REAL CLOSED FIELDS WITH NONSTANDARD ANALYTIC STRUCTURE

R. CLUCKERS ¹, L. LIPSHITZ ², AND Z. ROBINSON ²

Abstract. We consider the ordered field which is the completion of the Puiseux series field over \( \mathbb{R} \) equipped with a ring of analytic functions on \([-1, 1]^n\) which contains the standard subanalytic functions as well as functions given by \( t \)-adically convergent power series, thus combining the analytic structures from [DD] and [LR3]. We prove quantifier elimination and \( o \)-minimality in the corresponding language. We extend these constructions and results to rank \( n \) ordered fields \( \mathbb{R}_n \) (the maximal completions of iterated Puiseux series fields). We generalize the example of Hrushovski and Peterzil [HP] of a sentence which is not true in any \( o \)-minimal expansion of \( \mathbb{R} \) (shown in [LR3] to be true in an \( o \)-minimal expansion of the Puiseux series field) to a tower of examples of sentences \( \sigma_n \), true in \( \mathbb{R}_n \), but not true in any \( o \)-minimal expansion of any of the fields \( \mathbb{R}, \mathbb{R}_1, \ldots, \mathbb{R}_{n-1} \).

1. Introduction

In [LR3] it is shown that the ordered field \( K_1 \) of Puiseux series in the variable \( t \) over \( \mathbb{R} \), equipped with a class of \( t \)-adically overconvergent functions such as \( \sum_n (n+1)! (tx)^n \) has quantifier elimination and is \( o \)-minimal in the language of ordered fields enriched with function symbols for these functions on \([-1, 1]^n\). This was motivated (indirectly) by the observation of Hrushovski and Peterzil, [HP], that there are sentences true in this structure that are not satisfiable in any \( o \)-minimal expansion of \( \mathbb{R} \). This in turn was motivated by a question of van den Dries. See [HP] for details.

In [DMM1] it was observed that if \( K \) is a maximally complete, non–archimedean real closed field with divisible value group, and if \( f \) an element of \( \mathbb{R}[[\xi]] \) with radius of convergence \( > 1 \), then \( f \) extends naturally to an “analytic” function \( I^n \to K \), where \( I = \{ x \in K : -1 \leq x \leq 1 \} \). Hence if \( \mathcal{A} \) is the ring of real power series with radius of convergence \( > 1 \) then \( K \) has \( \mathcal{A} \)-analytic structure i.e. this extension preserves all the algebraic properties of the ring \( \mathcal{A} \). In particular the real quantifier

2000 Mathematics Subject Classification. Primary 03C64, 32P05, 32B05, 32B20; Secondary 03C10, 03C98, 03C60, 14P15.

Key words and phrases. \( o \)-minimality, subanalytic functions, quantifier elimination, Puiseux series.

¹ The author has been supported as a postdoctoral fellow by the Fund for Scientific Research - Flanders (Belgium) (F.W.O.) during the preparation of this paper.

² The authors have been supported in part by NSF grant DMS-0401175. They also thank the University of Leuven and the Newton Institute (Cambridge) for their support and hospitality.
elimination of [DD] works in this context so $K$ has quantifier elimination, is $o$–minimal in the analytic field language, and is even elementarily equivalent to $\mathbb{R}$ with the subanalytic structure. See [DMM2] and [DMM3] for extensions.

In Sections 2 and 3 below we extend the results of [LR3] by proving quantifier elimination and $o$–minimality for $K_1$ in a larger language that contains the overconvergent functions together with the usual analytic functions on $[-1, 1]^n$. In Section 4 we extend these results to a larger class of non-archimedean real-closed fields, including fields $\mathbb{R}_m$, of rank $m = 1, 2, 3, \ldots$, and in Section 5 we show that the idea of the example of [HP] can be iterated so that for each $m$ there is a sentence true in $\mathbb{R}_m$ but not satisfiable in any $o$–minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \cdots, \mathbb{R}_{m−1}$. In a subsequent paper we will give a more comprehensive treatment of both Henselian fields with analytic structure and real closed fields with analytic structure, see [CLR2].

2. Notation and Quantifier Elimination

In this section we establish notation and prove quantifier elimination (Theorem 2.16) for the field of Puiseux series over $\mathbb{R}$ in a language which contains function symbols for all the standard analytic functions on $[-1, 1]^n$ and all the $t$–adically overconvergent functions on this set.

Definition 2.1.

$$K_1 := \bigcup_n \mathbb{R}(\{t^{1/n}\}), \text{ the field of Puiseux series over } \mathbb{R},$$

$$K := \hat{K}_1, \text{ the } t\text{-adic completion of } K_1.$$ $K$ is a real closed, nonarchimedean normed field. We shall use $\| \cdot \|$ to denote the (nonarchimedean) $t$–adic norm on $K$, and $<$ to denote the order on $K$ that comes from the real closedness of $K$. We will use $| \cdot |$ to denote the corresponding absolute value, $|x| = \sqrt{x^2}$.

$$K^\circ := \{x \in K : \|x\| \leq 1\}, \text{ the finite elements of } K$$

$$K^{\circ\circ} := \{x \in K : \|x\| < 1\}, \text{ the infinitesimal elements of } K$$

$$K_{alg} := K[\sqrt{-1}], \text{ the algebraic closure of } K$$

$$\xi = (\xi_1, \cdots, \xi_n)$$

$$A_{n, \alpha} := \{f \in \mathbb{R}[[\xi]] : \text{ radius of convergence of } f > \alpha\}, 0 < \alpha \in \mathbb{R}$$

$$\mathcal{R}_{n, \alpha} := A_{n, \alpha} \otimes_{\mathbb{R}} K = (A_{n, \alpha} \otimes_{\mathbb{R}} K)^\wedge, \text{ where } ^\wedge \text{ stands for the t-adic completion}$$

$$\mathcal{R}_{n, \alpha}^\circ := A_{n, \alpha} \otimes_{\mathbb{R}} K^\circ = (A_{n, \alpha} \otimes_{\mathbb{R}} K^\circ)^\wedge$$

$$\mathcal{R}_n := \bigcup_{\alpha > 1} \mathcal{R}_{n, \alpha}$$

$$\mathcal{R} := \bigcup_n \bigcup_{\alpha > 1} \mathcal{R}_{n, \alpha}$$

$$I := [-1, 1] = \{x \in K : |x| \leq 1\}.$$

Remark 2.2. (i) $K$ has $\mathcal{R}$–analytic structure – in [DMM1] it is explained how the functions of $A_{n, \alpha}$ are defined on $I^n$. For example, if $f \in A_{1, \alpha}$, $\alpha > 1$, $a \in \mathbb{R}[−1, 1]$, $\beta \in K^{\circ\circ}$, then $f(a + \beta) := \sum_n f^{(n)}(a)\frac{\beta^n}{n!}$. The extension to functions in $\mathcal{R}_{n, \alpha}$ is clear from the completeness of $K$. We define these functions to be zero outside
This extension also naturally works for maximally complete fields and fields of LE-series, see also [DMM2] and [DMM3].

