MA 585 Midterm Exam, October 11, 2007

1. Adams, Brown and Clark are suspects for a murder.

   Adam says: "I didn't do it. The victim was an acquaintance of Brown's. But Clark hated him."

   Brown states: "I didn't do it. I didn't even know the guy. Besides, I was out of town all that week."

   Clark says: "I didn't do it. I saw both Adams and Brown downtown with the victim that day. One of them must have done it.

   Assume that the two innocent men are telling the truth, but the guilty man might not be. Who did it?

2. If K_1 and K_2 are two theories we say that K_1 and K_2 are *equivalent* if for every closed formula $\mathcal{C}$ we have $K_1 \vdash \mathcal{C}$ if and only if $K_2 \vdash \mathcal{C}$. We say that the axioms of K_2 are *independent* if for every $\mathcal{B} \in K_2$, $K_2 \backslash \{\mathcal{B}\} \vdash \mathcal{B}$. Show

   (a) If K_1 is finite (i.e. there are only finitely many nonlogical axioms in K_1) then there is a subset K_2 of K_1 that is independent and equivalent to K_1.

   (b) Give an example (in the propositional calculus) of a theory that has no independent equivalent subset.

   (c) Show that for any theory K_1 in a countable language there is an equivalent, independent theory K_2.

3. (a) State the Completeness Theorem

   (b) Carefully state (but do not prove) the two Lemmas used to prove the completeness theorem.

   (c) Briefly explain how the two lemmas are used to construct the desired model. (Do not prove that it is a model, just define it.)

4. An ordered set is called *well–ordered* if every non empty subset has a least element. Let WO be the theory of well ordered sets — i.e. the set of all closed formulas in the language $\{=, \leq\}$ that are true in all well–ordered sets. Show that WO has a model that is not well ordered.

5. (bonus question). The theory of divisible abelian groups is the theory of abelian groups together with the axioms, one for each $n \in \mathbb{N}$,

$$\forall x \; \exists y (\underbrace{y + \ldots + y}_{n \text{ times}} = x).$$

   Show that the theory of divisible abelian groups is complete.

   You may use the following fact: G is a divisible abelian group if and only if G is a vector space over the rationals $\mathbb{Q}$.