convergent power series over $\mathbf{Z}[[t]]$ (consisting of all $\sum_{i\in\mathbf{N}^n} a_i(t)x^i$ with $a_i(t) \in \mathbf{Z}[[t]]$ such that for each $m \geq 0$ there exists n' such that $a_i(t)$ belongs to (t^m) for each i with $i_1 + \ldots + i_n > n'$).

The purpose of these strictly convergent power series is to provide analytic functions in a uniform way, as follows. To each $f(x) = \sum_{i\in\mathbf{N}^n} a_i(t)x^i$ in $\mathbf{Z}[[t]]\langle x_1, \ldots, x_n\rangle$, $a_i(t) \in \mathbf{Z}[[t]]$, we associate for each $K \in \mathcal{A} \cup \mathcal{B}$ the K-analytic function

$$f_K : (K^\circ)^n \to K^\circ : x \mapsto \sum_{i\in\mathbf{N}^n} a_i(\pi_K)x^i.$$

Fix $f \in \mathbf{Z}[[t]]\langle x_1, \ldots, x_n \rangle$. As K varies in $\mathcal{A} \cup \mathcal{B}$, one has a family of numbers

$$a_K := \int\limits_{(K^\circ)^n} |f_K(x)||dx|, \tag{1.1}$$

with $|dx|$ the normalized Haar measure on K^n and $|y| = q_K^{-\mathrm{ord}\ y}$, and one would like to understand the dependence on K in a geometric way (see [25] for a context of this question). This is done using $\mathrm{Var}_{\mathbf{Z}}$, the collection of isomorphism classes of algebraic varieties over $\mathbf{Z}$ (i.e. reduced separated schemes of finite type over $\mathbf{Z}$), and, $\mathrm{Form}_{\mathbf{Z}}$, the collection of equivalence classes of formulas in the language of rings[3] with coefficients in $\mathbf{Z}$. For each finite field k, we consider the ring morphisms

$$\mathrm{Count}_k : \mathbf{Q}[\mathrm{Var}_{\mathbf{Z}}, \frac{1}{\mathbf{A}^1_{\mathbf{Z}}}, \{\frac{1}{1-\mathbf{A}^i_{\mathbf{Z}}}\}_{i<0}] \to \mathbf{Q}$$

which sends $Y \in \mathrm{Var}_{\mathbf{Z}}$ to $\sharp Y(k)$, the number of k-rational points on Y, and,

$$\mathrm{Count}_k : \mathbf{Q}[\mathrm{Form}_{\mathbf{Z}}, \frac{1}{\mathbf{A}^1_{\mathbf{Z}}}, \{\frac{1}{1-\mathbf{A}^i_{\mathbf{Z}}}\}_{i<0}] \to \mathbf{Q}$$

which sends $\varphi \in \mathrm{Form}_{\mathbf{Z}}$ to $\sharp\varphi(k)$, the number of k-rational points on φ, and where we also write $\mathbf{A}^\ell_{\mathbf{Z}}$ for the isomorphism class of the formula $x_1 = x_1 \wedge \ldots \wedge x_\ell = x_\ell$ (which has the set R^ℓ as R-rational points for any ring R), $\ell \geq 0$.

Using the work established in this paper, as well as results of Denef and Loeser [16], we establish Theorem 8.2, which is a generalization of the following.

Theorem 1.2.
(i) *There exists a (non-unique) element*

$$X \in \mathbf{Q}[\mathrm{Var}_{\mathbf{Z}}, \frac{1}{\mathbf{A}^1_{\mathbf{Z}}}, \{\frac{1}{1-\mathbf{A}^i_{\mathbf{Z}}}\}_{i<0}]$$

and a number N such that for each field $K \in \mathcal{A} \cup \mathcal{B}$ with $\mathrm{Char}\,\widetilde{K} > N$, *one has*

$$a_K = \mathrm{Count}_{\widetilde{K}}(X).$$

In particular, if $\mathrm{Char}\widetilde{K} > N$, *then a_K only depends on $\widetilde{K}$.*
(ii) *For fixed prime p and $n > 0$, there exists a (non-unique) element*

$$X_{p,n} \in \mathbf{Q}[\mathrm{Form}_{\mathbf{Z}}, \frac{1}{\mathbf{A}^1_{\mathbf{Z}}}, \{\frac{1}{1-\mathbf{A}^i_{\mathbf{Z}}}\}_{i<0}]$$

such that for each field $K \in \mathcal{A}_{p,n}$ one has

$$a_K = \mathrm{Count}_{\widetilde{K}}(X_{p,n}).$$

Note that (i) treats the case of big residue characteristic, while (ii) can be used for any fixed "small" residue characteristic. As one expects, for small residue characteristic, one get less information than in case (i), namely, in case (ii), only bounded ramification is allowed and formulas are used instead of varieties.

To prove Theorems 1.2 and 8.2 we calculate the a_K by inductively integrating variable by variable, in a uniform way, using analytic cell decomposition. By such decomposition, one can partition the domain of integration uniformly in $K \in \mathcal{A} \cup \mathcal{B}$, for big enough residue field characteristic, and prepare the integrand on the pieces in such a way that the integral with respect to a special variable becomes easy.

[3]Two formulas are equivalent in this language if they have the same R-rational points for every ring R.

There is possibly an alternative approach to prove Theorems 1.2 and 8.2 by using analytic embedded resolutions of $f = 0$ over (a ring of finite type over) $\mathbf{Z}[[t]]$, if such a resolution exists. We do not pursue this approach.

We comment on the non-uniqueness of X and $X_{p,n}$ in Theorem 1.2. By analogy to [6], [16] and using the results of this paper, one could associate unique objects, a motivic integral, to the data used to define a_K. Such objects would live in some quotient of $\mathbf{Q}[\mathrm{Form}_{\mathbf{Z}}, \frac{1}{\mathbf{A}_{\mathbf{Z}}^1}, \{\frac{1}{1-\mathbf{A}_{\mathbf{Z}}^i}\}_{i<0}]$, and the morphisms Count_k could factor through this quotient (at least for char k big enough). To establish uniqueness in some ring is beyond the scope of the present paper.

More generally, we consider

$$b_K(s) := \int\limits_{(K^\circ)^n} |f_{1K}(x)|^s |f_{2K}(x)| |dx|, \tag{1.3}$$

and similar integrals, when K varies over $\mathcal{A}\cup\mathcal{B}$, where f_1, f_2 are in $\mathbf{Z}[[t]]\langle x_1, \ldots, x_n\rangle$, and $s \geq 0$ is a real variable. In the generalization Theorem 8.2 of Theorem 1.2, we prove the rationality of $b_K(s)$ in q_K^s for $K \in \mathcal{B}$ of characteristic big enough and with q_K the number of elements in the residue field of K. For each fixed $K \in \mathcal{A}$ this rationality was proved by Denef and van den Dries in [18]. Here we thus prove uniformity, and hence rationality, results for the fields $\mathbf{F}_q((t))$ of big enough characteristic. Similarly as in the p-adic case, such integrals describe the generating power series with coefficients N_m obtained by counting points modulo t^m in $\mathbf{F}_q[[t]]/(t^m)$ satisfying analytic equations modulo t^m, cf. the work of Igusa and Denef, hence, we obtain uniform rationality results for these generating power series.

2. Analytic structures

Analytic structures, introduced in [19] (cf. [21] and [34]), are a framework for the model theory of analytic functions. This section contains an extensive elaboration of those ideas.

Model theory provides a convenient means to analyze algebraic properties that depend on parameters, and analytic structures are a way to extend model-theoretic techniques to the analytic setting. In particular, a cell decomposition for a family of functions of several variables is a partition of the domain into finitely many simple sets on each of which the behavior of the functions has a simple dependence on the value of the last variable. By assigning the other variables fixed values in a possibly non-standard field extension, the compactness theorem reduces the problem of cell decomposition for polynomial functions of several variables to that of obtaining a cell decomposition for polynomial functions of just one variable, at the expense of providing a uniform cell decomposition for all models of the theory. By expressing a power series as the product of a unit and a polynomial in the last variable, Weierstrass Preparation is used to reduce analytic questions to algebraic ones. An analytic structure provides a convenient framework for dealing with the parameters that arise in applying Weierstrass Preparation.

To make use of the Weierstrass data, the definition of a model-theoretic structure must be extended so that compositional and algebraic identities in the power series ring are preserved when the power series are interpreted as functions on the underlying field. In the case of polynomial rings, the interpretation of addition and multiplication in a model of the theory of rings already provides a natural homomorphism from the polynomial ring into the ring of functions on the underlying structure. Furthermore, if the underlying field is complete, the valued field structure also already provides a natural homomorphism from the ring of convergent power series into the ring of functions (that preserves not only algebraic, but compositional identities as well). But the fields over which we work may not be complete since we must work uniformly in all models of a given theory. Thus, to apply the Weierstrass techniques, our models must come equipped with a distinguished homomorphism from the ring of power series to the ring of functions. This is essentially the definition of analytic structure in Definition 2.7 and in [19]. (Note that, rather than a distinguished homomorphism, one could employ instead a first-order axiom scheme in which each power series identity is coded into an axiom, but that obscures the difference between the algebraic and analytic situations, where topological completeness, in some form, comes into play.)

As in [19], [21] and [34], in using Weierstrass techniques, one often introduces new parameters for certain ratios. Without a natural means of adjoining elements of a (possibly non-standard) model to the given coefficient ring of the power series ring, one is prevented from specializing the parameters, which complicates some computations. However, since the proof of the Weierstrass Division Theorem relies on completeness in the coefficient ring, adjoining elements of an arbitrary model to the coefficient ring is problematic. The methods of [32] were developed to analyze the commutative algebra of rings of separated power series, which are filtered unions of complete rings. Those ideas are applied in this section to show how to extend the coefficient ring (Theorem 2.13 and Definition 2.15) and ground field (Theorem 2.18) of a given analytic structure, which is how the present treatment of analytic structures differs from the previous ones. (Indeed, with minor modifications to the proofs, much of the theory of [32] applies to the rings $S_{m,n}(\sigma, K)$ introduced in Definition 2.15, and, although we prefer to give a self-contained treatment in this paper, would simplify the proofs of the results of Section 3.)

Finally, let K be a separated analytic A-structure as in Definition 2.7, so the power series in a ring $S_{m,n}(A)$ are interpreted as analytic functions on K in such a way as to preserve the algebraic and compositional identities of $S_{m,n}(A)$ and an extended power series ring $S_{m,n}(\sigma, K)$ is obtained from $S_{m,n}(A)$, as in Definition 2.15, by adjoining coefficients from the field K. It is important to note that, although the extended power series rings $S_{m,n}(\sigma, K)$ are much larger than the rings $S_{m,n}(A)$, the structure K has essentially the same first-order diagram in the extended language. Thus, although it is easier to work with the extended power series rings $S_{m,n}(\sigma, K)$, they have the same model theory as the smaller rings $S_{m,n}(A)$, which, in fact, is the point of introducing the extension.

Definition 2.1. Let E be a Noetherian ring that is complete and separated for the I-adic topology, where I is a fixed ideal of E. Let $(\xi_1, \dots, \xi_m)$ be variables, $m \geq 0$. The ring of *strictly convergent power series in* ξ *over* E (cf. [2], Section 1.4) is

$$T_m(E) = E\langle\xi\rangle := \{ \sum_{\nu \in \mathbb{N}^m} a_\nu \xi^\nu \colon \lim_{|\nu| \to \infty} a_\nu = 0 \}.$$

Let $(\rho_1, \dots, \rho_n)$ be variables, $n \geq 0$. The ring

$$S_{m,n}(E) := E\langle\xi\rangle[\![\rho]\!]$$

is a ring of *separated power series over* E (cf. [32], Section 2).

Remark 2.2.
(i) If the formal power series ring $E[\![\rho]\!]$ is given the ideal-adic topology for the ideal generated by I and (ρ), then $S_{m,n}(E)$ is isomorphic to $E[\![\rho]\!]\langle\xi\rangle$. Note that $S_{m,0} = T_m$ and $T_{m+n}(E)$ is contained in $S_{m,n}(E)$.
(ii) Observe that $E\langle\xi\rangle$ is the completion of the polynomial ring $E[\xi]$ in the $I \cdot E[\xi]$-adic topology and $E\langle\xi\rangle[\![\rho]\!]$ is the completion of the polynomial ring $E[\xi,\rho]$ in the J-adic topology, where J is the ideal of $E[\xi,\rho]$ generated by ρ and the elements of I.
(iii) The example of $A = \mathbf{Z}[[t]]$ and $I = (t)$ is the one used in the introduction to put a strictly convergent analytic structure on the p-adic fields and on the fields $\mathbf{F}_q((t))$.

The Weierstrass Division Theorem (cf. [32], Theorems 2.3.2 and 2.3.8) provides a key to the basic structure of the power series rings $S_{m,n}(E)$.

Definition 2.3. Let $f \in S_{m,n}(E)$. The power series f is *regular in* ξ_m *of degree* d if f is congruent, modulo the ideal $I + (\rho)$, to a monic polynomial in ξ_m of degree d, and f is *regular in* ρ_n *of degree* d if f is congruent, modulo the ideal $I + (\rho_1,\dots,\rho_{n-1})$ to $\rho_n^d \cdot g(\xi,\rho)$ for some unit g of $S_{m,n}(E)$.

Proposition 2.4 (Weierstrass Division)**.** *Let* $f, g \in S_{m,n}(E)$.

(i) *Suppose that* f *is regular in* ξ_m *of degree* d. *Then there exist uniquely determined elements* $q \in S_{m,n}(E)$ *and* $r \in S_{m-1,n}(E)[\xi_m]$ *of degree at most* $d-1$ *such that* $g = qf + r$. *If* $g \in J \cdot S_{m,n}$ *for some ideal* J *of* $S_{m-1,n}$, *then* $q, r \in J \cdot S_{m,n}$.

(ii) *Suppose that* f *is regular in* ρ_n *of degree* d. *Then there exist uniquely determined elements* $q \in S_{m,n}(E)$ *and* $r \in S_{m,n-1}(E)[\rho_n]$ *of degree at most* $d-1$ *such that* $g = qf + r$. *If* $g \in J \cdot S_{m,n}$ *for some ideal* J *of* $S_{m,n-1}$, *then* $q, r \in J \cdot S_{m,n}$.

Remark 2.5. By taking $n = 0$ in Proposition 2.4 (i), one obtains a Weierstrass Division Theorem for the $T_m(E)$.

Dividing ξ_m^d (respectively, ρ_n^d) by an element $f \in S_{m,n}$ regular in ξ_m (respectively, ρ_n) of degree d, as in [32], Corollary 2.3.3, we obtain the following corollary.

Corollary 2.6 (Weierstrass Preparation)**.** *Let* $f \in S_{m,n}(E)$.

(i) *If* f *is regular in* ξ_m *of degree* d, *then there exist: a unique unit* u *of* $S_{m,n}$ *and a unique monic polynomial* $P \in S_{m-1,n}[\xi_m]$ *of degree* d *such that* $f = u \cdot P$.

(ii) *If* f *is regular in* ρ_n *of degree* d, *then there exist: a unique unit* u *of* $S_{m,n}$ *and a unique monic polynomial* $P \in S_{m,n-1}[\rho_n]$ *of degree* d *such that* $f = u \cdot P$*; in addition,* P *is regular in* ρ_n *of degree* d.

Let the ring E and the ideal I be as in Definition 2.1. If K is a field containing E that is complete in a rank 1 valuation and I is contained in the maximal ideal $K^{\circ\circ}$ of the valuation ring K°, then $T_m(E)$ (respectively, $S_{m,n}(E)$) may be interpreted as a ring of analytic functions on the polydisc $(K^\circ)^m$ (respectively, $(K^\circ)^m \times (K^{\circ\circ})^n$), exactly as in [32]. The following definition permits an extension to more general valued fields K, for example, to ultraproducts of complete fields. In this more general setting, analytic properties usually derived employing the completeness of the domain can often be derived instead from Weierstrass Division (which relies on completeness in the coefficient ring).