(ii) \( R \) contains all the “standard” real analytic functions on \([-1,1]^n\) and all the \( t \)-adically overconvergent functions in the sense of [LR3].

(iii) The elements of \( A_{n,\alpha} \) in fact define complex analytic functions on the complex polydisc \( \{ x \in \mathbb{C} : |x_i| \leq \alpha \text{ for } i = 1, \ldots, n \} \), and hence the elements of \( R_{n,\alpha} \) define “\( K_{\text{alg}} \)-analytic” functions on the corresponding \( K_{\text{alg}} \)-polydisc \( \{ x \in K_{\text{alg}} : |x_i| \leq \alpha \text{ for } i = 1, \ldots, n \} \).

(iv) We could as well work with \( K_1 \) instead of \( K \). Then we must replace \( R_{n,\alpha} \) by
\[
\mathcal{R}'_{n,\alpha} := \bigcup_m A_{n,\alpha} \hat{\otimes}_R R((t^{1/m})) = \bigcup_m (A_{n,\alpha} \hat{\otimes}_R R((t^{1/m})))^\top).
\]

Lemma 2.3. Every nonzero \( f \in R_{n,\alpha} \) has a unique representation
\[
f = \sum_{i \in \mathbb{N}} f_i t^{\gamma_i}
\]
where the \( f_i \in A_{n,\alpha} \), \( f_i \neq 0 \), \( \gamma_i \in \mathbb{Q} \), the \( \gamma_i \) are increasing and, either the sum is finite, or \( \gamma_i \to \infty \) as \( i \to \infty \). The function \( f \) is a unit in \( R_{n,\alpha} \) exactly when \( f_0 \) is a unit in \( A_{n,\alpha} \).

Proof. Observe that if \( a \in K, a \neq 0 \) then \( a \) has a unique representation \( a = \sum_{i \in \mathbb{N}} a_i t^{\gamma_i} \), where the \( 0 \neq a_i \in \mathbb{R}, \gamma_i \in \mathbb{Q} \) and, either the sum is finite, or \( \gamma_i \to \infty \). If \( f_0 \) is a unit in \( A_{n,\alpha} \), then \( f \cdot f_0^{-1} \cdot t^{-\gamma_0} = 1 + \sum_{i=1}^{\infty} f_i \cdot f_0^{-1} \cdot t^{\gamma_i-\gamma_0} \) and \( \gamma_i-\gamma_0 > 0 \).

Definition 2.4. (i) In the notation of the previous definition, \( f_0 \) is called the top slice of \( f \). (ii) We call \( f \) regular in \( \xi_n \) of degree \( s \) at \( a \in [I \cap \mathbb{R}]^n \) if, in the classical sense, \( f_0 \) is regular in \( \xi_n \) of degree \( s \) at \( a \). (iii) We shall abuse notation and use \( \| \cdot \| \) to denote the \( t \)-adic norm on \( K \), and the corresponding gauss–norm on \( R_{n,\alpha} \), so, with \( f \) as in the above Lemma, \( \| f \| = \| t^{\gamma_0} \| \).

The standard Weierstrass Preparation and Division Theorems for \( A_{n,\alpha} \) extend to corresponding theorems for \( R_{n,\alpha} \).

Theorem 2.5. (Weierstrass Preparation and Division). If \( f \in R_{n,\alpha} \) with \( \| f \| = 1 \) is regular in \( \xi_n \) of degree \( s \) at \( 0 \), then there is a \( \delta \in \mathbb{R}, \delta > 0 \), such that there are unique \( A_1, \ldots, A_s, U \) satisfying
\[
f = [\xi_n^s + A_1(\xi')\xi_n^{s-1} + \ldots + A_s(\xi')]U(\xi)
\]
and
\[
A_1, \ldots, A_s, \in R_{n-1,\delta}, \text{ and } U \in R_{n,\delta} \text{ a unit.}
\]

Then automatically
\[
\| A_1 \|, \ldots, \| A_s \|, \| U \| \leq 1, \quad \| A_1(0) \|, \ldots, \| A_s(0) \| < 1, \quad \text{and } \| U(0) \| = 1.
\]

Furthermore, if \( g \in R_{n,\alpha} \) then there are unique \( Q \in R_{n,\delta} \) and \( R_0(\xi'), \ldots, R_{s-1}(\xi') \in R_{n-1,\delta} \), satisfying
\[
\| Q \|, \| R_s \| \leq \| g \|
\]
and
\[
g = Qf + R_0(\xi') + R_1(\xi')\xi_n + \ldots + R_{s-1}(\xi')\xi_n^{s-1}.
\]
Proof. We may assume that \( f = \sum_{\gamma \in I} f_\gamma t^\gamma \), where \( I \subset \mathbb{Q}^+, 0 \in I, I = I + I \) and \( I \) is well ordered. (We do not require that \( f_\alpha \neq 0 \) for all \( \alpha \in I \), but we do require \( f_0 \neq 0 \).) We prove the Preparation Theorem. The proof of the Division Theorem is similar. We shall produce, inductively on \( \gamma \in I \), monic polynomials \( P_\gamma [\xi_n] \) with coefficients from \( \mathcal{R}_{n-1, \delta} \), and units \( U_\gamma \in \mathcal{R}_{n, \delta} \) such that, writing \( \gamma' \) for the successor of \( \gamma \) in \( I \), we have
\[
f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'}
\]
and if \( \gamma < \beta \)
\[
P_\gamma \equiv P_\beta \text{ and } U_\gamma \equiv U_\beta \mod t^{\gamma'}.
\]
Using [GR], Theorem II.D.1 (p.80), or the proof on pp. 142-144 of [ZS], we see that there is a \( 0 < \delta \leq \alpha \) such that for every \( g \in A_{n, \delta} \) the Weierstrass data on dividing \( g \) by \( f_0 \) are in \( A_{n, \delta} \).

