

Comments on MA 58500 Midterm exam Fall 2009

1. Apply the proof of Proposition 1.5 in Mendelson to the truth table corresponding to the formula.

2(a). Let \mathcal{M} be an interpretation and suppose that the formula is false in \mathcal{M} . Then $(\exists x)(\exists y)(A_1^1(x) \Rightarrow A_2^1(y))$ is true in \mathcal{M} , and $(\exists x)(A_1^1(x) \Rightarrow A_2^1(x))$ is false in \mathcal{M} . Hence there are $a, b \in \mathcal{M}$ such that $(A_1^1(a) \Rightarrow A_2^1(b))$ is true in \mathcal{M} , and there is no $c \in \mathcal{M}$ such that $(A_1^1(c) \Rightarrow A_2^1(c))$, is true in \mathcal{M} . Hence either $(A_1^1(a)$ is false in \mathcal{M} or $A_2^1(b)$ is true in \mathcal{M} . In the first case $c = a$ makes $(A_1^1(c) \Rightarrow A_2^1(c))$ true, and in the second case $c = b$ makes $(A_1^1(c) \Rightarrow A_2^1(c))$ true. This is a contradiction. Hence the formula is true in every interpretation, i.e. the formula is valid.

2(b) is similar.

3. The theory $T_1 \cup T_2$ is inconsistent because there is no interpretation that satisfies all the axioms. Hence some finite subset of $T_1 \cup T_2$ is inconsistent. Hence there are finite subsets T'_1 and T'_2 of T_1 and T_2 respectively such that $T'_1 \cup T'_2$ is inconsistent. Since $T'_1 \subset T_1$, if $T'_1 \vdash \mathcal{C}$, then $T_1 \vdash \mathcal{C}$. Suppose the converse is not true, i.e. there is a formula \mathcal{C} such that $T_1 \vdash \mathcal{C}$ but $T'_1 \not\vdash \mathcal{C}$. Then $T'_1 \cup \{\neg \mathcal{C}\}$ is consistent. Hence it has a model \mathcal{M} . Clearly, \mathcal{M} is not a model of T_1 since $T_1 \vdash \mathcal{C}$ but $\mathcal{M} \models \neg \mathcal{C}$. Hence \mathcal{M} is a model of T_2 and hence also of T'_2 . But then \mathcal{M} is a model of $T'_1 \cup T'_2$ which is inconsistent. This contradiction proves that T'_1 is equivalent to T_1 . The proof that T'_2 is equivalent to T_2 is the same.

4. is in the book.

5. Let $T = GT \cup \{\mathcal{B}\} \cup \{c_i \neq c_j : i \neq j, i, j \in \mathbf{N}\}$, where GT is the axioms of group theory. Let T' be a finite subset of T , say $T' \subset GT \cup \{\mathcal{B}\} \cup \{c_i \neq c_j : i \neq j, i, j < k\}$. Any finite group with at least k elements is a model of T' . Hence T' is consistent, and hence so is T . Let \mathcal{G} be a model of T . Then T is an infinite group in which \mathcal{B} is true.