Definition 2.7 (cf. [19] and [34])**.** Let A be a Noetherian ring that is complete and separated with respect to the I-adic topology for a fixed ideal I of A. Let

$(K, \mathrm{ord}, \Gamma)$ be a valued field. A *separated analytic A-structure on K* is a collection of homomorphisms $\sigma_{m,n}$ from $S_{m,n}(A)$ into the ring of K°-valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ for each $m, n \in \mathbb{N}$ such that:

(i) $(0) \neq I \subset \sigma_0^{-1}(K^{\circ\circ})$, with $\sigma_0 := \sigma_{0,0}$,

(ii) $\sigma_{m,n}(\xi_i) =$ the i-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$, $i = 1, \dots, m$, and $\sigma_{m,n}(\rho_j) =$ the $(m+j)$-th coordinate function on $(K^\circ)^m \times (K^{\circ\circ})^n$, $j = 1, \dots, n$,

(iii) $\sigma_{m,n+1}$ extends $\sigma_{m,n}$, where we identify in the obvious way functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ with functions on $(K^\circ)^m \times (K^{\circ\circ})^{n+1}$ that do not depend on the last coordinate, and $\sigma_{m+1,n}$ extends $\sigma_{m,n}$ similarly.

A collection of homomorphisms σ_m from $T_m(A) = S_{m,0}(A)$ into the ring of K°-valued functions on $(K^\circ)^m$ is called a *strictly convergent analytic A-structure on K* if the homomorphisms $\sigma_{m,0} := \sigma_m$ satisfy the above three conditions (with $n = 0$).

In any case, we call A the *coefficient ring* of the analytic structure.

Here are some typical examples of valued fields with strictly convergent analytic A-structure. Take $A := \mathbb{Z}[[t]]$, where t is one variable, equipped with the (t)-adic topology. Then $(\mathbb{C}((t)), \mathrm{ord}_t, \mathbb{Z})$ carries a unique analytic A-structure determined by $\sigma_0(t) = t$. For each prime $p \in \mathbb{N}$, the valued field of p-adic numbers $(\mathbb{Q}_p, \mathrm{ord}_p, \mathbb{Z})$ carries a unique analytic A-structure determined by $\sigma_0(t) = p$. Similarly, $\sigma_0(t) = p$ determines a unique separated analytic structure on the non-discretely valued field $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, which yields a larger family of analytic functions than the corresponding strictly convergent analytic A-structure. The fields $\mathbf{F}_p((t))$ carry unique analytic A-structures determined by $\sigma_0(t) = t$. The latter (standard) analytic A-structures induce an analytic A-structure on any non-principal ultraproduct of the p-adic fields $\mathbb{Q}_p$, or $\mathbb{C}_p$, or $\mathbb{F}_p((t))$. Note that such fields carry analytic A-structure even though they are not complete.

By definition, analytic A-structures preserve the ring operations on power series, thus they preserve the Weierstrass Division data. It follows that analytic A-structures also preserve the operation of composition.

Proposition 2.8. *Analytic A-structures preserve composition. More precisely, if $f \in S_{m,n}(A)$, $\alpha_1, \dots, \alpha_m \in S_{M,N}(A)$, $\beta_1, \dots, \beta_n \in IS_{M,N}(A) + (\rho)$, where $S_{M,N}(A)$ contains power series in the variables (ξ, ρ) and I is the fixed ideal of A, then $g := f(\alpha, \beta)$ is in $S_{M,N}(A)$ and $\sigma(g) = (\sigma(f))(\sigma(\alpha), \sigma(\beta))$.*

Proof. By the Weierstrass Division Theorem, there are elements $q_i \in S_{m+M,n+N}(A)$ such that

$$f(\eta, \lambda) = g(\xi, \rho) + \sum_{i=1}^{m} (\eta_i - \alpha_i(\xi, \rho)) \cdot q_i + \sum_{j=1}^{n} (\lambda_j - \beta_j(\xi, \rho)) \cdot q_{m+j}.$$

Let $(x, y) \in (K^\circ)^M \times (K^{\circ\circ})^N$ and put $a_i := \sigma(\alpha_i)(x, y)$ and $b_j := \sigma(\beta_j)(x, y)$ for $i = 1, \dots, m$, $j = 1, \dots, n$. Clearly, $b \in (K^{\circ\circ})^n$. By plugging (a, x, b, y) into the above equation, the proposition follows. □

Next we show that the image of a power series is the zero function if, and only if, the image of each of its coefficients is zero. This employs parameterized Weierstrass Division which relies on the strong Noetherian property of Lemma 2.9 (cf. [18] Lemmas 1.4 and 4.12.) We refer to this as a *strong* Noetherian property because, implicit in Lemma 2.9 are the facts that not only are all the $f_{\mu,\nu}$ expressed as linear combinations of finitely many, but also that for "small" $f_{\mu,\nu}$ the coefficients in these linear combinations are also "small".

Lemma 2.9 ([32], Lemma 3.1.6)**.** *Let $F \in S_{m+M,n+N}(E)$ and write*

$$F = \sum f_{\mu,\nu}(\xi,\rho)\eta^{\mu}\lambda^{\nu}$$

with the $f_{\mu,\nu} \in S_{m,n}(E)$. Then there are: $d \in \mathbb{N}$ and units $G_{\mu,\nu}$ of $S_{m+M,n+N}(E)$ such that

$$F = \sum_{|\mu|+|\nu|\le d} f_{\mu,\nu}(\xi,\rho)\eta^{\mu}\lambda^{\nu}G_{\mu,\nu}(\xi,\eta,\rho,\lambda).$$

Proposition 2.10. *The image of a power series is the zero function if, and only if, the image of each of its coefficients is zero. More precisely,*

(i) *Let σ be a separated analytic A-structure on the valued field K. Then* $\ker\sigma_{m,n} = \ker\sigma_0 \cdot S_{m,n}(A)$.

(ii) *Let σ be a strictly convergent analytic A-structure on the valued field K. Then* $\ker\sigma_m = \ker\sigma_0 \cdot T_m(A)$.

Proof. (ii) The ring $\sigma_0(A)$ is a Noetherian ring that is complete and separated in the $\sigma_0(I)$-adic topology. The map σ induces a homomorphism $\pi_m\colon T_m(A) \to T_m(\sigma_0(A))$, and $\ker\pi_m = \ker\sigma_0 \cdot T_m(A)$. Thus, the homomorphism σ_m factors through $T_m(\sigma_0(A))$, yielding a strictly convergent $\sigma_0(A)$-analytic structure $\bar\sigma$ on K°. Hence, there is no loss in generality to assume that $\ker\sigma_0 = (0)$. Let $f \in T_m(A)\setminus\{0\}$; we must show that $\sigma_m(f)$ is not the zero function.

Observe that if $f \in T_{m-1}[\xi_m]$ is monic in ξ_m then $\sigma_m(f)$ is not the zero function. Indeed, write $f = \xi_m^d + \sum_{i=0}^{d-1}\xi_m^i a_i(\xi')$, where $\xi' = (\xi_1,\ldots,\xi_{m-1})$. Let $x \in (K^\circ)^{m-1}$; then $\sigma_m(f)(x,\xi_m) \in K^\circ[\xi_m]$ is monic of degree d. By Definition 2.7 (i), K is a non-trivially valued hence infinite field; thus $\sigma_m(f)$ is not the zero function.

Now let $f = \sum f_\mu \xi^\mu$ be any non-zero element of T_m. By Lemma 2.9, there are $d \in \mathbb{N}$ and units g_μ of T_m such that

$$f = \sum_{|\mu|\le d} f_\mu \xi^\mu g_\mu.$$

Let ν be the lexicographically largest index such that

$$\mathrm{ord}\sigma(f_\nu) = \min_{|\mu|\le d}\mathrm{ord}\sigma(f_\mu).$$

Let η_μ be new variables and put

$$F := \xi^\nu + \sum_{\substack{|\mu|\le d\\ \mu\ne\nu}} \eta_\mu \xi^\mu g_\nu^{-1} g_\mu.$$

Since the above sum is finite,

$$\sigma(f) = \sigma(f_\nu g_\nu)\sigma(F)(\xi, y_\mu),$$

where the $y_\mu := \frac{\sigma(f_\mu)}{\sigma(f_\nu)} \in K^\circ$. Since g_ν is a unit and $\sigma(f_\nu) \ne 0$, to show that $\sigma(f)$ is not the zero function, it suffices to show that $\sigma(F)(\xi, y_\mu)$ is not the zero function.

By the choice of ν, there is a polynomial change of variables φ, involving only the ξ, such that $F\circ\varphi$ is regular in ξ_m of some degree d. By Proposition 2.8, it is enough to show that $\sigma(F\circ\varphi)$ is not the zero function, which follows from Corollary 2.6 (i).

(i) As above, we may assume that $\ker \sigma_0 = (0)$. Let $f(\xi, \rho)$ be a nonzero element of $S_{m,n}$, let a be a nonzero element of the ideal I of A and put $g := f(\xi, a \cdot \rho)$. By Proposition 2.8, to show that $\sigma(f)$ is not the zero function, it suffices to show that $\sigma(g)$ is not the zero function. Since $\ker \sigma_0 = (0)$ and K° is an integral domain, so is A. Since a is a nonzero element of I, g is a nonzero element of T_{m+n}. It is then a consequence of part (ii) that $\sigma(g)$ is not the zero function. □

Next, we discuss how to extend the coefficient ring of a given analytic structure.

Definition 2.11. Let A and E be Noetherian rings that are complete and separated for the I-adic, respectively, J-adic, topologies, where I and J are fixed ideals of A, respectively, of E. Let K be a valued field with analytic A-structure $\{\sigma_{m,n}\}$ and analytic E-structure $\{\tau_{m,n}\}$. Suppose E is an A-algebra via the homomorphism $\varphi\colon A \to E$, and that $I \subset \varphi^{-1}(J)$. Note that φ extends coefficient-wise to a homomorphism $\varphi : S_{m,n}(A) \to S_{m,n}(E)$. The analytic structures σ and τ are called *φ-compatible* (or *compatible* when φ is understood) if, for all m and n, $\sigma_{m,n} = \tau_{m,n} \circ \varphi$.

It can be particularly useful to extend the coefficient ring of an analytic A-structure by adjoining finitely many parameters from the domain K. The coefficient rings of analytic structures are complete, and Lemma 2.12 permits us to define the appropriate completion of a finitely generated A-subalgebra of K°.

Lemma-Definition 2.12. (i) *Let K be a valued field with separated analytic A-structure $\{\sigma_{m,n}\}$ and let E be a finitely generated $\sigma_0(A)$-subalgebra of K°, say, generated by $a_1, \dots, a_m$. Then E is Noetherian. Let $b_1, \dots, b_n$ generate the ideal $E \cap K^{\circ\circ}$. The subset E^σ of K*

$$E^\sigma := \{\sigma(f)(a, b) : f \in S_{m,n}(A)\}$$

is independent of the choices of a and b. Moreover, E^σ is a Noetherian ring that is complete and separated with respect to the J-adic topology, where $J = (E^\sigma \cap K^{\circ\circ})$. Moreover, J is generated by b_j, $j = 1, \dots, n$.

(ii) *Let K be a valued field with strictly convergent analytic A-structure $\{\sigma_m\}$ and E be a finitely generated $\sigma_0(A)$-subalgebra of K°, say, generated by $a_1, \dots, a_m$. Then E is Noetherian. The subset E^σ of K*

$$E^\sigma := \{\sigma(f)(a) : f \in T_m(A)\}$$

is independent of the choice of a. Moreover, E^σ is a Noetherian ring that is complete and separated with respect to the J-adic topology, where $J = \sigma_0(I) \cdot E^\sigma$.

Proof. (i) Let E be generated by some tuple a' and $E \cap K^{\circ\circ}$ by b'. For some polynomials $p_i, q_{j,\ell} \in A[\xi]$,

$$a'_i = \sigma(p_i)(a),\ i = 1, \dots, m' \quad \text{and} \quad b'_j = \sum_{\ell=1}^{n} \sigma(q_{j,\ell})(a) b_\ell,\ j = 1, \dots, n'.$$

That E^σ is independent of the choice of a now follows from Proposition 2.8.

To prove the remainder of part (i), observe that the ideal J of E^σ is generated by $b_1, \dots, b_n$. Indeed, let $f \in S_{m,n}$ and write

$$f = f_0(\xi) + \sum_{\ell=1}^{r} e_\ell g_\ell(\xi) + \sum_{j=1}^{n} \rho_j h_j(\xi, \rho),$$

where $f_0 \in A[\xi]$ is a polynomial, the e_ℓ generate the ideal I of A, $g_\ell \in S_{m,0}$ and $h_j \in S_{m,n}$. Note that $\operatorname{ord} \sigma_0(e_\ell), \operatorname{ord} \sigma(\rho_j)(b) = \operatorname{ord} b_j > 0$ and that $\sigma_0(e_\ell)$ and the $\sigma(\rho_j)(b) = b_j$ belong to the ideal generated by b. Thus, $\operatorname{ord} \sigma(f)(a,b) > 0$ implies that $\operatorname{ord} \sigma(f_0)(a) > 0$. Since f_0 is a polynomial, $\sigma(f_0)(a) \in E$, and it follows that $\sigma(f_0)(a)$ must also belong to the ideal generated by b.

Now consider the A-algebra homomorphism

$$\varepsilon_{a,b} \colon S_{m,n}(A) \to E^\sigma \colon f \mapsto \sigma(f)(a,b).$$

Since $\varepsilon_{a,b}$ is clearly surjective and $S_{m,n}(A)$ is Noetherian, E^σ is Noetherian. By the above observation, the non-trivial ideal J is generated by the images of the ρ_j under $\varepsilon_{a,b}$. Since $S_{m,n}$ is complete in the (ρ)-adic topology, it follows from the Artin-Rees Theorem that the finitely generated $S_{m,n}$-module E^σ is complete and separated in the J-adic topology, as desired.

(ii) The proof is similar to part (i). □

Theorem 2.13, below, gives a basic example of extending the coefficient ring of an A-analytic structure to obtain a compatible analytic structure.

Theorem 2.13. (i) *Let K be a valued field with separated analytic A-structure $\{\sigma_{m,n}\}$. Let $E \subset K^\circ$ be a finitely generated A-subalgebra of K° and let E^σ be as in Definition 2.12 (i). Then σ induces a unique analytic E^σ-structure τ on K° such that σ and τ are compatible. Moreover, each $\tau_{m,n}$ is injective.*

(ii) *The analogous statement holds for K a valued field with strictly convergent analytic A-structure $\{\sigma_m\}$.*

Proof. (i) Let $f \in S_{M,N}(E^\sigma)$. By Lemma 2.12, J is generated by the $\sigma(\rho_j)(b)$, so there is some $F \in S_{m+M,n+N}(A)$, $F = \sum f_{\mu,\nu}(\xi,\rho)\eta^\mu\lambda^\nu$, such that

$$f = \sum \sigma(f_{\mu,\nu})(a,b)\eta^\mu\lambda^\nu.$$

Once the required homomorphisms τ are shown to exist, it follows by the Weierstrass Division Theorem as in the proof of Proposition 2.8, that

$$\tau_{m,n}(f)(\eta,\lambda) = \sigma_{m+M,n+N}(F)(a,\eta,b,\lambda); \tag{2.14}$$

i.e., that $\tau_{m,n}$ is uniquely determined by the conditions of Definition 2.11.