\( P_0 \) and \( U_0 \) are the classical Weierstrass data for \( f_0 \), i.e. \( f_0 = U_0 P_0 \), where \( P_0 \in A_{n-1, \delta} [\xi_n] \) is monic of degree \( s \), and \( U \in A_{n, \delta} \) is a unit. Suppose \( P_\gamma \) and \( U_\gamma \) have been found. Then
\[
f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'}
\]
so we have
\[
U_\gamma^{-1} \cdot f \equiv P_\gamma + g_\gamma t^{\gamma'} + o(t^{\gamma'}),
\]
where \( g_\gamma \in A_{n, \delta} \) and we write \( o(t^{\gamma'}) \) to denote terms of order \( > \gamma' \). By classical Weierstrass division we can write
\[
g_\gamma' = P_0 \cdot Q_\gamma' + R_\gamma',
\]
where \( Q_\gamma' \in A_{n, \delta} \) and \( R_\gamma' \in A_{n-1, \delta} [\xi_n] \) has degree \( < s \) in \( \xi_n \). Let
\[
P_\gamma' := P_\gamma + t^{\gamma'} R_\gamma'.
\]
Then
\[
U_\gamma^{-1} \cdot f = P_\gamma + t^{\gamma'} (P_0 Q_\gamma' + R_\gamma') + o(t^{\gamma'})
\]
\[
= (P_\gamma' + t^{\gamma'} P_\gamma Q_\gamma') + t^{\gamma'} (P_0 - P_\gamma' Q_\gamma') + o(t^{\gamma'})
\]
\[
= P_\gamma' (1 + t^{\gamma'} Q_\gamma') + o(t^{\gamma'}),
\]
since \( P_0 - P_\gamma = o(1) \), i.e. it has positive order. Take \( U_\gamma' := U_\gamma (1 + t^{\gamma'} Q_\gamma') \). The uniqueness of the \( A_1 \) and \( U \) follows from the same induction. \( \square \)

Remark 2.6. We remark, for use in a subsequent paper, that the argument of the previous proof works in the more general context that \( I \) is a well ordered subset of the value group \( \Gamma \) of a suitably complete field, for example a maximally complete field.

From the above proof or by direct calculation we have

Corollary 2.7. If \( g \in \mathcal{R}_1, \beta \in [-1, 1] \) and \( g(\beta) = 0 \) then \( \xi_1 - \beta \) divides \( g \) in \( \mathcal{R}_1 \).

Remark 2.8. Let \( f(\xi, \eta) \) be in \( \mathcal{R}_{m+n, \alpha} \). Then there are unique \( \overline{f}_\mu \) in \( \mathcal{R}_{m, \alpha} \) such that
\[
f(\xi, \eta) = \sum_{\mu} \overline{f}_\mu (\xi) \eta^\mu.
\]

The following Lemma is used to prove Theorem 2.10.
Lemma 2.9. Let \( f(\xi, \eta) = \sum_{\mu} \bar{f}_\mu(\xi)\eta^\mu \in R_{m+n,\alpha} \). Then the \( \bar{f}_\mu \in R_{m,\alpha} \) and there is an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \) and \( g_\mu \in R_{m+n,\beta}^o \) for \( |\mu| < d \), such that

\[
  f = \sum_{|\mu| < d} \bar{f}_\mu(\xi)g_\mu(\xi, \eta).
\]

Proof. We may assume that \( \|f\| = 1 \), and choose a \( \nu_0 \) such that \( \|\bar{f}_{\nu_0}\| = 1 \). Making an \( \mathbb{R} \)-linear change of variables, and shrinking \( \alpha \) if necessary, we may assume that \( \bar{f}_{\nu_0} \) is regular at 0 in \( \xi_m \) of degree \( s \), say. Write \( \xi' \) for \( (\xi_1, \ldots, \xi_{m-1}) \). By Weierstrass Division (Theorem 2.5) there is a \( \beta > 0 \) and there are \( Q(\xi, \eta) \in R_{m+n,\beta} \) and \( R(\xi, \eta) = R_0(\xi', \eta) + \cdots + R_{s-1}(\xi', \eta)\xi_m^{s-1} \in R_{m+n-1,\beta}[\xi_m] \) such that

\[
  f(\xi, \eta) = \bar{f}_{\nu_0}(\xi)Q(\xi, \eta) + R(\xi, \eta).
\]

By induction on \( m \), we may write

\[
  R_0 = \sum_{|\mu| < d} \bar{R}_{\nu_0}(\xi')g_\mu(\xi', \eta),
\]

for some \( d \in \mathbb{N} \), some \( \beta > 0 \) and \( g_\mu(\xi', \eta) \in R_{m+n-1,\beta}^o \). Writing \( R = \sum_{\nu} \bar{R}_\nu(\xi)\eta^\nu \), observe that each \( \bar{R}_\nu \) is an \( R_{m,\beta}^o \)-linear combination of the \( \bar{f}_\nu \), since, taking the coefficient of \( \eta^\nu \) on both sides of the equation \( f(\xi, \eta) = \bar{f}_{\nu_0}(\xi)Q(\xi, \eta) + R(\xi, \eta) \), we have

\[
  \bar{f}_\nu = \bar{f}_{\nu_0}Q + \bar{R}_\nu.
\]

Consider

\[
  f - \bar{f}_{\nu_0}Q - \sum_{|\mu| < d} \bar{R}_\mu(\xi)g_\mu(\xi', \eta) = S_1\xi_m + S_2\xi_m^2 + \cdots + S_{s-1}\xi_m^{s-1}
\]

\[
  = \xi_m[S_1 + S_2\xi_m + \cdots + S_{s-1}\xi_m^{s-2}]
\]

\[
  = \xi_m \cdot S, \text{ say,}
\]

where the \( S_i \in R_{m+n-1,\beta}^o \). Again, observe that each \( \bar{S}_\nu \) is an \( R_{m,\beta}^o \)-linear combination of the \( \bar{f}_\nu \). Complete the proof by induction on \( s \), working with \( S \) instead of \( R \). \( \square \)

Theorem 2.10. (Strong Noetherian Property). Let \( f(\xi, \eta) = \sum_{\mu} \bar{f}_\mu(\xi)\eta^\mu \in R_{m+n,\alpha} \). Then the \( \bar{f}_\mu \in R_{m,\alpha} \) and there is an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \) and units \( U_\mu(\xi, \eta) \in R_{m+n,\beta}^o \) for \( |\mu| < d \), such that

\[
  f = \sum_{\mu \in J} \bar{f}_\mu(\xi)\eta^\mu U_\mu(\xi, \eta),
\]

where \( J \) is a subset of \( \{0, 1, \ldots, d\}^n \).

Proof. It is sufficient to show that there are an integer \( d \), a set \( J \subset \{0, 1, \ldots, d\}^n \), and \( g_\mu \in R_{m+n,\beta}^o \) such that

\[
  f = \sum_{\mu \in J} \bar{f}_\mu(\xi)\eta^\mu g_\mu(\xi, \eta),
\]

(2.11)
since then, rearranging the sum if necessary, we may assume that each \( g_\mu \) is of the form \( 1 + h_\mu \) where \( h_\mu \in (\eta)R_{m+n,\beta}^\circ \). Shrinking \( \beta \) if necessary will guarantee that the \( g_\mu \) are units. But then it is in fact sufficient to prove (2.11) for \( f \) replaced by

\[
f_{I_i} := \sum_{\mu \in I_i} f_\mu(\xi)\eta^\mu
\]

for each \( I_i \) in a finite partition \( \{I_i\} \) of \( \mathbb{N}^n \).

