Comments on MA 585 Midterm exam, Fall 2007

1. was easy

2(a) Induction on the cardinality of K_1. If $|K_1| = 0$, then K_1 is independent. Suppose proved whenever $|K_1| \leq n$, and let K_1 with $|K_1| = n+1$ be given. If K_1 is independent take $K_2 = K_1$. If K_1 is not independent there is a $\mathcal{B} \in K_1$ such that $K_1 \setminus \{\mathcal{B}\} \vdash \mathcal{B}$. Let $K' = K_1 \setminus \{\mathcal{B}\}$. Then K' is equivalent to K_1 and $|K'| = n$. Hence, by the induction hypothesis, there is a $K_2 \subset K'$ with K_2 independent and K_2 equivalent to K' and hence to K_1.

2(b) Let $A_1, A_2, A_3, \ldots$ be proposition letters. Take

$$K = \{A_1,\ A_1 \wedge A_2,\ A_1 \wedge A_2 \wedge A_3,\ \cdots\}.$$

If $\mathcal{B}, \mathcal{C} \in K$ then either $\mathcal{B} \vdash \mathcal{C}$ or $\mathcal{C} \vdash \mathcal{B}$. Hence the only independent subsets of K are the singletons. Obviously $A_1 \wedge A_2 \wedge \cdots \wedge A_n \nvdash A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge A_{n+1}$.

2(c) Enumerate the axioms of K_1, say $K_1 = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \cdots\}$. Define an increasing sequence of theories K'_i as follows: K'_0 is empty. Suppose K'_n defined. Say $K'_n = \{\mathcal{C}_1, \cdots, \mathcal{C}_k\}$. If $K'_n \vdash \mathcal{B}_{n+1}$ then $K'_{n+1} = K'_n$. If $K'_n \nvdash \mathcal{B}_{n+1}$ then $K'_{n+1} = K'_n \cup \{\neg\mathcal{C}_1 \vee \neg\mathcal{C}_2 \vee \cdots \vee \neg\mathcal{C}_k \vee \mathcal{B}_{n+1}\}$. Then $K'_{n+1} \vdash \mathcal{B}_{n+1}$. Show that each K'_n is independent (exercise). Since the union is increasing $K_2 = \bigcup_n K'_n$ is independent and K_2 is equivalent to K_1. Show this (exercise).

3 was fairly well done, and is in the book.

4 Let $a_1, a_2, a_3, \cdots$ be infinitely many new constants. Consider

$$K = WO \cup \{a_2 < a_1, a_3 < a_2, a_4 < a_3, \cdots\}.$$

K is consistent since every finite subset has a model. By the Completeness Theorem, K has a model, M. In M the set $\{a_1, a_2, a_3, \cdots\}$ has no least element, so M is *not* well-ordered.

5 The theory DAG of divisible abelian groups has only infinite models as any model is a vector space over $\mathbf{Q}$. Let $\aleph$ be an uncountable cardinal. By the completeness theorem DAG has models of cardinality $\aleph$. A vector space over $\mathbf{Q}$ of cardinality $\aleph$ has dimension $\aleph$. Two vector spaces over $\mathbf{Q}$ of the same dimension are isomorphic. Hence all models of DAG of cardinality $\aleph$ are isomorphic. Hence, as in class, DAG is complete.