It remains to show that $\tau_{m,n}$ is well-defined by the assignment of equation 2.14. For that, it suffices to show for any $G \in S_{m+M,n+N}(A)$, $G = \sum g_{\mu,\nu}(\xi,\rho)\eta^\mu\lambda^\nu$, that if $\sum g_{\mu,\nu}(a,b)\eta^\mu\lambda^\nu$ is the zero power series of $S_{M,N}(E^\sigma)$, then $\sigma_{m+M,n+N}(G)(a,\eta,b,\lambda)$ is the zero function. By Lemma 2.9, there are: $d \in \mathbb{N}$ and power series $H_{\mu,\nu} \in S_{m+M,n+N}(A)$ such that

$$G = \sum_{|(\mu,\nu)| \le d} g_{\mu,\nu} H_{\mu,\nu}.$$

Then

$$\sigma(G)(a,\eta,b,\lambda) = \sum_{|(\mu,\nu)| \le d} \sigma(g_{\mu,\nu})(a,b)\sigma(H_{\mu,\nu})(a,\eta,b,\lambda) = 0,$$

as desired. Since E^σ is a subring of K°, the injectivity of τ is a consequence of Proposition 2.10. This proves part (i).

(ii) The proof of part (ii) is similar. □

For our purposes, it is useful to work with the ring of all separated (or strictly convergent) power series with parameters from K.

Definition 2.15. (i) Let K be a valued field with separated analytic A-structure $\{\sigma_{m,n}\}$. Let $\mathcal{F}(\sigma, K)$ be the collection of all finitely generated A-subalgebras $E \subset K^\circ$. Then $\mathcal{F}(\sigma, K)$ and $\{E^\sigma\}_{E\in\mathcal{F}(\sigma,K)}$ form direct systems of A-algebras in a natural way, where E^σ is as in Definition 2.12 (i). Put

$$S^\circ_{m,n}(\sigma, K) := \varinjlim_{E\in\mathcal{F}(\sigma,K)} E^\sigma\langle\xi\rangle[\![\rho]\!],$$

which is a K°-algebra. The *rings of separated power series with parameters from* K are then defined to be

$$S_{m,n}(\sigma, K) := K \otimes_{K^\circ} S^\circ_{m,n}(\sigma, K).$$

(ii) Let K be a valued field with separated analytic A-structure $\{\sigma_{m,n}\}$. Using the notation of (i) we define the *strictly convergent power series with parameters from* K to be

$$T_m(\sigma, K) := S_{m,0}(\sigma, K).$$

(iii) Let K be a valued field with strictly convergent analytic A-structure $\{\sigma'_m\}$. The rings $T_m(\sigma', K)$ of *strictly convergent power series with parameters from* K are defined similarly using Lemma 2.12 (ii). Using the same notation for the rings of strictly convergent power series with parameters from K arising from a separated analytic structure on K and from a strictly convergent analytic structure on K should not lead to confusion.

Remark 2.16. The rings $S^\circ_{m,n}(\sigma, K)$ (respectively, $T_m(\sigma, K)$) inherit Weierstrass Division, Theorem 2.4, and Weierstrass Preparation, Corollary 2.6, since they are direct unions of the rings $S_{m,n}(E^\sigma)$ (respectively, $T_m(E^\sigma)$) to which those results apply.

Just as it can be useful to extend the coefficient ring of an analytic structure, it is also useful to be able to extend the domain of an analytic structure. This requires the following proposition, which is proved exactly as [31], Lemma 3.3.

Proposition 2.17. *(i) Let K be a valued field with separated analytic A-structure; then K° is a Henselian valuation ring.*

(ii) Let K be a valued field with strictly convergent analytic A-structure such that $\operatorname{ord}(K^{\circ\circ})$ *has a minimal element γ, and $\gamma = \min \operatorname{ord}(\sigma_0(I))$. Then K° is a Henselian valuation ring.*

The following theorem permits us to work over any finite algebraic extension, or over the algebraic closure, of the domain of an analytic A-structure.

Theorem 2.18. *(i) Let K be a valued field with separated analytic A-structure σ. Then there is a unique extension of σ to a separated analytic A-structure τ on K_{alg}, the algebraic closure of K.*

(ii) Let K be a valued field with strictly convergent analytic A-structure such that $\operatorname{ord}(K^{\circ\circ})$ *has a minimal element γ, and $\gamma = \min \operatorname{ord}(\sigma_0(I))$. Then there is a unique extension of σ to a strictly convergent analytic A-structure τ on K_{alg}.*

(iii) Let K be as in part (ii); then there is a unique extension of σ to a separated analytic A-structure τ on K_{alg}.

Proof. Let $\alpha \in K^\circ_{alg}$ and let $P(t) = t^d + a_1t^{d-1} + \cdots + a_d$ be the minimal polynomal for α over K. Since by Proposition 2.17 (i), K° is Henselian, the coefficients a_i lie in K°; moreover, if $\alpha \in K^{\circ\circ}_{alg}$, then the a_i lie in $K^{\circ\circ}$. Now use Weierstrass division. □

Remark 2.19. Let K be a valued field with analytic A-structure that satisfies the conditions of either Theorem 2.18 (i) or (ii), and let L be an extension of K contained in K_{alg}. Then the arguments of Theorem 2.18 show that

$$S^\circ_{m,n}(\tau, L) = L^\circ \otimes_{K^\circ} S^\circ_{m,n}(\sigma, K) \quad \text{and} \quad S_{m,n}(\tau, L) = L \otimes_K S_{m,n}(\sigma, K).$$

Since the base change is faithfully flat, Remark 2.19 yields the following corollary.

Corollary 2.20. *Let K and L be as in Remark 2.19; then:*

(i) $S^\circ_{m,n}(\tau, L)$ (respectively, $S_{m,n}(\tau, L)$) is faithfully flat over $S^\circ_{m,n}(\sigma, K)$ (respectively, over $S_{m,n}(\sigma, K)$), and

(ii) if L is finite over K, then $S_{m,n}(\tau, L)$ is finite over $S_{m,n}(\sigma, K)$.

Similar statements hold for T_m.

3. Rational analytic functions in one variable

In this section, we develop the basis of a theory of analytic functions on a K-annulus (an irreducible R-domain in K°), when K carries a separated A-analytic structure, as it is needed for the proof of the cell decomposition of this paper. In particular, we show that given an analytic function f on a K-annulus, there is a partition of the annulus into finitely many annuli $\mathcal{U}$ such that $f|_{\mathcal{U}}$ is a rational function times a (very) strong unit (see Theorem 3.9). All the same results hold (with the same proofs) in the "standard" case where K is a complete non-Archimedean valued field and $S_{m,n}(\sigma, K)$ is replaced by $S_{m,n}(E, K)$ (with the notation of [34]). Hence, the results in this section also extend the affinoid results of [24], Sections 2.1 and 2.2, to the case that K is not algebraically closed and to the quasi-affinoid case (i.e., allowing strict as well as weak inequalities).

The results of this section require K to carry a separated A-analytic structure. Note, however, by Theorem 2.18 (iii), in the setting of this paper, a strictly convergent A-analytic structure on K can be extended uniquely to a separated A-analytic structure on K_{alg}.

A subsequent paper will give a complete treatment of the analytic geometry of the one-dimensional unit ball over K_{alg}, when K carries either a strictly convergent or separated analytic structure. This will include the analogue of the classical Mittag-Leffler Theorem (cf. [24], Theorems 2.2.6 and 2.2.9) over coefficient fields K that may be neither complete nor algebraically closed, both in the affinoid and quasi-affinoid setting. This will allow the exploration of more cell decompositions.

Definition 3.1. Let K be a Henselian valued field (with separated A-analytic structure) and let x be one variable.

(a) A K-*annulus formula* is a formula φ of the form

$$|p_0(x)| \square_0 \varepsilon_0 \wedge \bigwedge_{i=1}^{L} \varepsilon_i \square_i |p_i(x)|,$$

where the $p_i \in K^\circ[x]$ are monic and irreducible, the ε_i are in the divisible closure $\sqrt{|K| \setminus \{0\}}$ of $|K| \setminus \{0\}$, and $\square_i \in \{<, \leq\}$.[4] Define $\overline{\square}_i$ by $\{\square_i, \overline{\square}_i\} = \{<, \leq\}$. We require further that the "holes" $\{x \in K_{alg} : |p_i(x)| \overline{\square}_i \varepsilon_i\}$, $i = 1, \ldots, L$, all are

[4]Alternatively, we could require that $\varepsilon_i \in |K^\circ| \setminus \{0\}$ and allow the p_i to be powers of irreducible monic polynomials.

contained in the disc $\{x \in K_{alg} : |p_0(x)| \square_0 \varepsilon_0\}$ and that the holes corresponding to different indices i are disjoint.[5]

(b) The corresponding K-*annulus* is

$$\mathcal{U}_\varphi := \{x \in K_{alg} : \varphi(x)\}$$

(If $K_1 \supset K_{alg}$ then φ also defines an annulus in K_1. We shall also refer to this as $\mathcal{U}_\varphi$. No confusion will result.)

(c) a K-annulus formula φ, and the corresponding K-annulus $\mathcal{U}_\varphi$, is called *linear* if the p_i are all linear. (If $K = K_{alg}$ then all K-annulus formulas are linear.)

(d) a K-annulus formula φ, and the corresponding K-annulus $\mathcal{U}_\varphi$, is called *closed* (respectively *open*) if all the $\square_i$ are $\le$ (respectively $<$).

Lemma 3.2. *(i) Let $p \in K[x]$ be irreducible and let $\square \in \{<, \le\}$. Then for every $\delta \in \sqrt{|K \setminus \{0\}|}$ there is an $\varepsilon \in \sqrt{|K \setminus \{0\}|}$ such that for every $x \in K_{alg}$, $|p(x)| \square \varepsilon$ if, and only if, for some zero α of p, $|x - \alpha| \square \delta$.*

(ii) A K-annulus is a finite union of isomorphic (and linear) K_{alg}-annuli.

(iii) Any two K-discs (i.e. $L = 0$ in Definition 3.1) $\mathcal{U}_1$ and $\mathcal{U}_2$ are either disjoint or one is contained in the other.

(iv) For any two K-annuli $\mathcal{U}_1$ and $\mathcal{U}_2$, if $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$ then $\mathcal{U}_1 \cap \mathcal{U}_2$ is a K-annulus.

(v) The complement of a K-annulus is a finite union of K-annuli.

(vi) Every set of the form

$$\mathcal{U} = \left\{ x \in K_{alg}^\circ : |p_0(x)| \, \square_0 \, \varepsilon_0 \wedge \bigwedge_{i=1}^{s} \varepsilon_i \, \square_i |p_i(x)| \right\}$$

with the p_i irreducible over K is defined by a K-annulus formula.

Proof. Exercise □

Definition 3.3. Let φ be a K-annulus formula as in Definition 3.1 (a). Define the *ring of K-valued functions $\mathcal{O}_K(\varphi)$ on $\mathcal{U}_\varphi$* by

$$\mathcal{O}_K(\varphi) := S_{m+1,n}(\sigma, K)/(p_0^{l_0}(x) - a_0 z_0, p_1^{l_1}(x) z_1 - a_1, \ldots, p_L^{l_L}(x) z_L - a_L),$$

where $a_i \in K^\circ$, $|a_i| = \varepsilon_i^{l_i}$, $m+n = L+1$, $\{z_0, \ldots, z_L\}$ is the set $\{\xi_2, \ldots, \xi_{m+1}, \rho_1, \ldots, \rho_n\}$ and x is ξ_1 and z_i is a ξ or ρ variable depending, respectively, on whether $\square_i$ is $\le$ or $<$. Observe that each $f \in \mathcal{O}_K(\varphi)$ defines a function $\mathcal{U}_\varphi \to K_{alg}$ via the analytic structure on K, which by Theorem 2.18 extends uniquely to K_{alg}.

Remark 3.4. Let φ be a K-annulus formula and write $\mathcal{U}_\varphi = \dot{\bigcup} \mathcal{U}_i$ with the $\mathcal{U}_i$ isomorphic (linear) K_{alg}-annuli given by Lemma 3.2 (ii). Then one can prove

$$\mathcal{O}_K(\varphi) \hookrightarrow K_{alg} \otimes_K \mathcal{O}_K(\varphi) = \bigoplus_i \mathcal{O}_{K_{alg}}(\mathcal{U}_i).$$

This result is not needed here.

Definition 3.5. Let f be a unit in $\mathcal{O}_K(\varphi)$. Suppose that there is some $\ell \in \mathbb{N}$ and $c \in K$ such that $\left|f^\ell(x)\right| = |c|$ for all $x \in \mathcal{U}_\varphi$. Suppose also that there exists a polynomial $P(\xi) \in \widetilde{K}[\xi]$ such that $P\left(\left(\frac{1}{c} f^\ell(x)\right)^\sim\right) = 0$ for all $x \in \mathcal{U}_\varphi$, where $^\sim : K^\circ \to \widetilde{K}$ is the natural projection. Then call f a *strong unit*. Call f a *very strong* unit if moreover $|f(x)| = 1$ and $(f(x))^\sim = 1$ in $\mathcal{O}_K(\varphi)$.

[5] To give an example of a "hole", look at the K-annulus formula $|x| \le 1 \wedge 1/2 < |x|$, for which the hole is the ball $|x| \le 1/2$ around 0.

Lemma 3.6 (Normalization)**.** *(i) Let φ be a closed K-annulus formula. Then there is an inclusion*

$$S_{1,0}(\sigma, K) \hookrightarrow \mathcal{O}_K(\varphi),$$

which is a finite ring extension.

(ii) Let φ be an open K-annulus formula. Then there is an inclusion

$$S_{0,1}(\sigma, K) \hookrightarrow \mathcal{O}_K(\varphi),$$

which is a finite ring extension.

Proof. Apply a suitable Weierstrass automorphism, as in the classical case. □

The following two corollaries are proved exactly as in the classical case (cf. [2] Sections 3.8 and 5.2).

Corollary 3.7. *Let φ be a K-annulus formula that is either closed or open. Then*

(i) the Nullstellensatz holds for $\mathcal{O}_K(\varphi)$; i.e., the maximal ideals of $\mathcal{O}_K(\varphi)$ are K-algebraic.

(ii) $\mathcal{O}_K(\varphi)$ is an integral domain.

Corollary-Definition 3.8. *(i) If φ is a K-annulus formula that is either closed or open and ψ is any K-annulus formula with $\mathcal{U}_\varphi \subseteq \mathcal{U}_\psi$ then*

$$\mathcal{O}_K(\psi) \hookrightarrow \mathcal{O}_K(\varphi).$$

(ii) If φ is a K-annulus formula that is closed or open, the ring $\mathcal{O}_K(\varphi)$ depends only on $\mathcal{U}_\varphi$ and is independent of the formula φ. Hence, we can define $\mathcal{O}_K(\mathcal{U}_\varphi)$ as $\mathcal{O}_K(\varphi)$ for such φ.

Let U be a K-annulus. A K-annulus formula φ

$$|p_0(x)|\square_0\varepsilon_0 \wedge \bigwedge_{i=1}^{L} \varepsilon_i \square_i |p_i(x)|$$

is called a *good description* of U if $U = \mathcal{U}_\varphi$ and each p_i is of minimal degree. This condition implies that if $\deg q < \deg p_i$ then q has no zero in the hole defined by p_i; i.e., in the disc defined by the formula $|p_i(x)|\overline{\square}_i\varepsilon_i$. If φ is a good description of $\mathcal{U}_\varphi$, then we say that φ is a *good K-annulus formula*. Observe that each K-annulus has a good description. Moreover, by Corollary-Definition 3.8 (ii), if φ is a closed or open K-annulus formula, then replacing φ by a good description does not change the ring of analytic functions.

The main result of this section is the following.