By Lemma 2.9 there is an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \) and \( g_\mu \in R_{m+n,\beta}^\circ \) for \( |\mu| \leq d \), such that

\[
f = \sum_{|\mu| \leq d} f_\mu(\xi)g_\mu(\xi,\eta).
\]

Rearranging, we may assume for \( \nu, \mu \in \{1, \ldots, d\}^n \) that \( (g_\mu)_\nu \) equals 1 if \( \mu = \nu \) and that it equals 0 otherwise.

Focus on \( f_{I_1}(\xi, \eta) \), defined as above by

\[
f_{I_1}(\xi, \eta) = \sum_{\mu \in I_1} f_\mu(\xi)\eta^\mu
\]

with

\[
I_1 := \{0, \ldots, d\}^n \cup \{\mu: \mu_i \geq d \text{ for all } i\}
\]

and note that

\[
(2.12) f_{I_1}(\xi, \eta) = \sum_{|\mu| \leq d} f_\mu(\xi)g_{\mu,I_1}(\xi, \eta)
\]

with \( g_{\mu,I_1}(\xi, \eta) \in R_{m+n,\beta}^\circ \) defined by the corresponding sum

\[
g_{\mu,I_1}(\xi, \eta) = \sum_{\nu \in I_1} g_{\mu,\nu}(\xi)\eta^\nu.
\]

It is now clear that \( g_{\mu,I_1} \) is of the form \( \eta^\mu(1 + h_\mu) \) where \( h_\mu \in (\eta)R_{m+n,\beta}^\circ \).

One now proceeds by noting that \( f - f_{I_1} \) is a finite sum of terms of the form \( f_{I_j} \) for \( j > 1 \) and \( \{I_j\} \) a finite partition of \( \mathbb{N}^n \) and where each \( f_{I_j} \) for \( j > 1 \) is of the form \( \eta^j q(\xi, \eta') \) where \( \eta' = (\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_n) \) and \( q \) is in \( R_{m+n-1,\beta}^\circ \). These terms can be handled by induction on \( n \).

\[\square\]

**Definition 2.13.** For \( \gamma \in K_{alg}^\circ \), let \( \gamma^\circ \) denote the closest element of \( \mathbb{C} \), i.e. the unique element \( \gamma - \gamma^\circ \in K_{alg}^\circ \).

**Lemma 2.14.** Let \( f \in R_1 \). If \( f(\gamma) = 0 \) then \( f_0(\gamma^\circ) = 0 \) (\( f_0 \) is the top slice of \( f \)). Conversely, if \( \beta \in \mathbb{R} \) (or \( \mathbb{C} \)) and \( f_0(\beta) = 0 \) there is a \( \gamma \in K_{alg} \) with \( \gamma^\circ = \beta \) and \( f(\gamma) = 0 \). Indeed, \( f_0 \) has a zero of order \( n \) at \( \beta \in \mathbb{C} \) if, and only if, \( f \) has \( n \) zeros \( \gamma \) (counting multiplicity) with \( \gamma^\circ = \beta \).

**Proof.** Use Weierstrass Preparation and [BGR] Proposition 3.4.1.1. \[\square\]

**Corollary 2.15.** A nonzero \( f \in R_{1,\alpha} \) has only finitely many zeros in the set \( \{x \in K_{alg}: |x| \leq \alpha\} \). Indeed, there is a polynomial \( P(x) \in K[x] \) and a unit \( U(x) \in R_{1,\alpha} \) such that \( f(x) = P(x) \cdot U(x) \).
Proof. Observe that $f_0$ has only finitely many zeros in $\{ x \in \mathbb{C} : |x| \leq \alpha \}$, that non-real zeros occur in complex conjugate pairs, and that $f$ is a unit exactly when $f_0$ has no zeros in this set, i.e. when $f_0$ is a unit in $A_{1,\alpha}$, and use Lemma 2.14. □

Theorem 2.16. (Quantifier Elimination Theorem). Denote by $L$ the language $\langle +, \cdot, -1, 0, 1, <, R \rangle$ where the functions in $R_n$ are interpreted to be zero outside $I^n$. Then $K$ admits quantifier elimination in $L$.

Proof. This is a small modification of the real quantifier elimination of [DD] as in [DMM1], using the Weierstass Preparation Theorem and the Strong Noetherian Property above. Crucial is that, in Theorems 2.5 and 2.10, as in [DD], $\delta$ is a positive real number so one can use the compactness of $[-1, 1]^n$ in $\mathbb{R}^n$. □

3. $O$–MINIMALITY

In this section we prove the o–minimality of $K$ in the language $L$. Let $\alpha > 1$. As we remarked above, each $f \in A_{n,\alpha}$ defines a function from the poly-disc $(I_{c,\alpha})^n \rightarrow \mathbb{C}$, where $I_{c,\alpha} := \{ x \in \mathbb{C} : |x| \leq \alpha \}$, and hence each $f \in R_{n,\alpha}$ defines a function from $(I_{K_{alg},\alpha})^n \rightarrow K_{alg}$, where $I_{K_{alg},\alpha} := \{ x \in K_{alg} : |x| \leq \alpha \}$. In general $A_{n,\alpha}$ is not closed under composition. However, if $F(\eta_1, \ldots, \eta_m) \in R_{n,\alpha}$, $G_j(\xi) \in R_{n,\beta}$ for $j = 1, \ldots, m$ and $|G_j(x)| \leq \alpha$ for all $x \in (I_{K_{alg},\beta})^n$, then $F(G_1(\xi), \ldots, G_m(\xi)) \in R_{n,\beta}$.

This is clear if $F \in A_{n,\alpha}$ and the $G_j \in A_{n,\beta}$. The general case follows easily.

For $c, r \in K$, $r > 0$, we denote the “closed interval” with center $c$ and radius $r$ by

$$I(c, r) := \{ x \in K : |x - c| \leq r \}$$

and for $c, \delta, \varepsilon \in K$, $0 < \delta < \varepsilon$, we denote the “closed annulus” with center $c$, inner radius $\delta$ and outer radius $\varepsilon$ by

$$A(c, \delta, \varepsilon) := \{ x \in K : \delta \leq |x - c| \leq \varepsilon \}.$$

On occasion we will consider $I(c, r)$ as a disc in $K_{alg}$ and $A(c, \delta, \varepsilon)$ as an annulus in $K_{alg}$, replacing $K$ by $K_{alg}$ in the definitions. No confusion should result. Note that these discs and annuli are defined in terms of the real-closed order on $K$, not the non–archimedean absolute value $\| \cdot \|$, and hence are not discs or annuli in the sense of [BGR],[LR3] or [FP], which we will refer to as affinoid discs and annuli.

