1(a) Number of integers in the set = 322
Number divisible by 17 = 18
Number divisible by 19 = 16
Number divisible by both = 0
Number divisible by neither 17 nor 19 = 322
− 16 − 18 = 288

1(b) Number of integers in the set = 5490
Number divisible by 17 = \( \lfloor \frac{5490}{17} \rfloor = 322 \)
Number divisible by 19 = \( \lfloor \frac{5490}{19} \rfloor = 288 \)
Number divisible by both 17 and 19 = \( \lfloor \frac{5490}{17 \times 19} \rfloor = 16 \)
Number divisible by neither 17 nor 19 = 5490 − 322 − 288 + 16 = 4896

(2) \(100023 = 3 \times 30669 + 8016 \)
30669 = \( 3 \times 8016 + 6621 = 4 \times 8016 − 1395 \)
8016 + 5 \times 1395 + 1041 = 6 \times 1395 − 354
1041 = \( 3 \times 354 + 18 \)
354 = \( 16 + 3 \)
g.c.d = 3

(3) If \( n \) is even there are \( 2^{n/2} \) since we can fill in the first \( n/2 \) digits arbitrarily and then the last \( n/2 \) digits are determined.
If \( n \) is odd there are \( 2^{n−1/2} \) since we can fill in the first \( n−1/2 \) digits arbitrarily, we can put a 0 or a 1 in the middle position, and then the last \( n−1/2 \) digits are determined.

4. It is sufficient to show that (i) \( n^9 − n \equiv 0 \mod 2 \), (ii) \( n^9 − n \equiv 0 \mod 3 \), and (iii) \( n^9 − n \equiv 0 \mod 5 \)
(i) is clear since \( n^9 − n \) is even when \( n \) is even and is even when \( n \) is odd.
(ii) If \( n \equiv 0 \mod 3 \) then \( n^9 − n \equiv 0 \mod 3 \) and if \( n \not\equiv 0 \mod 3 \) then \( n^2 ≡ 1 \mod 3 \) hence \( n^9 − n \equiv (n^2)^4 \cdot n − n \equiv n − n \equiv 0 \mod 3 \).
(iii) is similar to (ii) using that if \( n \not\equiv 0 \mod 5 \) then \( n^4 \equiv 1 \mod 5 \).

5. \((x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n−k} y^k.\)

6. An integer \( n = a_k2^k + a_{k−1}2^{k−1} + \cdots + a_12 + a_0 \) in binary notation is divisible by 3 exactly when
\[ \sum_{i \text{ even}} a_i - \sum_{i \text{ odd}} a_i \]
is divisible by 3. To prove this observe that \( 2 ≡ −1 \mod 3 \) so \( n ≡ \sum_{i=0}^{k} a_i2^i ≡ \sum_{i=0}^{k} a_i(-1)^i \mod 3.\)

7. (a) This is immediate: either there are two people who know each other or all four do not know each other.
(b) Let \( a \) be one of the 6 people. Let \( A \) be the set of people \( a \) knows, and let \( B \) be the set of people \( a \) does not know.
By the pigeon-hole principle either \( |A| ≥ 3 \) or \( |B| ≥ 3 \). If \(|A| ≥ 3 \) then either there are 2 people in \( A \) who know each other in which case they, together with \( a \) form a set of 3 people all of whom know each other, or all the people in \( A \) do not know each other, in which case we have a set of at least 3 people all of whom do not know each other. The case that \( |B| ≥ 3 \) is similar.
(c) Let \( a \) be one of the 10 people. Let \( A \) be the set of people \( a \) knows, and let \( B \) be the set of people \( a \) does not know. So \(|A| + |B| = 9 \).
Case (i) \(|A| ≥ 4 \). If 2 people in \( A \) know each other then they, together with \( a \) give a set of 3 people all of whom know each other. If no 2 people in \( A \) know each other then \( A \) is a set of at least 4 people all of whom do not know each other.
Case (ii) \(|A| ≤ 3 \). Then \(|B| ≥ 6 \), so by (b) either there are 3 people in \( B \) all of whom know each other and we are done, or, also by (b), there are 3 people in \( B \) all of whom do not know each other, and these 3 together with \( a \) give a set of 4 people all of whom do not know each other.