Theorem 3.9. *Let φ be a K-annulus formula and let $f \in \mathcal{O}_K(\varphi)$. Then there are: finitely many K-annulus formulas φ_i, each either closed or open, such that φ is equivalent to the disjunction of the φ_i, rational functions $R_i \in \mathcal{O}_K(\varphi_i)$ and very strong units $E_i \in \mathcal{O}_K(\varphi_i)$ such that for each i,$f|_{\mathcal{U}_{\varphi_i}} = R_i E_i$.*

Proof. By Lemma 3.11, the theorem follows from Propositions 3.14 and 3.15, below. □

The decomposition in Theorem 3.9 will be given in terms of two types of annuli, thin annuli and Laurent annuli, defined as follows.

Definition 3.10. (i) A linear K-annulus is called *thin* if it is of the form

$$\{x \in K_{alg} : |x - a_0| \leq \varepsilon \text{ and for } i = 1, \dots, n,\ |x - a_i| \geq \varepsilon\}$$

for some $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $\varepsilon \leq 1$, and $a_i \in K^\circ$.

In general, a K-annulus is called *thin* if it can be written as a union of isomorphic (and linear) K_{alg}-annuli $\mathcal{U}_i$ (as in Lemma 3.2 (ii)), such that each $\mathcal{U}_i$ is a thin K_{alg}-annulus.

A K-annulus formula is called *thin* when it defines a thin K-annulus.

(ii) A K-annulus $\mathcal{U}$ of the form

$$\{x \in K_{alg} : \varepsilon_1 < |p(x)| < \varepsilon_0\},$$

where $p \in K[x]$ is irreducible is called a *Laurent annulus.*

Note that any open K-annulus with one hole is Laurent.

Lemma 3.11. *Every K-annulus formula is equivalent to a finite disjunction of K-annulus formulas that are either thin or Laurent.*

Proof. Define the *complexity* of a K-annulus formula

$$|p_0(x)| \,\square_0 \varepsilon_0 \wedge \bigwedge_{i=1}^{L} \varepsilon_i \square_i \,|p_i(x)|$$

to be $\sum_{i=1}^{L} \deg p_i$. The proof is by induction on complexity. The base case is easy.

By removing thin annuli, we may assume that the remaining set is an open K-annulus. If there is only one hole, the annulus is Laurent, and we are done. Assume that there are at least two holes. Two closed holes are said to *abut* when their radii are equal to the distance between the centers. After removing the largest possible Laurent annuli surrounding each hole, we may assume that all holes are closed, and at least two abut. Now, removing a thin annulus lowers the complexity. □

Lemma 3.12. *Consider the following K-annulus formula, φ:*

$$|p_0(x)| \leq \varepsilon_0 \wedge \bigwedge_{i=1}^{L} \varepsilon_i \leq |p_i(x)|\,.$$

Suppose that φ is a good, thin K-annulus formula. Let $\nu_{ij} \in \mathbb{N}$, $i = 1, 2$, $j = 1, \dots, L$. Suppose $f_i \in K[x]$ satisfy $\deg f_i < \deg p_j$ for all j such that $\nu_{ij} > 0$ and suppose that $\nu_{1j} \neq \nu_{2j}$ for some j. Then

$$\left\| \frac{f_1}{\prod_j p_j^{\nu_{1j}}} + \frac{f_2}{\prod_j p_j^{\nu_{2j}}} \right\|_{\sup} = \max \left\{ \left\| \frac{f_1}{\prod_j p_j^{\nu_{1j}}} \right\|_{\sup}, \left\| \frac{f_2}{\prod_j p_j^{\nu_{2j}}} \right\|_{\sup} \right\},$$

with $\|\cdot\|_{\sup}$ the supremum norm on $\mathcal{U}_\varphi$.

Proof. This reduces easily to the linear case, which is treated in the proof of [23], Theorem 2.2.6. □

Lemma 3.13. *When φ is a thin K-annulus formula, the supremum norm $\|\cdot\|_{\sup}$ is a valuation on $\mathcal{O}_K(\varphi)$.*

Proof. Let the notation be as in the statement of Lemma 3.12. By that lemma and the definition of $\mathcal{O}_K(\varphi)$, this reduces to showing that

$$\left\| \frac{f_1}{\prod p_j^{\nu_{1j}}} \cdot \frac{f_2}{\prod p_j^{\nu_{2j}}} \right\|_{\sup} = \left\| \frac{f_1}{\prod p_j^{\nu_{1j}}} \right\|_{\sup} \cdot \left\| \frac{f_2}{\prod p_j^{\nu_{2j}}} \right\|_{\sup},$$

which is immediate. □

Proposition 3.14. *Let φ be a thin K-annulus formula. Then for each $f \in \mathcal{O}_K(\varphi)$ there is a rational function $R \in \mathcal{O}_K(\varphi)$ and a very strong unit $E \in \mathcal{O}_K(\varphi)$ such that $f = R \cdot E$.*

Proof. By the Nullstellensatz, Corollary 3.7, there is a monic polynomial $f_0 \in K^\circ[x]$, with zeros only in $\mathcal{U}_\varphi$, and an $f' \in \mathcal{O}_K(\varphi)$ such that $f = f_0 \cdot f'$, and f' is a unit of $\mathcal{O}_K(\varphi)$. Thus we may assume that f is a unit. We may also assume that the supremum norm $\|f\|_{\sup}$ on $\mathcal{U}_\varphi$ equals 1. By Lemma 3.13, this implies that $\|g\|_{\sup} = 1$, where $g \in \mathcal{O}_K(\varphi)$ satisfies $gf = 1$. Thus, by Lemma 3.12 and the definition of $\mathcal{O}_K(\varphi)$, there is a rational function $\hat{f} \in \mathcal{O}_K(\varphi)$ such that $\left\|f - \hat{f}\right\|_{\sup} < 1$ and $\hat{f}$ is a unit of $\mathcal{O}_K(\varphi)$. Since $\hat{f}$ is a unit, we may write $f = \hat{f} \cdot E$. We have $\|E - 1\|_{\sup} < 1$, so E is a very strong unit. □

Proposition 3.15. *Let φ be a Laurent K-annulus formula, and let $f \in \mathcal{O}_K(\varphi)$. There are finitely many K-annulus formulas φ_i, each either thin or Laurent, such that $\mathcal{U}_\varphi = \cup_i \mathcal{U}_{\varphi_i}$, and for each i, there are rational functions $R_i \in \mathcal{O}_K(\varphi_i)$ and very strong units $E_i \in \mathcal{O}_K(\varphi_i)$ such that $f|_{U_{\varphi_i}} = R_i \cdot E_i$.*

Proof. Write

$$f = \sum_{i \in \mathbb{Z}} a_i(x) p^i,$$

where p is the polynomial that occurs in φ and the a_i are polynomials of degree less than the degree of p, by using Euclidian division for polynomials. By Lemma 2.9, there are only finitely many a_i that can be dominant (in the sense of the proof of Proposition 2.10) on any sub-annulus of $\mathcal{U}_\varphi$. There is a partition of $\mathcal{U}_\varphi$ into a finite collection of thin sub-annuli and Laurent sub-annuli such that each Laurent sub-annulus is either of lower complexity or on the Laurent sub-annulus, each of the finitely many dominant a_i is a strong unit (only the fact that it is a unit is used). The thin sub-annuli are handled by Proposition 3.14. The Laurent sub-annuli of lower complexity are treated by induction and the remaining Laurent sub-annuli are treated as in [30], Theorem 3.3. □

Lemma 3.16. *Let*

$$R(x) = x^{n_0} \prod_{i=1}^{s} p_i(x)^{n_i} \in K(x),$$

where the $p_i \in K^\circ[x]$ are monic, irreducible and mutually prime and the $n_i \in \mathbf{Z}$. Let $\varepsilon \in \sqrt{|K| \setminus \{0\}}$, let $\square \in \{<, \leq\}$, and let

$$\mathcal{U} := \{x \in K^\circ_{alg} : |R(x)| \,\square\, \varepsilon\}.$$

There are finitely many K-annuli $\mathcal{U}_i$, $i = 1, \ldots, L$, such that $\mathcal{U} = \bigcup_{i=1}^{L} \mathcal{U}_i$ and for each i, $R(x)|_{\mathcal{U}_i} \in \mathcal{O}_K(\mathcal{U}_i)$.

Proof. Induction on s. For each i, let $\alpha_i \in K^\circ_{alg}$ be a zero of p_i, and let $a_i := |\alpha_i| \in \sqrt{|K|}$. Since p_i is irreducible, a_i is independent of which zero of p_i is chosen. Let

$$d_{ij} := \min\{|\alpha - \beta| : p_i(\alpha) = 0 = p_j(\beta)\} \in \sqrt{|K|}.$$

Hence d_{ij} is the smallest distance between a zero of p_i and a zero of p_j. We consider several cases.

(Case 1) There are i, j with $a_i < a_j$. Choose $\gamma \in \sqrt{|K|}$ with $a_i < \gamma < a_j$. Let $\mathcal{U}_1 := \mathcal{U} \cap \{x : |x| \leq \gamma\}$ and $\mathcal{U}_2 := \mathcal{U} \cap \{x : |x| \geq \gamma\}$. Then on $\mathcal{U}_1$, p_j is a strong unit and on $\mathcal{U}_2$, we have that $p_i = E \cdot x^{deg(p_i)}$, E a strong unit.

(Case 2) $a_i = \gamma$ for all i and $d_{12} < d_{13}$, say. Choose $\delta \in \sqrt{|K|}$ so that $d_{12} < \delta < d_{13}$ and by Lemma 3.2 (i), choose $\gamma' \in \sqrt{|K|}$ so that

$$|p_1(x)| \leq \gamma' \longleftrightarrow \bigvee_{\substack{\alpha \text{ such that} \\ p_1(\alpha)=0}} |x - \alpha| \leq \delta.$$

On $\mathcal{U}_3 := \mathcal{U} \cap \{x : |p_1(x)| \leq \gamma'\}$, p_3 is a strong unit. On $\mathcal{U}_4 := \mathcal{U} \cap \{x : \gamma' \leq |p_1(x)|\}$, we have that $p_2(x)^{n_1} = E \cdot p_1(x)^{n_2}$, E a strong unit, for suitable $n_1, n_2 \in \mathbb{N}$.

(Case 3) $a_i = \gamma$ for all i and $d_{ij} = \delta$ for all $i \neq j$. Then $\delta \leq \gamma$. Choose γ_i such that

$$\{x : |p_i(x)| < \gamma_i\} = \bigcup_{\substack{\alpha \text{ such that} \\ p_i(\alpha)=0}} \{x : |x - \alpha| < \delta\}$$

On $\mathcal{U}_{5i} := \{x : |p_i(x)| < \gamma_i\}$, each p_j with $j \neq i$ is a strong unit. On $\mathcal{U}_{6i} := \{x : |p_i(x)| = \gamma_i\}$, each p_i is of constant size γ_i. On $\mathcal{U}_7 := \{x : |p_1(x)| > \gamma_1\}$ there are $d_i \in \mathbf{Q}_+$ such that $|p_i(x)| = |p_1(x)|^{d_i}$. (For each zero α_i of p_i there is a zero α_1 of p_1 so that for all $x \in \mathcal{U}_7$ we have $|x - \alpha_i| = |x - \alpha_1|$.) □

4. A-analytic languages and quantifier elimination

In this section we recall the notion of languages of Denef-Pas and we introduce the notion of A-analytic languages, suitable for talking about valued fields with A-analytic structure. Further, we specify the theories that we will consider, and we establish the corresponding quantifier elimination results in equicharacteristic zero and in mixed characteristic with bounded ramification.

For K a valued field, I an ideal of K°, write $\mathrm{res}_I : K^\circ \to K^\circ/I$ for the natural projection. An *angular component modulo I* is a map $\overline{\mathrm{ac}}_I : K \to K^\circ/I$ such that the restriction to $K^\times$ is a multiplicative homomorphism to $(K^\circ/I)^\times$, the restriction to $(K^\circ)^\times$ coincides with the restriction to $(K^\circ)^\times$ of res_I, and such that $\overline{\mathrm{ac}}_I(0) = 0$ (for R a ring, $R^\times$ is the group of units of R). The importance of the maps $\overline{\mathrm{ac}}_I$ for some applications is explained in Remark 8.3.

Fix a sequence of positive numbers $(n_p)_p$, indexed by the prime numbers and write $\mathbf{N}_0 := \{x \in \mathbf{Z} : x > 0\}$. We consider structures

$$(K, \{K^\circ/I_m\}_{m \in \mathbf{N}_0}, \mathrm{ord}(K^\times)),$$

where K is a Henselian valued field of characteristic zero with valuation ring K°, additively written valuation[6] $\mathrm{ord} : K^\times \to \mathrm{ord}(K^\times)$, angular component maps $\overline{\mathrm{ac}}_m$

[6]The problem that ord is not defined globally on K is easily settled and the reader may choose a way to do so. For example, the reader may choose a value of ord(0) in the value group and treat the cases that the argument or ord equals zero always separately, or, the reader may add a symbol $+\infty$ to the language $\mathbf{L}_{\mathrm{Ord}}$ that is bigger than any element of the value group, and make the natural changes.

modulo I_m, a constant $t_K \in K$, and ideals I_m of K° for $m \in \mathbf{N}_0$, satisfying the following properties

(I) I_1 is the maximal ideal of K°,
(II) either $I_m = I_1$ for all $m \in \mathbf{N}_0$ and $t_K = 1$, or, $\mathrm{ord}(K^\times)$ has a minimal positive element, $I_m = I_1^m$ for all $m \in \mathbf{N}_0$, and t_K is either 1 or an element of K° with minimal positive valuation such that $\overline{\mathrm{ac}}_m(t_K) = 1$ for all $m > 0$,
(III) if the residue field $\widetilde{K}$ of K has characteristic $p > 0$, then, $t_K \neq 1$ (hence, $I_2 \neq I_1$), and the ramification is bounded by $\mathrm{ord}(p) \leq n_p$.

Let $\mathcal{K}((n_p)_p)$ be the class of these structures. We call the sorts Val for the valued field sort, Res_m for the m-th residue ring K°/I_m for $m \in \mathbf{N}_0$, more generally Res for the disjoint union of the Res_m, and Ord for the value group sort.

For Val we use the language $\mathbf{L}_{\mathrm{Val}} = (+, -, ^{-1}, \cdot, 0, 1, t_0)$ of fields with an extra constant symbol t_0, interpreted in K as t_K. Let $\mathbf{L}_{\mathrm{Ord},0} = (+, -, \leq, 0)$ be the language of ordered groups.

For Res we define the language $\mathbf{L}_{\mathrm{Res},0}$ as the language having the ring language and a constant symbol t_m for each sort Res_m and natural projection maps $\pi_{mn} : \mathrm{Res}_m \to \mathrm{Res}_n$ giving commutative diagrams with the maps $\mathrm{res}_m := \mathrm{res}_{I_m}$ and res_n for $m \geq n$. If $I_2 = I_1$, the t_m are interpreted as 1. If $I_2 \neq I_1$, t_m denotes the image under res_m of an element x with $\overline{\mathrm{ac}}_m(x) = 1$ and $\mathrm{ord}(x)$ the minimal positive element.