For $I = I(c, r)$, $A = A(c, \delta, \varepsilon)$ as above, we define the rings of analytic function on $I$ and $A$ as follows:

$$O_I := \left\{ f \left( \frac{x - c}{r} \right) : f \in R_1 \right\}$$

$$O_A := \left\{ g \left( \frac{\delta}{x - c} \right) + h \left( \frac{x - c}{\varepsilon} \right) : g, h \in R_1, \ g(0) = 0 \right\}.$$

The elements of $O_I$ (respectively, $O_A$) are analytic functions on the corresponding $K_{alg}$ disc (respectively, annulus) as well.

Remark 3.1. (i) Elements of $O_A$ are multiplied using the relation $\frac{\delta}{\varepsilon} \cdot \frac{c - x}{c} = \frac{\delta}{\varepsilon}$ and the fact that $|\frac{\delta}{\varepsilon}| < 1$. Indeed, let $g(\xi_1) = \sum_i a_i \xi_1^i$, $h(\xi_2) = \sum_j b_j \xi_2^j$. Then,
using ξ_1 ξ_2 = \frac{4}{ε} we have
\[ g \cdot h = \sum_{j<i} a_i b_j (\frac{δ}{ε})^j \xi_1^j + \sum_{i<j} a_i b_j (\frac{δ}{ε})^{j-i} \xi_2^{j-i} \]
\[ = f_1(\xi_1) + f_2(\xi_2). \]
If g, h ∈ A_1, and \( \frac{δ}{ε} \in \mathbb{R} \) then f_1, f_2 ∈ A_1, and this extends easily to the case g, h ∈ R_1, and \( \frac{δ}{ε} \in K^0 \). Lemma 3.6 will show that in fact the only case of an annulus that we must consider is when \( \frac{δ}{ε} \in K^{oo} \).

(ii) We define the gauss-norm on O_I by \( ||f(\frac{ε-δ}{ε})|| := \|f(ε)\| \), and on O_A by \( ||g(\frac{δ}{ε}) + h(\frac{ε-δ}{ε})|| := \max\{\|g(ε)\|, \|h(ε)\|\} \). It is clear that the gauss-norm equals the supremum norm.

(iii) If \( f \in O_I \) then \( ||(\frac{ε-δ}{ε})f|| = \|f\| \). If \( f \in O_A \) then \( ||(\frac{ε-δ}{ε})f|| \leq \|f\| \) and if \( \frac{δ}{ε} \in K^{oo} \), (i.e. is infinitesimal) then \( \|g(\frac{ε-δ}{ε})\| = \|\frac{δ}{ε}\| \|g(\frac{δ}{ε})\| < \|g(\frac{ε-δ}{ε})\| \) and \( ||(\frac{ε-δ}{ε})f|| = ||h(\frac{ε-δ}{ε})|| \).

(iv) If \( \|f\| < 1 \) then \( 1 - f \) is a unit in O_A. (In fact it is a strong unit – a unit u satisfying \( \|1 - u\| < 1 \).)

**Definition 3.2.** (i) We say that \( f \in O_{I(0,1)} = R_1 \) has a zero close to \( a \in I(0,1) \) if \( f \), as a K_{alg} function defined on the K_{alg} disc \( \{x| |x| < α\} \) for some \( α > 1 \) has a K_{alg}–zero \( b \) with \( a - b \) infinitesimal in K_{alg}. We say \( f \) has a zero close to \( I(0,1) \) if it has a zero close to \( a \) for some \( a \in I(0,1) \). For an arbitrary interval \( I = I(c,r) \) we say that \( f = F(\frac{ε-δ}{ε}) \in O_I \) has a zero close to \( a \in I \) if \( F \) has a zero close to \( \frac{α-ε}{ε} \in I(0,1) \), and that \( f \) has a zero close to \( I(c,r) \) if \( F \) has a zero close to \( I(0,1) \).

(ii) For \( 0 < a, b \in K^0 \) write \( a \sim b \) if \( \frac{a}{b} \), \( \frac{b}{a} \in K^0 \) and write \( a \ll b \) if \( \frac{a}{b} \in K^{oo} \).

(iii) Let \( X \) be an interval or an annulus, and let \( f \) be defined on a superset of \( X \). We shall write \( f \in O_X \) to mean that there is a function \( g \in O_X \) such that \( f|_X = g \).

**Lemma 3.3.** If \( f \in O_{I(c,r)} \) has no zero close to \( I(c,r) \), then there is a cover of \( I(c,r) \) by finitely many closed intervals \( I_j = I(c_j,r_j) \) such that \( \frac{1}{r} \in O_{I_j} \) for each \( j \).

**Proof.** It is sufficient to consider the case \( I(c,r) = I(0,1) \). Cover \( I(0,1) \) by finitely many intervals \( I(c_j,r_j) \) such that \( f \) has no K_{alg}–zero in the K_{alg}–disc \( I(c_j,r_j) \). Finally use Corollary 2.15. □

**Remark 3.4.** The function \( f(x) = 1 + x^2 \) has no zeros close to \( I(0,1) \). It is not a unit in \( O_{I(0,1)} \), but it is a unit in both \( O_{I(-\frac{1}{2}, \frac{1}{2})} \) and \( O_{I(\frac{1}{2}, \frac{3}{2})} \). The function \( g(x) = x \) is not a unit in \( O_{I(\delta, ε)} \) for any \( 0 < δ \in K^{oo} \) and \( 0 < ε \in K^0 \). It is of course a unit in \( O_{A(0,δ,ε)} \). The function \( \frac{1}{g} = \frac{1}{x} \) is in \( A(0,δ,1) \) but is not in \( O_I \) for \( I = I(\frac{1}{x} - \delta + \frac{1}{x}, \frac{1}{x}) \), for any \( δ \in K^{oo} \). Thus we see that if \( X_1 \subset X_2 \) are annuli or intervals it does not necessarily follow that \( O_{X_2} \subset O_{X_1} \). However the following are clear. If \( I_1 \subset I_2 \) are intervals, then \( O_{I_2} \subset O_{I_1} \). If \( 0 < δ \in K^{oo}, 0 < r \in K^0 \) and \( A = A(0,δ,1), I = I(\frac{1}{x} - \delta + \frac{1}{x}), \) then \( O_{A} \subset O_{I} \). If \( A_1 \subset I \) are annuli that have the same center, then \( O_{A_2} \subset O_{A_1} \). If \( 0 < δ < c \) and \( 0 < r < \frac{ε}{δ} \) for some
1 < α ∈ ℝ, and I = I(c, r) ⊂ A(0, δ, 1) = A, then Ω_A ⊂ Ω_I. (Writing x = c − y, |y| ≤ r we see that \( \frac{δ}{x} = \frac{δ}{c-y} = \frac{δfc}{1-y/c} + \frac{δ}{c} \sum \left( \frac{y}{c} \right)^k = \frac{δ}{c} \sum \left( \frac{y}{c} \right)^k \).

Restating Corollary 2.15 we have

**Corollary 3.5.** If \( f \in Ω_I(c, r) \) there is a polynomial \( P \in K[ξ] \) and a unit \( U \in Ω_I(c, r) \) such that \( f(ξ) = P(ξ) \cdot U(ξ) \).

**Lemma 3.6.** If \( ε < Nδ \) for some \( N ∈ ℕ \), then there is a covering of \( A(c, δ, ε) \) by finitely many intervals \( I_j \) such that for every \( f ∈ Ω_A(c, δ, ε) \) and each \( j, f ∈ Ω_{I_j} \).

**Proof.** Use Lemma 3.3 or reduce directly to the case \( ε = 1, 0 < δ = r ∈ ℝ \) and the two intervals \([-1, 1] = I(-1+\frac{r}{2}, 1-r) \) and \([1, 1] = I(\frac{1+r}{2}, 1-r) \).

**Lemma 3.7.** Let \( f ∈ A(c, δ, ε) \). There are finitely many intervals and annuli \( X_j \) that cover \( A(c, δ, ε) \), polynomials \( P_j \) and units \( U_j ∈ Ω_X \) such that for each \( j \) we have \( f|_{X_j} = (P_j \cdot U_j)|_{X_j} \).

**Proof.** By the previous lemma, we may assume that \( c = 0, ε = 1 \) and \( δ ∈ K^{∞} \) (i.e. \( δ \) is infinitesimal, say \( δ = t^γ \) for some \( γ > 0 \)). Let

\[
 f(x) = g(\frac{δ}{x}) + h(x) \quad \text{with} \quad g(ξ), h(ξ) ∈ R_1, \; g(0) = 0.
\]

and

\[
g(ξ) = \sum_{i=0}^{∞} t^{α_i}ξ^{n_i}g_i(ξ),
\]

with \( n_i > 0, g_i(0) ≠ 0, g_i ∈ A_{1, α} \) for some \( α > 1 \). Observe that

\[
xg(\frac{δ}{x}) = \sum_{i=0}^{∞} (t^{α_i}δ)(\frac{δ}{x})^{n_i-1}g_i(\frac{δ}{x}).
\]

Hence (see Remark 3.1) for suitable \( n ∈ ℕ \), absorbing the constant terms into \( h \), we have

\[
x^n f(x) = \overline{g}(\frac{δ}{x}) + \overline{h}(x)
\]

where \( \overline{g}(0) = 0 \) and \( \|\overline{g}\| < \|\overline{h}\| \). (For use in Section 4, below, note that this argument does not use that \( K \) is complete or of rank 1.) Multiplying by a constant, we may assume that \( \|\overline{h}\| = 1 \). Let

\[
\overline{g}(ξ) = \sum_{i=0}^{∞} t^{β_i}ξ^{m_i}\overline{g}_i(ξ), \quad \text{with} \; m_i > 0 \quad \text{and} \; β_i > 0 \quad \text{for each} \; i,
\]

and

\[
\overline{h}(ξ) = ξ^{k_0}\overline{h}_0(ξ) + \sum_{i=1}^{∞} t^{γ_i}\overline{h}_i(ξ), \quad \text{with} \; \overline{h}_0(0) ≠ 0 \quad \text{and the} \; γ_i > 0.
\]

Since \( \|\overline{g}\| = \|t^{β_0}\| < 1 \) there is a \( δ' \) with \( δ ≤ δ' ∈ K^{∞} \) (i.e. \( \|δ'\| < 1 \)) such that \( \|t^{γ_i}\| < \|(δ')^{k_0}\| \) and \( \|\overline{g}\| < \|(δ')^{k_0}\| \). Splitting off some intervals of the form \( I(-1+\frac{r}{2}, 1-r) = [-1, 1] \) or \( I(\frac{1+r}{2}, 1-r) = [r, 1] \) for \( r > 0 \), \( r ∈ ℝ \) (on which the
result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume that \( \overline{h}_0 \), is a unit in \( A_{1,\alpha} \), for some \( \alpha > 1 \). So

\[
x^n f(x) = \overline{h}(\frac{\delta}{x}) + \overline{h}(x)
\]

\[
= \overline{h}_0(x)x^{k_0} \left[ 1 + \sum_{i=1}^{\infty} \left( \frac{\delta'}{x} \right)^{k_0} t^{\gamma_i} \frac{h_i(x)}{\overline{h}_0(x)} + \frac{1}{\overline{h}_0(x)} \left( \frac{\delta'}{x} \right)^{k_0} \frac{\delta' \delta''}{x} \right].
\]

By our choice of \( \delta' \) and Remark 3.1 the quantity in square brackets is a (strong) unit. Hence we have taken care of an annulus of the form \( A(0, \delta', 1) \) for some \( \delta' \) with \( |\delta| \leq |\delta'| \) and \( |\delta'| < 1 \).

Observe that the change of variables \( y = \frac{\delta}{x} \) interchanges the sets \( \{ x: ||x|| = ||\delta|| \text{ and } \delta \leq ||x|| \} \) and \( \{ y: ||y|| = 1 \text{ and } |y| \leq 1 \} \). Hence, as above, there is a \( \delta'' \in K^{\times} \) with \( ||\delta|| < ||\delta''|| \) and a covering of the annulus \( \delta \leq ||x|| \leq \delta'' \) by finitely many intervals and annuli with the required property.

It remains to treat the annulus \( \delta'' \leq ||x|| \leq \delta' \). Using the terminology of [LR3], observe that on the much bigger affinoid annulus \( ||\delta''|| \leq ||x|| \leq ||\delta|| \) the function \( f \) is strictly convergent, indeed even overconvergent. Hence, as in [LR3], on this affinoid annulus we can write

\[ f = \frac{P(x)}{x^k} \cdot U(x) \]

where \( P(x) \) is a polynomial and \( U(x) \) is a strong unit (i.e. \( ||U(x) - 1|| < 1 \)). □

**Corollary 3.8.** If \( X \) is an interval or an annulus and \( f \in \mathcal{O}_X \), then the set \( \{ x \in X: f(x) \geq 0 \} \) is semialgebraic (i.e. a finite union of (closed) intervals).