Fix expansions $\mathbf{L}_{\mathrm{Ord}}$ of $\mathbf{L}_{\mathrm{Ord},0}$ and $\mathbf{L}_{\mathrm{Res}}$ of $\mathbf{L}_{\mathrm{Res},0}$.[7] To this data we associate the *language* $\mathcal{L}_{\mathrm{DP}} = \mathcal{L}_{\mathrm{DP}}(\mathbf{L}_{\mathrm{Res}}, \mathbf{L}_{\mathrm{Ord}})$ *of Denef-Pas* defined as

$$(\mathbf{L}_{\mathrm{Val}}, \mathbf{L}_{\mathrm{Res}}, \mathbf{L}_{\mathrm{Ord}}, \{\overline{\mathrm{ac}}_m\}_{m \in \mathbf{N}_0}, \mathrm{ord}).$$

Fix a language $\mathcal{L}_{\mathrm{DP}}$ of Denef-Pas, an $\mathbf{L}_{\mathrm{Ord}}$-theory $\mathbf{T}_{\mathrm{Ord}}$, and an $\mathbf{L}_{\mathrm{Res}}$-theory $\mathbf{T}_{\mathrm{Res}}$. For such data, let $\mathcal{T}_{\mathrm{DP}} = \mathcal{T}_{\mathrm{DP}}(\mathcal{L}_{\mathrm{DP}}, \mathbf{T}_{\mathrm{Res}}, \mathbf{T}_{\mathrm{Ord}}, (n_p)_p)$ be the $\mathcal{L}_{\mathrm{DP}}$-theory of all structures

$$(K, \{K^\circ/I_m\}_{m \in \mathbf{N}_0}, \mathrm{ord}(K^\times))$$

in $\mathcal{K}((n_p)_p)$ which are $\mathcal{L}_{\mathrm{DP}}$-structures such that $\{K^\circ/I_m\}_{m \in \mathbf{N}_0}$ is a model of $\mathbf{T}_{\mathrm{Res}}$, and $\mathrm{ord}(K^\times)$ is a model of $\mathbf{T}_{\mathrm{Ord}}$.

The sorts Res_m for $m \in \mathbf{N}_0$ and Ord are called *auxiliary sorts*, and Val is the main sort.

Now we come to the notion of A-analytic languages. Fix a Noetherian ring A that is complete and separated for the I-adic topology for a fixed ideal I of A. Define the *separated A-analytic language*

$$\mathcal{L}_{S(A)} := \mathcal{L}_{\mathrm{DP}} \cup_{m,n \geq 0} S_{m,n}(A)$$

and the *strictly convergent A-analytic language*

$$\mathcal{L}_{T(A)} := \mathcal{L}_{\mathrm{DP}} \cup_{m \geq 0} T_m(A),$$

where $S_{m,n}(A)$ and $T_m(A)$ are as in Definition 2.1.

For $\mathcal{T}_{\mathrm{DP}}$ as before, we define the $\mathcal{L}_{S(A)}$-theory

$$\mathcal{T}_{S(A)} := \mathcal{T}_{\mathrm{DP}} \cup (\mathrm{IV})_S$$

and the $\mathcal{L}_{T(A)}$-theory

$$\mathcal{T}_{T(A)} := \mathcal{T}_{\mathrm{DP}} \cup (\mathrm{IV})_T,$$

[7] The reason that one can work with arbitrary one-sorted expansions of $\mathbf{L}_{\mathrm{Ord},0}$ and $\mathbf{L}_{\mathrm{Res},0}$ is that all results about $\mathcal{T}_{S(A)}$ and $\mathcal{T}_{T(A)}$ are relative to the Ord- and Res-sorts.

where

(IV)$_S$ the Val-sort is equipped with a separated analytic A-structure and each symbol f of $S_{m,n}(A)$ is considered as a function $\mathrm{Val}^{m+n} \to \mathrm{Val}$ by extending f by zero outside its domain $\{(x,y) \in \mathrm{Val}^{m+n} : \mathrm{ord}(x_i) \geq 0,\ \mathrm{ord}(y_j) > 0\}$,

(IV)$_T$ the Val-sort is equipped with a strictly convergent analytic A-structure and each symbol f of $T_m(A)$ is considered as a function $\mathrm{Val}^m \to \mathrm{Val}$ by extending f by zero outside its domain $\{x \in \mathrm{Val}^m : \mathrm{ord}(x_i) \geq 0\}$. Moreover, the value group has a minimal positive element and this is the order of some constant of $T_0(A)$.

Observe that there exist $\mathcal{L}_{S(A)}$-terms, resp. $\mathcal{L}_{T(A)}$-terms, which yield all kinds of restricted division as considered in [29], [33] and [34].

Write res_m for the natural projection from the valuation ring to Res_m, for each $m > 0$. Note that each map res_m is definable without Val-quantifiers, since it sends $x \in \mathrm{Val}$ to $\overline{\mathrm{ac}}_m(x)$ when $\mathrm{ord}(x) = 0$, and to $\overline{\mathrm{ac}}_m(1+x) - 1$ when $\mathrm{ord}(x) > 0$.

Later on, in Theorem 7.5, we will use definitional expansions to reveal the term structure of definable functions; these expansions are defined as follows. For each language $\mathcal{L}$, write $\mathcal{L}^*$ for the expansion

$$\mathcal{L}^* := \mathcal{L} \cup \{(\cdot,\cdot,\cdot)_e^{1/m},\ h_{m,e}\}_{m>0,\ e\geq 0}, \tag{4.1}$$

with m, e integers. Then, each model of $\mathcal{T}_{\mathrm{DP}}$ extends uniquely to an $\mathcal{L}^*_{\mathrm{DP}}$-structure, axiomatized as follows:

(V) $(\cdot,\cdot,\cdot)_e^{1/m}$ is the function $\mathrm{Val} \times \mathrm{Res}_{2e+1} \times \mathrm{Ord} \to \mathrm{Val}$ sending (x, ξ, z) to the (unique) m-th root y of x with $\overline{\mathrm{ac}}_{e+1}(y) = \pi_{2e+1,e+1}(\xi)$ and $\mathrm{ord}(y) = z$, whenever $\xi^m = \overline{\mathrm{ac}}_{2e+1}(x)$, $m \neq 0$ in Res_{e+1}, and $mz = \mathrm{ord}(x)$, and to 0 otherwise;

(VI) $h_{m,e}$ is the function $\mathrm{Val}^{m+1} \times \mathrm{Res}_{2e+1} \to \mathrm{Val}$ sending $(a_0, \dots, a_m, \xi)$ to the unique y satisfying $\mathrm{ord}(y) = 0$, $\overline{\mathrm{ac}}_{e+1}(y) = \pi_{2e+1,e+1}(\xi)$, and $\sum_{i=0}^m a_i y^i = 0$, whenever ξ is a unit, $\mathrm{ord}(a_i) \geq 0$, $f(\xi) = 0$, and
$$\pi_{2e+1,e+1}(f'(\xi)) \neq 0,$$
with $f(\eta) = \sum_{i=0}^m \mathrm{res}_{2e+1}(a_i)\eta^i$ and f' its derivative, and to 0 otherwise.

Sometimes we will use the property, for ℓ either zero or a prime number,

(VII)$_\ell$ the residue field has characteristic ℓ.

The following result extends quantifier elimination results of van den Dries [19] and Pas in [38] and [39]. Theorem 4.2 for the theory $\mathcal{T}_{\mathrm{DP}}$ can be compared with results obtained by Kuhlmann [26]. In [26], the language for the auxiliary sorts is less explicit than in this paper.

Theorem 4.2 (Quantifier elimination)**.** *Let $\mathcal{T}$ be one of the theories $\mathcal{T}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$, and let $\mathcal{L}_{\mathcal{T}}$ be respectively $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{L}_{S(A)}$, or $\mathcal{L}_{T(A)}$. Then $\mathcal{T}$ admits elimination of quantifiers of the valued field sort. Moreover, every $\mathcal{L}_{\mathcal{T}}$-formula $\varphi(x, \xi, \alpha)$, with x variables of the valued field-sort, ξ variables of the residue rings sorts and α variables of the value group sort, is $\mathcal{T}$-equivalent to a finite disjunction of formulas of the form*

$$\psi(\overline{\mathrm{ac}}_\ell f_1(x), \dots, \overline{\mathrm{ac}}_\ell f_k(x), \xi) \wedge \theta(\mathrm{ord}\, f_1(x), \dots, \mathrm{ord}\, f_k(x), \alpha), \tag{4.3}$$

with ψ an $\mathbf{L}_{\mathrm{Res}}$-formula, θ an $\mathbf{L}_{\mathrm{Ord}}$-formula, and $f_1 \dots, f_k$ $\mathcal{L}_{\mathcal{T}}$-terms.

Proof of Theorem 4.2. If one knows the quantifier elimination statement, the statement about the form of the formulas follows easily, cf. [19].

The quantifier elimination statement for $\mathcal{T}_{\rm DP}$ is proved together with the cell decomposition Theorem 7.4 for $\mathcal{T}_{\rm DP}$.

The statement for the analytic theories follows in a nowadays standard way from the result for $\mathcal{T}_{\rm DP}$ and the Weierstrass division as developed in Section 2, cf. the proof of Theorem 3.9 of [19] in the strictly convergent case, and Theorem 4.2 of [34] in the separated case. □

5. Terms in one variable

The results of section 3 have strong implications for terms in the languages $\mathcal{L}_{S(A)}$ and $\mathcal{L}_{T(A)}$ in one valued field variable.

Using Theorem 3.9, Propositions 3.14 and 3.15, Lemma 3.16, and induction on the complexity of terms, we obtain the following.

Theorem 5.1. *Let $\tau(x)$ be an $\mathcal{L}_{S(A)}$-term of the valued field sort (cf. section 4) in one valued field variable x, and K be a $\mathcal{T}_{S(A)}$-model. Then there is a finite set $S\subset K^\circ_{alg}$ and a finite collection of disjoint K-annuli $\mathcal{U}_i$, each open or closed, and for each i, an $F_i\in\mathcal{O}_K(\mathcal{U}_i)$ such that $K^\circ_{alg}=\bigcup\mathcal{U}_i$ and*

$$\tau\mid_{\mathcal{U}_i\setminus S}=F_i\mid_{\mathcal{U}_i\setminus S}.$$

Moreover, we can ensure that

$$F_i=R_iE_i$$

where R_i is a rational function over K and $E_i\in\mathcal{O}_K(\mathcal{U}_i)$ is a very strong unit.

The analogue of Theorem 5.1 for $\mathcal{L}_{T(A)}$-models is the following:

Corollary 5.2. *Let $\tau(x)$ be an $\mathcal{L}_{T(A)}$-term of the valued field sort in the valued field variable x, let $n>0$ be an integer. Let K and K' be $\mathcal{T}_{T(A)}$-models, such that K is a submodel of K'. Suppose that the value group of K and of K' have minimal positive elements and suppose that there exists v in A such that both these minimal elements are equal to the value of $\sigma_0(v)$ (thus, K' is an unramified field extension of K). Then there is a finite set $S\subset K'$ and a finite collection of disjoint closed K-annulus formulas φ_i, and for each i, an $F_i\in\mathcal{O}_K(\mathcal{U}_{\varphi_i})$, such that*

$$K'^\circ=\bigcup\mathcal{U}_i'$$

and

$$\tau\mid_{\mathcal{U}_i'\setminus S}=F_i\mid_{\mathcal{U}_i'\setminus S},$$

where $\mathcal{U}_i'=\{x\in K':\varphi_i(x)\}$. Moreover, we can ensure that

$$F_i=R_iE_i$$

and

$$E_i\equiv 1 \bmod \sigma_0(v)^n$$

hold on $\mathcal{U}_i'$, where R_i is a rational function over K and $E_i\in\mathcal{O}_K(\mathcal{U}_i)$ is a very strong unit.

Proof. Since K' is an unramified field extension of K, we can replace strict inequalities by weak inequalities in the data given by Theorem 5.1, by using the element $\sigma_0(v)$. □

6. Definable assignments

We elaborate on the terminology of [16] and [6] on definable assignments and definable subassignments. By some authors, definable assignments are just called "formulas", or "definable sets", and definable morphisms are often called "definable functions".

Let $\mathcal{T}$ be a multisorted theory formulated in some language $\mathcal{L}$, where some of the sorts are auxiliary sorts and the other sorts are main sorts. Let $\mathrm{Mod}(\mathcal{T})$ be the category whose objects are models of $\mathcal{T}$ and whose morphisms are elementary embeddings. By a $\mathcal{T}$*-assignment* X we mean a $\mathcal{T}$-equivalence class of $\mathcal{L}$-formulas φ, where we say that φ and φ' are $\mathcal{T}$-equivalent if they have the same set of $\mathcal{M}$-rational points for each $\mathcal{M}$ in $\mathrm{Mod}(\mathcal{T})$. Knowing a $\mathcal{T}$-assignment X is equivalent to knowing the functor from $\mathrm{Mod}(\mathcal{T})$ to the category of sets which sends $\mathcal{M} \in \mathrm{Mod}(\mathcal{T})$ to the set $\varphi(\mathcal{M})$ for any $\mathcal{L}$-formula φ in X, and we will identify $\mathcal{T}$-assignments with these functors.

The usual set theoretic operations can be applied to $\mathcal{T}$-assignments, for example, for two $\mathcal{T}$-assignments X, Y, $X \subset Y$ has the natural meaning and if $X \subset Y$ call X a $\mathcal{T}$-subassignment of Y. Similarly, for $X, Y \subset Z$ $\mathcal{T}$-assignments, $X \cup Y$, $X \cap Y$, and $X \setminus Y$ have the obvious meaning. Cartesian products have the obvious meaning and notation. We refer to [6] and [16] for more details on the general theory of assignments and $\mathcal{T}$-assignments.

For $\mathcal{T}$-assignments X, Y, a collection of functions $f_{\mathcal{M}} : X(\mathcal{M}) \to Y(\mathcal{M})$ for each $\mathcal{M} \in \mathrm{Mod}(\mathcal{T})$ is called a $\mathcal{T}$*-morphism* from X to Y if the functor sending $\mathcal{M} \in \mathrm{Mod}(\mathcal{T})$ to the graph of $f_{\mathcal{M}}$ is a $\mathcal{T}$-assignment. A $\mathcal{T}$-morphism $f : X \to Y$ such that $f_{\mathcal{M}}$ is a bijection for each $\mathcal{M} \in \mathrm{Mod}(\mathcal{T})$ is called a $\mathcal{T}$-isomorphism.

Definition 6.1. By a $\mathcal{T}$*-parameterization* of a $\mathcal{T}$-assignment X, we mean a $\mathcal{T}$-isomorphism $f : X \to Y \subset X \times R$ with R a Cartesian product of auxiliary sorts, such that $\pi \circ f : X \to X$ is the identity on X, with π the projection.

Example 6.2. If $\mathcal{T}$ is one of the theories $\mathcal{T}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$, write $\mathrm{Val}^{\ell_1} \times \mathrm{Res}_n^{\ell_2} \times \mathrm{Ord}^{\ell_3}$, for the $\mathcal{T}$-assignment which sends a model

$$(K, \{K^\circ/I_m\}_{m>0}, \mathrm{ord}(K^\times))$$

to

$$K^{\ell_1} \times (K^\circ/I_n)^{\ell_2} \times \mathrm{ord}(K^\times)^{\ell_3},$$

for any $n > 0, \ell_i \geq 0$. We recall that, for such $\mathcal{T}$, the sorts Res_m and Ord are called auxiliary sorts. For such $\mathcal{T}$, the map $\mathrm{Val} \to \mathrm{Val} \times \mathrm{Res}_1 : x \mapsto (x, \overline{\mathrm{ac}}_1(x))$ is an example of a $\mathcal{T}$-parameterization.