**Proof.** This is an immediate corollary of 3.5 and 3.7 since units don’t change sign on intervals and since an annulus has two intervals as connected components. □

**Definition 3.9.** For \( c = (c_1, \ldots, c_n), r = (r_1, \ldots, r_n) \) we define the poly–interval \( I(c, r) := \{ x \in K^n: |x_i - c_i| \leq r_i, i = 1, \ldots, n \} \). This also defines the corresponding polydisc in \( (K_{alg})^n \). The ring of analytic functions on this poly–interval (or polydisc) is

\[ \mathcal{O}_{I(c, r)} := \left\{ f \left( \frac{x_1 - c_1}{r_1}, \ldots, \frac{x_n - c_n}{r_n} \right): f \in \mathcal{R}_n \right\}. \]

**Lemma 3.10.** Let \( \alpha, \beta > 1, F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha} \) and \( G_j(\xi_1, \ldots, \xi_n) \in \mathcal{R}_{n,\beta} \) with \( ||G_j|| \leq 1 \) for \( j = 1, \ldots, m \). Let \( X = \{ x \in [-1,1]^n: |G_j(x)| \leq 1 \text{ for } j = 1, \ldots, m \} \). There are (finitely many) \( c_i = (c_{i1}, \ldots, c_{in}) \in \mathbb{R}^n, \varepsilon_i \in \mathbb{R}, \varepsilon_i > 0 \text{ with } \varepsilon_i < |c_{ij}| \) if \( c_{ij} \neq 0 \) such that the (poly) intervals \( I_i = I(c_i, \varepsilon_i) = \{ x \in K: |x_j - c_{ij}| < \varepsilon_i \text{ for } j = 1, \ldots, n \} \) cover \( X \), \( |G_j(x)| < \alpha \) for all \( x \in I_i, j = 1, \ldots, m \), and there are \( H_i \in \mathcal{O}_{I_i} \) such that

\[ F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}. \]

**Proof.** Use the compactness of \( [-1,1]^n \cap \mathbb{R}^n \) and the following facts. If \( ||G_j|| = 1 \) then \( |G_j(x) - G_{j0}(x)| \) is infinitesimal for all \( x \in K^{\circ} \), where \( G_{j0} \) is the top slice of \( G_j \). If \( ||G_j|| < 1 \) then \( |G_j(x)| \in K^{\circ \circ} \) for all \( x \in [-1,1]^n \). □
Corollary 3.11. (i) Let $I$ be an interval, $F(\eta_1, \ldots, \eta_m) \in R_m$, and $G_j \in O_I$ for $j = 1, \ldots, m$. Then there are finitely many intervals $I_i$ covering $I$ and functions $H_i \in O_{I_i}$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}$$

(ii) Let $A$ be an annulus, $F(\eta_1, \ldots, \eta_m) \in R_m$, and $G_j \in O_A$ for $j = 1, \ldots, m$. Then there are finitely many $X_i$, each an interval or an annulus, covering $A$ and $H_i \in O_{X_i}$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{X_i} = H_i|_{X_i}.$$

Proof. Part (i) reduces to Lemma 3.10 once we see that if $||G_j|| > 1$ we can use Corollary 2.15 to restrict to the subintervals of $I$ around the zeros of $G_j$ on which $|G_j| \leq C$ for some $1 < C \in \mathbb{R}$. On the rest of $I$, $F(G_1, \ldots, G_m)$ is zero.

For (ii), we may assume that $A = A(0, \delta, 1)$ with $\delta$ infinitesimal and that $G_j(x) = G_{j_1}(\frac{x}{2}) + G_{j_2}(x)$. As in (i), we may reduce to the case that $||G_j|| \leq 1$, using Lemma 3.7 instead of Corollary 2.15, and using Lemma 3.6 and Remark 3.4. Apply Lemma 3.10 to the functions $F$ and $G_j(\xi_1, \xi_2) = G_{j_1}(\xi_1) + G_{j_2}(\xi_2)$. The case $c = (0, 0)$ gives us the annulus $|\frac{x}{2}| \leq \varepsilon$, $|x| \leq \varepsilon$ i.e. $\frac{\varepsilon}{2} \leq |x| \leq \varepsilon$. The case $c = (0, c_2)$ with $c_2 \neq 0$ gives us $\frac{\varepsilon}{2} \leq |x| \leq \varepsilon$ with $\varepsilon < |c_2|$. This is equivalent to $|x - c_2| \leq \varepsilon$ since $\varepsilon, c_1 \in \mathbb{R}$, and hence equivalent to $-\varepsilon \leq x - c_2 \leq \varepsilon$ or $c_2 - \varepsilon \leq x \leq c_2 + \varepsilon$ which is an interval bounded away from 0. The case $c = (c_1, 0)$ gives us $\frac{\varepsilon}{c_1} - |c_1| \leq \varepsilon$, $|x| \leq \varepsilon$ (since $\varepsilon < |c_1|$) which is equivalent to $\frac{\varepsilon}{c_1} - |c_1| \leq \varepsilon$ or $c_1 - \varepsilon \leq \frac{\varepsilon}{x} \leq c_1 + \varepsilon$ or (considering the case $c_1 > 0$, the case $c_1 < 0$ is similar) $\frac{\sqrt{\frac{\varepsilon}{x}}}{c_1 + \varepsilon} \leq x \leq \frac{\sqrt{\frac{\varepsilon}{x}}}{c_1 - \varepsilon}$ which is part of an annulus that can be reduced to intervals using Lemma 3.6. The case $c = (c_1, c_2)$ with both $c_1, c_2 \neq 0$ is vacuous since either $x$ or $\frac{x^2}{2}$ is infinitesimal and $\varepsilon < |c_1|, |c_2|$.

Lemma 3.12. Let $X$ be an interval or an annulus and let $f, g \in O_X$. There are finitely many subintervals and subannuli $X_i \subset X$, $i = 1, \ldots, \ell$ such that

$$\{ x \in X : |f(x)| \leq |g(x)| \} \subset \bigcup X_i \subset X,$$

and, except at finitely many points,

$$\frac{f}{g}|_{X_i} \in O_{X_i}.$$
Theorem 3.14. \( K \) is \( o \)-minimal in \( \mathcal{L} \).

4. Further Extensions

In this section we give extensions of the results of Sections 2 and 3 and the results of [LR3].

Let \( G \) be an (additive) ordered abelian group. Let \( t \) be a symbol. Then \( t^G \) is a (multiplicative) ordered abelian group. Following the notation of [DMM1] and [DMM2] (but not [DMM3] or [LR2]) we define \( \mathbb{R}(t^G) \) to be the maximally-complete valued field with additive value group \( G \) (or multiplicative value group \( t^G \)) and residue field \( \mathbb{R} \). So

\[
\mathbb{R}(t^G) := \left\{ \sum_{g \in I} a_gt^g : a_g \in \mathbb{R} \text{ and } I \subset G \text{ well-ordered} \right\}.
\]

We shall be a bit sloppy about mixing the additive and multiplicative valuations. \( I \subset G \) is well-ordered exactly when \( t^I \subset t^G \) is reverse well-ordered. The field \( K \) of Puiseux series, or its completion, is a proper subfield of \( \mathbb{R}_1 := \mathbb{R}(t^G) \). Considering \( G = \mathbb{Q}^m \) with the lexicographic ordering, we define

\[
\mathbb{R}_m := \mathbb{R}(t^{\mathbb{Q}^m}).
\]

It is clear that if \( G_1 \subset G_2 \) as ordered groups, then \( \mathbb{R}(t^{G_1}) \subset \mathbb{R}(t^{G_2}) \) as valued fields. Also, \( \mathbb{R}(t^{G_1}) \) is Henselian and, if \( G \) is divisible, then \( \mathbb{R}(t^G) \) is real-closed.