7. Cell decomposition

In this section we state and prove an analytic cell decomposition theorem for $\mathcal{T}$-assignments with $\mathcal{T}$ one of the theories $\mathcal{T}_{S(A)}$ or $\mathcal{T}_{T(A)}$, see Theorem 7.4 below. Theorem 7.4 generalizes cell decompositions of [38], [39], [14], [4], and [6] and provides what is needed for the applications to analytic integrals in the next section. Also for the theory $\mathcal{T}_{\mathrm{DP}}$ we obtain a cell decomposition, cf. Theorem 7.4, which generalizes and refines the cell decompositions of Pas [38], [39] in several ways: in equicharacteristic zero, also angular components of higher order (i.e., modulo powers of the maximal ideal) are allowed; we can take the centers of the cells to be $\mathcal{L}^*_{\mathrm{DP}}$-terms; we can partition any definable set into cells adapted to any given definable function[8]; in mixed characteristic, we allow for any value group with a least positive element[9]. In [6], a notion of cells is introduced that is more general than the one in [38]; we base the definition of cells below on this notion of [6]. Summarizing, our cell decomposition holds for the theories introduced in section 4, which includes both the analytic and the algebraic cases, in equicharacteristic zero with quite general angular components, and in mixed characteristic as long as the degree of ramification is bounded. Crucial to the proof of analytic cell decomposition are Propositions 3.14 and 3.15, Theorem 5.1, and Corollary 5.2. The proof of the analytic cell decomposition seems to require all the work of the previous sections. As a second main result of this section, we prove the fundamental result that $\mathcal{L}$-definable functions are, after parameterization using auxiliary sorts, given by $\mathcal{L}^*$-terms, where $\mathcal{L}$ is either $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{L}_{S(A)}$, or $\mathcal{L}_{T(A)}$, see Theorem 7.5; this result is motivated by notes of van den Dries [20].

Fix $\mathcal{T}$ to be $\mathcal{T}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$, and let $\mathcal{L}_{\mathcal{T}}$ be respectively $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{L}_{S(A)}$, or $\mathcal{L}_{T(A)}$. We come to the definition of $\mathcal{T}$-cells. The usage of parameterizations in Definition 7.1 is necessary in view of Theorem 7.4, as is shown by the example of the subset X of K^2 defined by $\mathrm{ord}(z^2-y)>\mathrm{ord}(y)$, for which a parametrization is essentially given by $(y,z)\mapsto(\overline{\mathrm{ac}}(z),\mathrm{ord}(z),y,z)$. (More generally, for such $\mathcal{T}$, the following definition makes sense as well for the $\mathcal{L}_{\mathcal{T}}\cup K$-theory $\mathcal{T}(K)$, where K is the valued field of a model of $\mathcal{T}$, $\mathcal{L}_{\mathcal{T}}\cup K$ is $\mathcal{L}_{\mathcal{T}}$ together with constant symbols for the elements of K, and $\mathcal{T}(K)$ the theory $\mathcal{T}$ together with the diagram of K.)

Definition 7.1. Let C be a $\mathcal{T}$-assignment, $k>0$ an integer, and $\alpha : C\to \mathrm{Ord}$, $\xi : C\to \mathrm{Res}_k$, and $c : C\to \mathrm{Val}$ $\mathcal{T}$-morphisms, such that ξ takes values in the multiplicative units of the Res_k-sort. The $\mathcal{T}$-1-cell $Z_{C,\alpha,\xi,c}$ with base C, order α, angular component ξ, and center c is the $\mathcal{T}$-subassignment of $C\times\mathrm{Val}$ defined by

$$y\in C\ \wedge\ \mathrm{ord}\,(z-c(y))=\alpha(y)\ \wedge\ \overline{\mathrm{ac}}_k(z-c(y))=\xi(y),$$

where $y\in C$ and $z\in\mathrm{Val}$. Similarly, if c is a $\mathcal{T}$-morphism $c : C\to\mathrm{Val}$, we define the $\mathcal{T}$-0-cell $Z_{C,c}\subset C\times\mathrm{Val}$ with base C and center c as the $\mathcal{T}$-subassignment of $C\times\mathrm{Val}$ defined by

$$y\in C\wedge z=c(y).$$

More generally, $Z\subset S\times\mathrm{Val}$ with Z and S $\mathcal{T}$-assignments will be called a $\mathcal{T}$-1-cell, resp. a $\mathcal{T}$-0-cell, if there exists a $\mathcal{T}$-parameterization

$$\lambda : Z\to Z_C\subset S\times R\times\mathrm{Val},$$

for some Cartesian product R of auxiliary sorts and some $\mathcal{T}$-1-cell $Z_C=Z_{C,\alpha,\xi,c}$, resp. $\mathcal{T}$-0-cell $Z_C=Z_{C,c}$.

We shall call the data $(\lambda, Z_{C,\alpha,\xi,c})$, resp. $(\lambda, Z_{C,c})$, sometimes written for short (λ, Z_C), a *$\mathcal{T}$-presentation* of the $\mathcal{T}$-cell Z.

Definition 7.2. A $\mathcal{T}$-morphism $f : Z\subset S\times\mathrm{Val}\to R$ with Z a $\mathcal{T}$-cell, S a $\mathcal{T}$-assignment, and R a Cartesian product of auxiliary sorts, is called *$\mathcal{T}$-prepared* if there exist a $\mathcal{T}$-presentation $\lambda : Z\mapsto Z_C$ of Z onto a cell Z_C with base C and a $\mathcal{T}$-morphism $g : C\to R$ such that $f=g\circ\pi\circ\lambda$, with $\pi : Z_C\to C$ the projection.

[8]The cells used by Pas are neither suitable for the partition of definable sets, nor for the preparation of definable functions. This problem has been addressed in [6].

[9]In mixed characteristic, Pas [39] allows for the integers as value group only.

Example 7.3. Let Z be the $\mathcal{T}$-cell $\mathrm{Val} \setminus \{0\}$. The $\mathcal{T}_{\mathrm{DP}}$-morphism $Z \to \mathrm{Ord} : x \mapsto \mathrm{ord}\, x^2$ is $\mathcal{T}$-prepared since with the $\mathcal{T}$-presentation

$$\lambda : Z \to \mathrm{Ord} \times \mathrm{Res}_1 \times \mathrm{Val} \setminus \{0\} : x \mapsto (\mathrm{ord}\, x, \overline{\mathrm{ac}}\, x, x)$$

and the map $g : \mathrm{Ord} \times \mathrm{Res}_1 \to \mathrm{Ord} : (z, \eta) \mapsto 2z$, one has $f = g \circ \pi \circ \lambda$.

The following two theorems lay the technical foundations for analytic motivic integration, the first one to calculate the integrals, the second one to prove a change of variables formula, cf. [6] for the algebraic setting.

Theorem 7.4 (Cell decomposition). *Let $\mathcal{T}$ be $\mathcal{T}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$, and let $\mathcal{L}_{\mathcal{T}}$ be respectively $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{L}_{S(A)}$, or $\mathcal{L}_{T(A)}$. Let X be a $\mathcal{T}$-subassignment of $S \times \mathrm{Val}$ and let $f : X \to R$ be a $\mathcal{T}$-morphism with R a Cartesian product of auxiliary sorts, S a $\mathcal{T}$-assignment. Then there exists a finite partition of X into $\mathcal{T}$-cells Z such that each of the restrictions $f|_Z$ is $\mathcal{T}$-prepared. Moreover, this can be done in such a way that for each occurring cell Z one can choose a presentation $\lambda : Z \to Z_C$ onto a cell Z_C with center c, such that c is given by an $\mathcal{L}^*_{\mathcal{T}}$-term, where $\mathcal{L}^*_{\mathcal{T}}$ is defined by (4.1).*

The following is a fundamental result on the term-structure of definable functions. The statement of Theorem 7.5 for fields of the form $k((t))$, uniform in the field k of characteristic zero, and ideals $I_2 = I_1$ was announced in [6] and will be proved completely here.

Theorem 7.5 (Term structure of definable morphisms). *Let $\mathcal{T}$ be $\mathcal{T}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$, and let $\mathcal{L}_{\mathcal{T}}$ be respectively $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{L}_{S(A)}$, or $\mathcal{L}_{T(A)}$. Let $f : X \to Y$ be a $\mathcal{T}$-morphism. Then there exist a $\mathcal{T}$-parameterization $g : X \mapsto X'$ and a tuple h of $\mathcal{L}^*_{\mathcal{T}}$-terms in variables running over X' and taking values in Y such that $f = h \circ g$. (See (4.1) and 6.1 for the definitions.)*

The following notion is only needed for the proof of quantifier elimination in the language $\mathcal{T}_{\mathrm{DP}}$, cf. similar proofs in [11], [14], [38], and [39].

Definition 7.6. An $\mathcal{L}_{\mathrm{DP}}$-definable function $h : X \to \mathrm{Val}$, with X a Cartesian product of sorts, is called strongly definable if for each Val-quantifier free $\mathcal{L}_{\mathrm{DP}}$-formula $\varphi(v, y)$, with y a tuple of variables running over arbitrary sorts, v a Val-variable, and x running over X, there exists a Val-quantifier free $\mathcal{L}_{\mathrm{DP}}$-formula $\psi(x, y)$ such that

$$\varphi(h(x), y)$$

is $\mathcal{T}_{\mathrm{DP}}$-equivalent with

$$\psi(x, y).$$

The next Lemma yields $\mathcal{L}^*_{\mathrm{DP}}$-terms picking a specific root when Hensel's Lemma implies that there exists a unique such one.

Lemma 7.7. *Let Z be a $\mathcal{T}_{\mathrm{DP}}$-1-cell with $\mathcal{T}_{\mathrm{DP}}$-presentation $\mathrm{id} : Z \to Z = Z_{C,\alpha,\xi,c}$ with ξ taking values in Res_{e+1} and such that $c = 0$ on C. Let x run over C, and y over Val. Let $n > 0$ and $f(x, y) = \sum_{i=0}^{n} a_i(x) y^i$ be a polynomial in y with $\mathcal{L}^*_{\mathrm{DP}}$-terms $a_i(x)$ as coefficients, such that $a_n(x)$ is nowhere zero on C. Suppose that for (x, y) in Z*

$$\min_i \mathrm{ord}\, a_i(x) y^i = \mathrm{ord}\, a_{i_0}(x) y^{i_0} \quad \textit{for some fixed } i_0 \geq 1,$$

and

$$\operatorname{ord} f'(x,y) \le \operatorname{ord} t_0^e a_{i_0}(x) y^{i_0-1},$$

and that there exists a $\mathcal{T}_{\mathrm{DP}}$-morphism $d : C \to \mathrm{Val}$ whose graph lies in Z and satisfies

$$f(x, d(x)) = 0.$$

Then, d is the unique such morphism and, after a $\mathcal{T}_{\mathrm{DP}}$-parametrization of C, d can be given by an $\mathcal{L}^*_{\mathrm{DP}}$-term. Moreover, if the a_i are strongly definable, then the function d is strongly definable (see Definition 7.6).

Proof. The uniqueness of d follows from Hensel's Lemma, cf. [38] and [39]. Consider the $\mathcal{T}_{\mathrm{DP}}$-parametrization

$$g : C \to C \times \mathrm{Res}_{2e+1} \times \mathrm{Ord} : x \mapsto (x, \overline{\mathrm{ac}}_{2e+1}\, d(x), \operatorname{ord} d(x)).$$

We prove that, piecewise, d can be given by a term after the $\mathcal{T}_{\mathrm{DP}}$-parametrization g; at the end we will glue the pieces together. Note that $\alpha(x) = \operatorname{ord} d(x)$. We first prove that there exists a Val-term b such that $\operatorname{ord} b(g(x)) = \alpha(x)$. Let $f_{I_x}(x,y)$ be the polynomial $\sum_{i\in I_x} a_i(x)y^i$, with

$$I_x := \{i \in \{0,\dots,n\} : \operatorname{ord} a_i(x)y^i \le \operatorname{ord} t_0^{2e+1} a_{i_0}(x) y^{i_0}\}.$$

Note that I_x only depends on x, since the valuation of y for $(x,y) \in Z$ only depends on x. We work piecewise to find b. First we work on the piece where $\gcd(I_x) = 1$. After partitioning further to ensure that the quotients

$$a_i(x)y^i / a_j(x)y^j$$

have constant order on Z for $i, j \in I_x$, one readily verifies that there exists an $\mathcal{L}^*_{\mathrm{DP}}$-term b such that $\operatorname{ord} b(g(x)) = \alpha(x)$ (for this, the constant symbol t_0 is needed). Now work on the part $\gcd(i \in I_x) = \ell$ for some $\ell > 1$. One obtains, by induction on the degree, an $\mathcal{L}^*_{\mathrm{DP}}$-term h such that

$$\sum_{i\in I_x} a_i(x) h(g(x))^{i/\ell} = 0, \quad \operatorname{ord} h(g(x)) = \ell\alpha(x), \quad \text{and} \quad \overline{\mathrm{ac}}_{e+1} h(g(x)) = \xi(x)^\ell.$$

By the conditions of the lemma, $\ell \ne 0$ in Res_{e+1}. Defining the term $b(x,\eta,a)$ as $(h(x,\eta,a))^{1/\ell}$, one verifies that $\operatorname{ord} b(g(x)) = \alpha(x)$ for all $x \in C$ with $\gcd(i \in I_x) = \ell$.

Now one has $d(x) = \tau(g(x))$ with τ the term

$$\tau(x,\eta,a) := b h_{n,e}\Big(\frac{a_0}{a_{i_0} b^{i_0}}, \frac{b a_1}{a_{i_0} b^{i_0}}, \dots, \frac{b^n a_n}{a_{i_0} b^{i_0}}, \eta\overline{\mathrm{ac}}_{2e+1}(1/b)\Big).$$

One can glue s pieces together using extra parameters contained in the definable subassignment $A := \{\xi \in \mathrm{Res}_1^s : \sum_i \xi_i = 1 \wedge (\xi_i = 0 \vee \xi_i = 1)\}$ to index the pieces, by noting that for each element a in A there exists a definable morphism $A \to \mathrm{Val}$, given by an $\mathcal{L}^*_{\mathcal{T}}$-term, which is the characteristic function of $\{a\}$. The fact that d is strongly definable when the a_i are will be proved in the proof of Theorem 7.4. □

The following is a refinement of both Theorem 3.1 of [38] and Theorem 3.1 of [39], the refinements being the same as the list of algebraic refinements in the introduction of section 7.

Theorem 7.8. *Let $f(x,y)$ be a polynomial in y with $\mathcal{L}^*_{\mathrm{DP}}$-terms in $x=(x_1,\dots,x_m)$ as coefficients, x running over a $\mathcal{T}_{\mathrm{DP}}$-assignment S. Then there exists an integer ℓ and a finite partition of $S\times \mathrm{Val}$ into $\mathcal{T}_{\mathrm{DP}}$-cells Z with presentation $\lambda : Z\to Z_C$ such that Z_C has an $\mathcal{L}^*_{\mathrm{DP}}$-term c as center, and such that, if we write*

$$f(x,y)=\sum_i a_i(z)(y-c(z))^i,$$

for $(x,y)\in Z$ and $(z,y)=\lambda(x,y)$, then we have

$$\operatorname{ord} f(x,y)\le \min_i \operatorname{ord} t_0^{\ell} a_i(z)(y-c(z))^i$$

for all $(x,y)\in Z$. Here, $\operatorname{ord}(x)\le\operatorname{ord}(0)$ always holds by convention. If one restricts to the theory $\mathcal{T}_{\mathrm{DP}}\cup$(VII)$_0$, one can take $\ell=0$ and one can choose cells whose angular components take values in Res_1.

Proof. The equicharacteristic 0 result induces the analogous result for big enough residue field characteristic, leaving only finitely many residue characteristics and ramification degrees to treat separately.

We give a proof for $\mathcal{T}=\mathcal{T}_{\mathrm{DP}}\cup(\mathrm{VII})_p$ with $p>0$. For $\mathcal{T}=\mathcal{T}_{\mathrm{DP}}\cup(\mathrm{VII})_0$ one can use the same proof with $e=0$, $\ell_0=0$, and $k=1$.

Let f be of degree d in y and proceed by induction on d. Let $f'(x,y)$ be the derivative of f with respect to y. Applying the induction hypothesis to f', we find a partition of $S\times\mathrm{Val}$ into cells. By replacing S we may suppose that these cells have the identity mapping as presentation.

First consider a 0-cell $Z=Z_{C,c}$ in the partition. Then we can write $f(x,y)=f(x,c(x))=\tau(x)$ for some $\mathcal{L}^*_{\mathrm{DP}}$-term τ, for $(x,y)\in Z$, and the theorem follows.

Next consider a 1-cell $Z=Z_{C,\alpha,\xi,c}$ in the partition. Write $f(x,y)=\sum_i a_i(x)(y-c(x))^i$ for $(x,y)\in Z$. There is some ℓ_0 such that for all $(x,y)\in Z$

$$\operatorname{ord} f'(x,y)\le \min_i \operatorname{ord} t_0^{\ell_0} i a_i(x)(y-c(x))^{i-1}.$$

We may suppose that $a_i(x)$ is either identically zero or else never zero on C for each i. Put $I=\{i:\forall x\in C\ a_i(x)\neq 0\}$ and $J=\{(i,j)\in I\times I: i>j\}$. We may suppose that J is nonempty since the case $J=\varnothing$ is trivial. Put $\Theta:=\{<,>,=\}^J$. For $\theta=(\square_{ij})\in\Theta$, put

$$C_\theta=\{x\in C:\forall (i,j)\in J\ i\alpha(x)+\operatorname{ord} a_i(x)\,\square_{ij}\, j\alpha(x)+\operatorname{ord} a_j(x)\}.$$

Ignoring the C_θ which are empty, this gives a partition of C and hence a partition of Z into cells $Z_\theta=Z_{C_\theta,\alpha|_{C_\theta},\xi|_{C_\theta},c|_{C_\theta}}$. Fix $\theta\in\Theta$. We may suppose that $Z=Z_\theta$. The case that $\operatorname{ord} a_0(x)<i\alpha(x)+\operatorname{ord} a_i(x)$ for all $i\ge 1$ and all $x\in C$ follows trivially. Hence, we may suppose that there exists $i_0>0$ such that $i_0\alpha(x)+\operatorname{ord} a_{i_0}(x)\le i\alpha(x)+\operatorname{ord} a_i(x)$ for all $x\in C$ and all i. Put $e:=\ell_0+\operatorname{ord} i_0$. Let ξ take values in Res_k. By either enlarging e and ℓ_0 or enlarging k, we may suppose that $k=e+1$. Define the subassignments $B_i\subset Z$ by

$$B_1:=\{(x,y)\in Z:$$
$$\forall z\,\big(\operatorname{ord}(z-y)>\alpha(x)+\operatorname{ord} t_0^e\ \to\ \operatorname{ord} f(x,z)\le i_0\alpha(x)+\operatorname{ord} t_0^{2e}a_{i_0}(x)\big)\}$$

and $B_2:=Z\setminus B_1$. If B_i is nonempty, it is equal to the cell $Z_{C_i,\alpha|_{C_i},\xi|_{C_i},c|_{C_i}}$ for some $\mathcal{T}_{\mathrm{DP}}$-definable assignment $C_i\subset C$. Moreover, the B_i can be described without using new Val-quantifiers, by using the maps $\overline{\mathrm{ac}}_m$ for big enough m. On B_1 the theorem holds with $\ell=2e$. On B_2, by Lemma 7.7, there exists a unique definable function

$d: C_2 \to \mathrm{Val}$ such that, for each x in C_2, $(x, d(x))$ lies in B_2 and $f(x, d(x)) = 0$. Again by Lemma 7.7, we may suppose that $d(x)$ is given by an $\mathcal{L}^*_{\mathrm{DP}}$-term. Let D be the $\mathcal{T}_{\mathrm{DP}}$-definable assignment $\{j \in \mathrm{Ord} : j > 0\}$. For

$$C_2' := C_2 \times D \times (\mathrm{Res}_1 \setminus \{0\}),$$
$$\beta : C_2' \to \mathrm{Ord} : (x, j, z) \mapsto \alpha(x) + e + j,$$
$$\eta : C_2' \to \mathrm{Res}_1 \setminus \{0\} : (x, j, z) \mapsto z,$$
$$d' : C_2' \to \mathrm{Val} : (x, j, z) \mapsto d(x),$$

consider the cell $Z_{C_2',\beta,\eta,d'}$ and its projection π to $C_2 \times \mathrm{Val}$. One checks that B_2 is the disjoint union of the 1-cell $\pi(Z_{C_2',\beta,\eta,d'})$, with presentation π^{-1}, and the 0-cell $Z_{C_2,d}$. Moreover, if one writes $f(x, y) = \sum b_i(z)(y - d'(z))^i$ for (z, y) in $Z_{C_2',\beta,\eta,d'}$ and $(x, y) = \pi(z, y)$, one has $\mathrm{ord}\, f(x, y) = \mathrm{ord}\, b_1(z)(y - d'(z))$, which can be seen using a Taylor expansion of f around d'. This finishes the proof. □

Remark 7.9.

(i) By Theorem 7.8, one can probably add arbitrary angular components modulo I_j for a collection of nonzero ideals $I_{j\,j \in J}$ to $\mathcal{L}_{\mathrm{DP}}$, $\mathcal{T}_{S(A)}$, or $\mathcal{T}_{T(A)}$. When one enlarges the language $\mathbf{L}_{\mathrm{Res}}$ to the the full induced language, (which can be richer in the analytic than in the algebraic case), one can probably obtain a form of quantifier elimination and cell decomposition. Most likely, one also gets a similar term structure result (even without introducing new $(\cdot,\cdot,\cdot)_j^{1/m}$ or $h_{m,j}$ for the new ideals). This should follow from Theorem 7.8.

(ii) Similar proofs should hold to show that, if one restricts to the theory $\mathcal{T} \cup (\mathrm{VII})_0$ in Theorem 7.5, one can take for h in Theorem 7.5 a tuple of $\mathcal{L}^\diamond_{\mathcal{T}}$-terms, with $\mathcal{L}^\diamond_{\mathcal{T}}$ the language

$$\mathcal{L}^\diamond := \mathcal{L} \cup_{m>0} \{(\cdot,\cdot,\cdot)_0^{1/m},\ h_{m,0}\}.$$

Proof of Theorems 7.4 and 4.2 for $\mathcal{T}_{\mathrm{DP}}$. First suppose that $X = X_0 = \mathrm{Val}^{m+1}$ and that $f = f_0$ is the map

$$f_0 : \mathrm{Val}^{m+1} \to \mathrm{Res}_n^\ell \times \mathrm{Ord}^\ell : x \mapsto (\overline{\mathrm{ac}}_n(g_i(x, t)), \mathrm{ord}(g_i(x, t)))_i, \tag{7.10}$$

with $g_i(x_1, \ldots, x_m, t)$ polynomials over $\mathbf{Z}$, $m \geq 0$, $\ell, n > 0$, $i = 1, \ldots, \ell$. By Theorem 7.8, the result for $\ell = 1$ follows rather immediately. It is from this partial result for $\ell = 1$ that one deduces the final statement of Lemma 7.7 in the same way as this is proved in [38] and [39]. We will not recall this proof of the final statement of Lemma 7.7.

By induction on ℓ, we may suppose that the result holds for $G_1 := (\overline{\mathrm{ac}}_n\, g_i, \mathrm{ord}\, g_i)_{i=1}^{\ell-1}$ and for $G_2 := (\overline{\mathrm{ac}}_n\, g_\ell, \mathrm{ord}\, g_\ell)$. This gives us two finite partitions $\{Z_{ij}\}$ such that G_i is prepared on Z_{ij} for each j and $i = 1, 2$. Choose $Z_1 := Z_{1j}$ and $Z_2 := Z_{2j'}$. It is enough to partition $Z_1 \cap Z_2$ into cells such that f_0 is prepared on these cells. If Z_1 or Z_2 is a 0-cell, this is easy, so we may suppose that Z_1 is a 1-cell with presentation $\lambda_1 : Z_1 \to Z_{C_1} = Z_{C_1,\alpha_1,\xi_1,c_1}$ and Z_2 a 1-cell with presentation $\lambda_2 : Z_2 \to Z_{C_2} = Z_{C_2,\alpha_2,\xi_2,c_2}$. We may suppose that $\pi(Z_1) = \pi(Z_2)$ with $\pi : X \to S$ the projection, that ξ_1 and ξ_2 take values in Res_k, and that $Z_{C_i} \subset Z \times R$ with R a fixed product of auxiliary sorts for $i = 1, 2$. Under these suppositions it follows from the non archimedean property that $Z_1 \cap Z_2$ is already a cell on which the function f_0 is $\mathcal{T}_{\rm DP}$-prepared, where one can use the presentation

$$\lambda_{12} : Z_1 \cap Z_2 \to \lambda_{12}(Z_1 \cap Z_2) \subset Z \times R \times \{0,1\} :$$
$$(x,t) \mapsto \begin{cases} (\lambda_1(x,t),0) & \text{if } \alpha_1 \geq \alpha_2, \\ (\lambda_2(x,t),1) & \text{else,} \end{cases}$$

and where we write α_i for $\alpha_i(\lambda_i(x,t))$. Here, $\lambda_{12}(Z_1 \cap Z_2)$ has as center $c_1 d_0 + c_2 d_1$, where d_i is the $\mathcal{L}^*_{\rm DP}$-term from Res_1 to Val which is the characteristic function of $\{i\}$ for $i = 0, 1$.

By exploiting the proof of this partial result for general ℓ, one can ensure that all occurring centers are strongly definable and that there are no Val-quantifiers introduced in the process of the cell decomposition. From this partial result for general ℓ one deduces quantifier elimination for $\mathcal{T}_{\rm DP}$ in the language $\mathcal{L}_{\rm DP}$ as in [38] and [39].

Now let $f : X \to R$ be a general $\mathcal{T}$-morphism with R a Cartesian product of auxiliary sorts and X an arbitrary $\mathcal{T}$-assignment. Let $f_1, \ldots, f_t$ be all the polynomials in the Val-variables, say, $x_1, \ldots, x_{m+1}$ occurring in the formulas describing the X and f, where we may suppose that these formulas do not contain quantifiers over the valued field sort. Apply the above case of cell decomposition to the polynomials f_i. This yields a partition of Val^{m+1} into cells Z_i with presentations $\lambda_i : Z_i \to Z_{C_i}$ and centers c_i. Write $x = (x_1, \ldots, x_{m+1})$ for the Val-variables, $\xi = (\xi_j)$ for the Res-variables and $z = (z_j)$ for the Ord-variables on Z_{C_i}. If Z_i is a 1-cell, we may suppose that for (x, ξ, z) in Z_{C_i} we have $\mathrm{ord}(x_{m+1} - c_i) = z_1$ and $\overline{\mathrm{ac}}_n(x_{m+1} - c_i) = \xi_1$, by changing the presentation of Z_i if necessary (that is, by adding more Ord-variables and Res-variables). By changing the presentation as before if necessary, we may also assume that

$$\mathrm{ord} f_j(x) = z_{k_j},$$
$$\overline{\mathrm{ac}}_n f_j(x) = \xi_{l_j},$$

for (x, ξ, z) in the 1-cell Z_{C_i}, where the indices k_j and l_j only depend on j and i.

Since the condition $f(x) = 0$ is equivalent to $\overline{\mathrm{ac}}_n(f(x)) = 0$, we may suppose that, in the formulas describing X and f the only terms involving Val-variables are of the forms $\mathrm{ord}\, f_j(x)$ and $\overline{\mathrm{ac}}_n\, f_j(x)$. Combining this with the above description of $\mathrm{ord} f_j(x)$ and $\overline{\mathrm{ac}}_n f_j(x)$, one sees that the value of f only depends on variables running over the bases of the cells. Hence, f is $\mathcal{T}_{\rm DP}$-prepared on these cells.

□

Proof of Theorem 7.4 for $\mathcal{T}_{S(A)}$ and $\mathcal{T}_{T(A)}$. Let $\mathcal{M}$ be a model of $\mathcal{T}$, a a Val-tuple of $\mathcal{M}$, $\mathcal{M}_a$ the $\mathcal{L}_\mathcal{T}$-substructure of $\mathcal{M}$ generated by a, K_a the valued field of $\mathcal{M}_a$, $\mathcal{L}_\mathcal{T}(K_a)$ the language $\mathcal{L}_\mathcal{T}$ together with constant symbols for the elements of K_a, and $\mathcal{T}(K_a)$ the $\mathcal{L}_\mathcal{T}(K_a)$-theory of all models of $\mathcal{T}$ containing the structure $\mathcal{M}_a$. Let $\mathcal{L}_{\rm DP}(K_a)$ be the language $\mathcal{L}_{\rm DP}$ together with constant symbols for the elements of K_a and $\mathcal{T}_{\rm DP}(K_a)$ the $\mathcal{L}_{\rm DP}(K_a)$-theory of all $\mathcal{T}_{\rm DP}$-models containing the structure $\mathcal{M}_a$.

First we prove some special cases of Theorem 7.4 for the theory $\mathcal{T}(K_a)$. Suppose first that $X = \mathrm{Val}$ and that f is the map

$$f : X \to \mathrm{Res}_n^\ell \times \mathrm{Ord}^\ell : x \mapsto (\overline{\mathrm{ac}}_n(g_i(x)), \mathrm{ord}(g_i(x)))_i,$$

with the g_i $\mathcal{L}_\mathcal{T}(K_a)$-terms in the variable x for $i = 1, \ldots, \ell$. In the case that $\mathcal{T}$ is $\mathcal{T}_{S(A)}$, apply Theorem 5.1 to the terms g_i and to the terms $g_i(x^{-1})$. In the case that $\mathcal{T}$ is $\mathcal{T}_{T(A)}$, there is a $\mathcal{L}_{\mathcal{T}}$-term which presents a valued field element with minimal positive valuation by $(\mathrm{IV})_T$, hence we can apply Corollary 5.2 to the terms g_i and to the terms $g_i(x^{-1})$. In this way we find a finite partition of $X_0 := \{x \in \mathrm{Val} : \mathrm{ord}(x) \geq 0\}$ into $\mathcal{L}_{\mathrm{DP}}(K_a)$-$\mathcal{T}(K_a)$-assignments X_j given by K_a-annulus formulas φ_j, rational functions $h_{ij}(x)$ with coefficients in K_a, a polynomial $F(x)$ over K_a, and very strong units $U_{ij} \in \mathcal{O}_{K_a}(\varphi_j)$, such that for all i, j and all $x \in X_j$

$$F(x) = 0 \ \vee \ g_i(x) = U_{ij}(x)h_{ij}(x),$$

where we mean by an $\mathcal{L}_{\mathrm{DP}}(K_a)$-$\mathcal{T}(K_a)$-assignment a $\mathcal{T}(K_a)$-assignment which can be defined by an $\mathcal{L}_{\mathrm{DP}}(K_a)$-formula. If $I_1 \neq I_2$, the separated analytic structure collapses to a strictly convergent analytic structure, and thus, by Corollary 5.2, we can even assume that $\overline{\mathrm{ac}}_n(U_{ij}) = 1$ on X_j. Up to the transformation $x \mapsto x^{-1}$, we can partition $X_1 := \{x \in \mathrm{Val} : \mathrm{ord}(x) < 0\}$ in a similar way. Now apply Theorem 7.4 for the theory $\mathcal{T}_{\mathrm{DP}}(K_a)$ to the $\mathcal{L}_{\mathrm{DP}}(K_a)$-$\mathcal{T}(K_a)$-assignments X_{jk} and the functions $x \in X_{jk} \mapsto (\overline{\mathrm{ac}}(F(x)), \overline{\mathrm{ac}}_n\, h_{ij}(x), \mathrm{ord}\, h_{ij}(x))_i$ to refine the partition and to finish the proof for X and f of the above form.