We shall continue to use \( < \) for the corresponding order on \( \mathbb{R}(t^G) \).
In analogy with Section 2, we define

**Definition 4.1.**

\[ R_{n,\alpha}(G) := A_{n,\alpha} \otimes^* \mathbb{R}((t^G)) \]
\[ R_n(G) := \bigcup_{\alpha>1} R_{n,\alpha}(G) \]
\[ R(G) := \bigcup_n \bigcup_{\alpha>1} R_{n,\alpha}(G). \]

As in Section 2, the elements of \( R_n(G) \) define functions from \( I^n \subset \mathbb{R}((t^G))^n \) to \( \mathbb{R}((t^G)) \). Indeed this interpretation is a ring endomorphism. In other words, the field \( \mathbb{R}((t^G)) \) has analytic \( R(G) \)-structure. (See [CLR1] and especially [CLR2] for more about fields with analytic structure.) The elements of \( R_n(G) \) are interpreted as zero on \( \mathbb{R}((t^G))^n \setminus I^n \).

**Theorem 4.2.** The Weierstrass Preparation Theorem (Theorem 2.5) and the Strong Noetherian Property (Theorem 2.10) hold with \( R \) replaced by \( R(G) \).

**Proof.** Only minor modifications to the proofs of Theorems 2.5 and 2.10 are needed. □

The arguments of Sections 2 and 3 show

**Theorem 4.3.** If \( G \) is divisible then \( \mathbb{R}((t^G)) \) admits quantifier elimination and is \( o \)-minimal in \( L_G \).

**Corollary 4.4.** If \( G_1 \subset G_2 \) are divisible, then \( \mathbb{R}((t^{G_1})) \prec \mathbb{R}((t^{G_2})) \) in \( L_{G_1} \).

We shall show in Section 5 that, though for \( m < n \) we have \( \mathbb{R}_m \prec \mathbb{R}_n \) in \( L_{Q^n} \), there is a sentence of \( L_{Q^n} \) that is true in \( \mathbb{R}_n \) but is not true in any \( o \)-minimal expansion of \( \mathbb{R}_m \).

The results of [LR3] also extend to this more general setting.

**Definition 4.5.** We define the ring of *strictly convergent power series over \( \mathbb{R}((t^G)) \) as

\[ \mathbb{R}((t^G))^* := \{ \sum_{g \in I} a_g(t^g) : a_g(\xi) \in \mathbb{R}[\xi] \text{ and } I \text{ well ordered} \}, \]

and the subring of *overconvergent power series over \( \mathbb{R}((t^G)) \) as

\[ \mathbb{R}((t^G)^* \langle \langle \xi \rangle \rangle) := \{ f : f(\gamma \xi) \in \mathbb{R}((t^G))^* \langle \xi \rangle \text{ for some } \gamma \in \mathbb{R}((t^G)), \| \gamma \| > 1 \}, \]
\[ \mathcal{R}(G)_{\text{over}} := \bigcup_n \mathbb{R}((t^G))^* \langle \{ \xi_1, \ldots, \xi_n \} \rangle, \]
and the corresponding overconvergent language as
\[ \mathcal{L}_{G,\text{over}} := (+, \cdot, ^{-1}, 0, 1, <, \mathcal{R}(G)_{\text{over}}). \]

As in [LR3] we have

**Theorem 4.6.** If \( G \) is divisible then \( \mathbb{R}((t^G)) \) admits quantifier elimination and is \( o \)-minimal in \( \mathcal{L}_{G,\text{over}} \).

Of course the \( o \)-minimality follows immediately from Theorem 4.6.

5. **Extensions of the Example of Hrushovski and Peterzil** [HP]

In this section we show that with minor modifications, the idea of the example of [HP] can be iterated to give a nested family of examples. This relates to a question of Hrushovski and Peterzil whether there exists a small class of \( o \)-minimal structures such that any sentence, true in some \( o \)-minimal structure, can be satisfied in an expansion of a model in the class. Combining with expansions with the exponential function, one perhaps can elaborate the tower of examples further.

Consider the functional equation
\[ (*) \quad F(\beta z) = \alpha z F(z) + 1, \]
and suppose that \( F \) is a “complex analytic” solution for \(|z| \leq 1\). By this we mean that, writing \( z = x + \sqrt{-1} y \), \( F(z) = f(x, y) + \sqrt{-1} g(x, y), F(z) \) is differentiable as a function of \( z \). This is a definable condition on the two “real” functions, \( f, g \) of the two “real” variables \( x, y \). Then
\[
F(z) = \sum_{k=0}^{\infty} a_k z^k
\]
where
\[
a_k = \frac{\alpha^k}{\beta^{k+1}}.
\]
(\( \alpha \), and \( \beta \) are parameters).

By this we mean that for each \( n \in \mathbb{N} \) there is a constant \( A_n \) such that
\[ (** \quad |F(z) - \sum_{k=0}^{n} a_k z^k| \leq A_n |z^{n+1}| \]
is true for all \( z \) with \(|z| \leq 1\). Indeed, by [PS] Theorem 2.50, one can take \( A_n = C \cdot 2^{n+1} \), for \( C \) a constant independent of \( n \).

Consider the following statement: \( F(z) \) is a complex analytic function (in the above sense) on \(|z| \leq 1\) that satisfies (\( * \)). The number \( \beta > 0 \) is within the radius of convergence of the function \( f(z) = \sum_{n=1}^{\infty} (n-1)! z^n \) and \( \alpha > 0 \).

This statement is not satisfiable by any functions in any \( o \)-minimal expansion of the field of Puiseux series \( K_1 \), or the maximally complete field \( \mathbb{R}_1 = \mathbb{R}((t^\gamma)) \), because if it were, we would have \( \| \beta \| = \| t^\gamma \|, \| \alpha \| = \| t^\delta \| \), for some 0 < \( \gamma, \delta \in \mathbb{Q} \), and for suitable choice of \( n \) \( (**) \) would be violated. On the other hand, if we choose
\( \alpha, \beta \in \mathbb{R}_2 \) with \( \text{ord}(\alpha) = (1, 0) \) and \( \text{ord}(\beta) = (0, 1) \) then \( \sum a_k z^k \in \mathbb{R}_2(\langle z \rangle)^* \) satisfies the statement on \( \mathbb{R}_2 \).

This process can clearly be iterated to give, in the notation of Section 4,

**Proposition 5.1.** For each \( m \) there is a sentence of \( \mathcal{L}_{Q_m} \) true in \( \mathbb{R}_m \) but not satisfiable in any \( \alpha \)-minimal expansion of \( \mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1} \).

**References**


[LR1] L. Lipshitz and Z. Robinson, Rings of separated power series, Asterisque **264**.