Next we suppose that $X = \mathrm{Val}$ and f is an arbitrary $\mathcal{T}(K_a)$-morphism $f : X \to \mathrm{Res}_n^\ell \times \mathrm{Ord}^\ell$. Apply Theorem 4.2 to obtain a formula φ without Val-quantifiers, as in (4.3), which describes the graph of f. Then, let $g_1, \ldots, g_r$ be the $\mathcal{L}_{\mathcal{T}}(K_a)$-Val-terms occurring in φ. Applying the previous case to the terms g_i, the case of this f easily follows, cf. the analogous step in the proof of Theorem 7.4 for $\mathcal{T}_{\mathrm{DP}}$.

Next we suppose that $X = \mathrm{Val}^{m+1}$ and that f is an arbitrary $\mathcal{T}$-morphism $f : X \to \mathrm{Res}_n^\ell \times \mathrm{Ord}^\ell$, $\ell, m > 0$. In this case, the theorem is reduced by a compactness argument to the case $m = 0$, as follows. Suppose that for every candidate $\mathcal{T}$-cell decomposition of X into $\mathcal{T}$-cells A_i, with $\mathcal{L}^*_{\mathcal{T}}$-terms as centers, and $\mathcal{T}$-prepared functions $g_i : A_i \to \mathrm{Res}_n^\ell \times \mathrm{Ord}^\ell$, this data is not the data of a cell decomposition of X which prepares f. This is equivalent to saying that for each such candidate cell decomposition there exists a model with valued field K (with A-analytic structure) and $a \in K^m$ such that either the fibers of the $\mathcal{T}$-cells A_i above a (under the projection $\mathrm{Val}^{m+1} \to \mathrm{Val}^m$) are not a $\mathcal{T}(a)$-cell decomposition of Val, or the fibers of the functions g_i above a do not coincide with f on the fiber of A_i above a. Then, by compactness, there exists a model with valued field K' and $a' \in K'^m$ such that Val can not be partitioned into $\mathcal{T}(K_a)$-cells on which the fibers of the functions g_i above a are prepared, which contradicts the previous case for $X = \mathrm{Val}$. Moreover, this construction ensures that we can work with $\mathcal{L}^*_{\mathcal{T}}$-terms as centers of the cells.

Finally, the general Theorem follows from this case similarly as the general case is obtained in the proof of Theorem 7.4 for $\mathcal{T}_{\mathrm{DP}}$.

□

Proof of Theorem 7.5. By working componentwise it is enough to prove the theorem for $Y = \mathrm{Val}$. Let $\mathrm{Graph}(f) \subset X \times \mathrm{Val}$ be the $\mathcal{T}$-assignment which is the graph of f, and suppose it is described by an $\mathcal{L}_{\mathcal{T}}$-formula φ of the form given by the quantifier elimination Theorem 4.2; let g_j be the $\mathcal{L}_{\mathcal{T}}$-terms occurring in this formula. Apply Theorem 7.4 for $\mathcal{T}$ to the terms g_i. Doing so, all occurring centers of the cells are given by $\mathcal{L}^*_{\mathcal{T}}$-terms. For each occurring cell Z_i, let Z'_i be $\lambda_i^{-1}(\mathrm{Graph}(c_i)) \cap \mathrm{Graph}(f)$, where λ_i is the presentation of Z_i and c_i its center. Clearly each Z'_i is a 0-cell with presentation the restriction of λ_i to Z'_i. It follows from the special form of φ (as given by the application of Theorem 4.2) that the cells Z_i' form a cell decomposition of Graph(f) and one concludes that the restriction of f to each of finitely many pieces in a partition of X satisfies the statement. Now the Theorem follows by gluing the pieces together using extra parameters as in the proof of Lemma 7.7.

□

8. Applications to analytic motivic integration

Let $\mathcal{O}_F$ be the ring of integers of a number field F. Let $\mathcal{A}_F$ be the class of all finite field extensions of all p-adic completions of F, and $\mathcal{B}_F$ the class of all local fields of positive characteristic which are algebras over $\mathcal{O}_F$. For a fixed prime p and integer $n > 0$, let $\mathcal{A}_{F,p,n}$ be the subset of $\mathcal{A}_F$ consisting of all fields with residue field of characteristic p and with degree of ramification fixed by $\mathrm{ord}_p(p) = n$.

For $K \in \mathcal{A}_F \cup \mathcal{B}_F$ write K° for the valuation ring, π_K for a uniformizer, $\widetilde{K}$ for the residue field, and q_K for $\sharp\widetilde{K}$. By $T_m(\mathcal{O}_F[[t]])$ denote the ring of strictly convergent power series in m variables over $\mathcal{O}_F[[t]]$. For each K in $\mathcal{A}_F \cup \mathcal{B}_F$ and each power series $f = \sum_{i\in\mathbf{N}^m} a_i(t)X^i$ in $T_m(\mathcal{O}_F[[t]])$ define the analytic function

$$f_K : (K^\circ)^m \to K^\circ : x \mapsto \sum_{i\in\mathbf{N}^m} a_i(\pi_K)x^i,$$

and extend this by zero to a function $K^m \to K$.

In the terminology of section 2, we have thus fixed the strictly convergent analytic $\mathcal{O}_F[[t]]$-structure on all the fields $K \in \mathcal{A}_F \cup \mathcal{B}_F$.

Let $\mathcal{L}^F$ be the language $\mathcal{L}_{T(\mathcal{O}_F[[t]])}$ with $\mathbf{L}_{\mathrm{Ord}}$ the Presburger language $\mathbf{L}_{\mathrm{Pres}} = (+, -, 0, 1, \leq, \{\equiv \bmod n\}_n)$ and $\mathbf{L}_{\mathrm{Res}}$ the language $\mathbf{L}_{\mathrm{Res},0}$ (cf. section 4). Define the $\mathcal{L}^F$-theory $\mathcal{T}^F$ as $\mathcal{T}_{T(A)}$ together with the axiom $t_0 \neq 1$ (that is, with higher order angular components, see section 4), and axioms describing the congruence relations modulo n in the natural way.

Let W be an $\mathcal{L}^F$-formula with m free valued field variables and no other free variables. (Note that W determines a $\mathcal{T}^F$-assignment in the sense of section 6, but this is not needed here.) For each $K \in \mathcal{A}_F \cup \mathcal{B}_F$, we obtain a set $W_K \subset K^m$ by interpreting the formula W in the natural way. In a similar way, a $\mathcal{T}^F$-morphism f from W to the valued field[10] determines a function $f_K : W_K \to K$.

Suppose now that the set W_K is contained in $(K^\circ)^m$ for each $K \in \mathcal{A}_F \cup \mathcal{B}_F$. Fix $\mathcal{T}^F$-morphisms f_1 and f_2 from W to the valued field, such that the images of the f_{iK} lie in K° for each $K \in \mathcal{A}_F \cup \mathcal{B}_F$.

For each $K \in \mathcal{A}_F \cup \mathcal{B}_F$ and $s \geq 0$ a real number, we consider

$$a_K(s) := \int_{W_K} |f_{1K}(x)|^s |f_{2K}(x)||dx|. \tag{8.1}$$

It is well known by work of Denef and van den Dries [18] that, for each fixed $K \in \mathcal{A}_F$, $a_K(s)$ is a rational function in q_K^{-s}. We prove that also for fixed $K \in \mathcal{B}_F$ with big enough characteristic, $a_K(s)$ is a rational function in q_K^{-s}. This is known in the semialgebraic case by a ultraproduct argument, cf.[9] or [10], but was not

[10] By this we mean an $\mathcal{L}^F$-formula φ such that the set described by φ in any model $\mathcal{M}$ of $\mathcal{T}^F$ is the graph of a function from the $\mathcal{M}$-rational points on W to the valued field of $\mathcal{M}$, cf. section 6.

known before for the analytic case. In fact, we give a geometric meaning to how the a_K vary for $K \in \mathcal{A}_F$, and, when the characteristic is big enough, also for $K \in \mathcal{B}_F$.

Let $\mathrm{Var}_{\mathcal{O}_F}$ denote the collection of isomorphism classes of algebraic varieties over $\mathcal{O}_F$ and let $\mathrm{Form}_{\mathcal{O}_F}$ be the collection of equivalence classes of formulas[11] in the language of rings with coefficients in $\mathcal{O}_F$. Define the rings

$$\mathcal{M}(\mathrm{Var}_{\mathcal{O}_F}) := \mathbf{Q}[T, T^{-1}, \mathrm{Var}_{\mathcal{O}_F}, \frac{1}{\mathbf{A}^1_{\mathcal{O}_F}}, \{\frac{1}{1-\mathbf{A}^b_{\mathcal{O}_F}T^a}\}_{(a,b)\in J}]$$

and

$$\mathcal{M}(\mathrm{Form}_{\mathcal{O}_F}) := \mathbf{Q}[T, T^{-1}, \mathrm{Form}_{\mathcal{O}_F}, \frac{1}{\mathbf{A}^1_{\mathcal{O}_F}}, \{\frac{1}{1-\mathbf{A}^b_{\mathcal{O}_F}T^a}\}_{(a,b)\in J}],$$

with $J = \{(a,b)\in \mathbf{Z}^2 : a \geq 0,\ b < 0\}$, and where we write $\mathbf{A}^\ell_{\mathcal{O}_F}$ for the isomorphism class of the formula $x_1 = x_1 \wedge \ldots \wedge x_\ell = x_\ell$ (which has the set R^ℓ as R-rational points for any ring R over $\mathcal{O}_F$), $\ell \geq 0$.

For each finite field k over $\mathcal{O}_F$ with q_k elements, we write Count_k for the ring morphisms

$$\mathrm{Count}_k : \mathcal{M}(\mathrm{Var}_{\mathcal{O}_F}) \to \mathbf{Q}[q_k^{-s}, q_k^s, \{\frac{1}{1-q_k^{-as+b}}\}_{(a,b)\in J}]$$

and

$$\mathrm{Count}_k : \mathcal{M}(\mathrm{Form}_{\mathcal{O}_F}) \to \mathbf{Q}[q_k^{-s}, q_k^s, \{\frac{1}{1-q_k^{-as+b}}\}_{(a,b)\in J}]$$

which send T to q_k^{-s}, $Y \in \mathrm{Var}_{\mathcal{O}_F}$ to the number of k-rational points on Y and $\varphi \in \mathrm{Form}_{\mathcal{O}_F}$ to the number of k-rational points on φ.

We prove the following generalization of Theorem 1.2:

Theorem 8.2.

(i) *There exists a (non-unique) element $X \in \mathcal{M}(\mathrm{Var}_{\mathcal{O}_F})$ and a number N such that for each $K \in \mathcal{A}_F \cup \mathcal{B}_F$ with $\mathrm{Char}\widetilde{K} > N$ one has*

$$a_K(s) = \mathrm{Count}_{\widetilde{K}}(X).$$

In particular, for $K \in \mathcal{A}_F \cup \mathcal{B}_F$ with $\mathrm{Char}\widetilde{K}$ big enough, $a_K(s)$ only depends on $\widetilde{K}$.

(ii) *For fixed prime p and $n > 0$ there exists a (non-unique) element $X_{p,n} \in \mathcal{M}(\mathrm{Form}_{\mathcal{O}_F})$ such that for each $K \in \mathcal{A}_{F,p,n}$ one has*

$$a_K(s) = \mathrm{Count}_{\widetilde{K}}(X_{p,n}).$$

Proof of Theorem 8.2. The Cell Decomposition Theorem 7.4 together with the Quantifier Elimination Theorem 4.2 translates the calculation of the $a_K(s)$ in a nowadays standard way into calculations of the form

$$\sum_i \mathrm{Count}_{\widetilde{K}}(\varphi_i) \sum_{z\in S_i\subset \mathbf{Z}^m} q_K^{-\alpha_i(z)-s\beta_i(z)}$$

with $\alpha_i,\ \beta_i : S_i \to \mathbf{N}$ Presburger functions on the Presburger sets S_i, and the φ_i $\mathbf{L}_{\mathrm{ring}}$-formulas[12]. One such expression works for $K \in \mathcal{A}_F \cup \mathcal{B}_F$ with residue field characteristic big enough, and one needs another such expression for each class

[11]Two formulas are equivalent in this language if they have the same R-rational points for every ring R over $\mathcal{O}_F$.

[12]Here, we use that, for any $\mathbf{L}_{\mathrm{Res}}$-formula φ, there exists an $\mathbf{L}_{\mathrm{ring}}$-formula ψ, such that for all $K \in \mathcal{A}_F \cup \mathcal{B}_F$ the number of $K^\circ/(\pi_K^m K^\circ)$-rational points on φ is the same as the number of $\widetilde{K}$-rational points on ψ. For this, use the definable bijection $K^\circ/(\pi_K^m K^\circ) \to \widetilde{K}^m$.

$\mathcal{A}_{F,p,n}$. Using Lemma 3.2 of [13] on the summation of such Presburger functions and their exponentials, it follows that there exist a number N_0 and objects X_0, $X_{p,n}$ of $\mathcal{M}(\mathrm{Form}_{\mathcal{O}_F})$ such that

(i) for each $K \in \mathcal{A}_F \cup \mathcal{B}_F$ with $\mathrm{Char}\widetilde{K} > N_0$ one has
$$a_K(s) = \mathrm{Count}_{\widetilde{K}}(X_0),$$

(ii) for each $K \in \mathcal{A}_{F,p,n}$ one has
$$a_K(s) = \mathrm{Count}_{\widetilde{K}}(X_{p,n}).$$

By Theorem 6.4.1 of [17], one can associate virtual motives Y_i to the isomorphism classes of the formulas φ_i occurring in X_0. Since each Y_i belongs to the image of the natural ring morphism from (a certain localisation of) the Grothendieck ring of varieties over F into a ring of virtual motives (cf. Theorem 6.4.1 of [17]), there exist N_1 and $A_i \in \mathcal{M}(\mathrm{Var}_{\mathcal{O}_F})$ such that the number of k-points on Y_i is equal to $\mathrm{Count}_k(A_i)$ for all finite fields k of characteristic $> N_1$. One then easily finds $X \in \mathcal{M}(\mathrm{Var}_{\mathcal{O}_F})$ for which the Theorem holds for $N = \max(N_0, N_1)$. □

Remark 8.3. The extension of the Denef-Pas cell decomposition to the theory $\mathcal{T}_{\mathrm{DP}}$ which allows for $I_2 \neq I_1$, i.e. , which allows for higher order angular components $\overline{\mathrm{ac}}_n$ also in equicharacteristic zero, gives a new view on the foundational work by Denef and Loeser on geometric motivic integration [15] and subsequent work. More precisely, using $\overline{\mathrm{ac}}_n$ to define a broader but analogous class of semialgebraic sets than in [15], stable sets, cylinders, and weakly stable sets would be semialgebraic, which is not the case in [15].

To illustrate this, consider the arc space $\mathcal{L}(\mathbf{A}^1_{\mathbf{Q}})$ of the affine line, $\pi_2 : \mathcal{L}(\mathbf{A}^1_{\mathbf{Q}}) \to \mathcal{L}_1(\mathbf{A}^1_{\mathbf{Q}}) \cong \mathbf{A}^2_{\mathbf{Q}}$ its natural projection on the arcs modulo t^2, and a constructible subset X of $\mathcal{L}_1(\mathbf{A}^1_{\mathbf{Q}})$. Then in general, the cylinder $\pi_2^{-1}(X)$ is not a semialgebraic subset of $\mathcal{L}(\mathbf{A}^1_{\mathbf{Q}})$ in the sense of [15], but with $\overline{\mathrm{ac}}_2$ one can solve this problem, by slightly generalizing the definition of semialgebraic sets in [15].

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