REAL CLOSED FIELDS WITH NONSTANDARD AND
STANDARD ANALYTIC STRUCTURE

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ABSTRACT. We consider the ordered field which is the completion of the Puiseux series field over \( \mathbb{R} \) equipped with a ring of analytic functions on \([-1, 1]^n\) which contains the standard subanalytic functions as well as functions given by \( t \)-adically convergent power series, thus combining the analytic structures from [DD] and [LR3]. We prove quantifier elimination and \( o \)-minimality in the corresponding language. We extend these constructions and results to rank \( n \) ordered fields \( \mathbb{R}_n \) (the maximal completions of iterated Puiseux series fields). We generalize the example of Hrushovski and Peterzil [HP] of a sentence which is not true in any \( o \)-minimal expansion of \( \mathbb{R} \) (shown in [LR3] to be true in an \( o \)-minimal expansion of the Puiseux series field) to a tower of examples of sentences \( \sigma_n \), true in \( \mathbb{R}_n \), but not true in any \( o \)-minimal expansion of any of the fields \( \mathbb{R}, \mathbb{R}_1, \ldots, \mathbb{R}_{n-1} \).

1. Introduction

In [LR3] it is shown that the ordered field \( K_1 \) of Puiseux series in the variable \( t \) over \( \mathbb{R} \) equipped with a class of \( t \)-adically overconvergent functions such as \( \sum_n (n+1)!((tx)^n \) has quantifier elimination and is \( o \)-minimal in the language of ordered fields enriched with function symbols for these functions on \([-1, 1]^n\). This was motivated (indirectly) by the observation of Hrushovski and Peterzil, [HP], that there are sentences true in this structure that are not satisfiable in any \( o \)-minimal expansion of \( \mathbb{R} \). This in turn was motivated by a question of van den Dries. See [HP] for details.

In [DMM1] it was observed that if \( K \) is a maximally complete, non-archimedean real closed field with divisible value group, and if \( f \) an element of \( \mathbb{R}[[\xi]] \) with radius of convergence \( > 1 \), then \( f \) extends naturally to an “analytic” function \( I^n \rightarrow K \), where \( I = \{ x \in K : -1 \leq x \leq 1 \} \). Hence if \( \mathcal{A} \) is the ring of real power series with radius of convergence \( > 1 \) then \( K \) has \( \mathcal{A} \)-analytic structure i.e. this extension preserves all the algebraic properties of the ring \( \mathcal{A} \). In particular the real quantifier

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elimination of [DD] works in this context so $K$ has quantifier elimination, is $o$–minimal in the analytic field language, and is even elementarily equivalent to $\mathbb{R}$ with the subanalytic structure. See [DMM2] and [DMM3] for extensions.

In Sections 2 and 3 below we extend the results of [LR3] by proving quantifier elimination and $o$–minimality for $K_1$ in a larger language that contains the overconvergent functions together with the usual analytic functions on $[-1,1]^n$. In Section 4 we extend these results to a larger class of non-archimedean real-closed fields, including fields $\mathbb{R}_m$, of rank $m = 1, 2, 3, \ldots$, and in Section 5 we show that the idea of the example of [HP] can be iterated so that for each $m$ there is a sentence true in $\mathbb{R}_m$ but not satisfiable in any $o$–minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \cdots, \mathbb{R}_{m-1}$. In a subsequent paper we will give a more comprehensive treatment of both Henselian fields with analytic structure and real closed fields with analytic structure, see [CL].

2. Notation and Quantifier Elimination

In this section we establish notation and prove quantifier elimination (Theorem 2.16) for the field of Puiseux series over $\mathbb{R}$ in a language which contains function symbols for all the standard analytic functions on $[-1,1]^n$ and all the $t$–adically overconvergent functions on this set.

**Definition 2.1.**

\[ K_1 := \bigcup_n \mathbb{R}(t^{1/n}), \text{ the field of Puiseux series over } \mathbb{R}, \]
\[ K := \hat{K}_1, \text{ the } t^-\text{adic completion of } K_1. \]

$K$ is a real closed, nonarchimedean normed field. We shall use $\| \cdot \|$ to denote the (nonarchimedean) $t$–adic norm on $K$, and $<$ to denote the order on $K$ that comes from the real closedness of $K$. We will use $| \cdot |$ to denote the corresponding absolute value, $|x| = \sqrt{x^2}$.

\[ K^o := \{ x \in K : \| x \| \leq 1 \}, \text{ the finite elements of } K \]
\[ K^{oo} := \{ x \in K : \| x \| < 1 \}, \text{ the infinitesimal elements of } K \]
\[ K_{alg} := K[\sqrt{-1}], \text{ the algebraic closure of } K \]
\[ A_{n,\alpha} := \{ f \in \mathbb{R}[[\xi_1, \ldots, \xi_n]]: \text{ radius of convergence of } f > \alpha \}, 0 < \alpha \in \mathbb{R} \]
\[ R_{n,\alpha} := A_{n,\alpha} \otimes_{\mathbb{R}} K = (A_{n,\alpha} \otimes_{\mathbb{R}} K)^\sim, \text{ where } \sim \text{ stands for the } t\text{-adic completion} \]
\[ R_{n,\alpha}^o := A_{n,\alpha} \otimes_{\mathbb{R}} K^o = (A_{n,\alpha} \otimes_{\mathbb{R}} K^o)^\sim \]
\[ R_n := \bigcup_{\alpha > 1} R_{n,\alpha} \]
\[ R := \bigcup_{n \geq 1} R_n \]
\[ I := [-1,1] = \{ x \in K : |x| \leq 1 \}. \]

**Remark 2.2.** (i) $K$ has $R$–analytic structure – in [DMM1] it is explained how the functions of $A_{n,\alpha}$ are defined on $I^n$. For example, if $f \in A_{1,\alpha}, \alpha > 1, a \in \mathbb{R}\cap[-1,1], \beta \in K^{oo}$, then $f(a + \beta) := \sum_n f^{(n)}(a)\beta^n$. The extension to functions in $R_{n,\alpha}$ is clear from the completeness of $K$. We define these functions to be zero outside
$[-1, 1]^n$. This extension also naturally works for maximally complete fields and fields of LE-series, see also [DMM2] and [DMM3].

(ii) $\mathcal{R}$ contains all the “standard” real analytic functions on $[-1, 1]^n$ and all the $t$–adically overconvergent functions in the sense of [LR3].

(iii) The elements of $A_{n, \alpha}$ in fact define complex analytic functions on the complex polydisc $\{x \in \mathbb{C}: |x_i| \leq \alpha \text{ for } i = 1, \ldots, n\}$, and hence the elements of $\mathcal{R}_{n, \alpha}$ define “$K_{alg}$–analytic” functions on the corresponding $K_{alg}$–polydisc $\{x \in K_{alg}: |x_i| \leq \alpha \text{ for } i = 1, \ldots, n\}$.

(iv) We could as well work with $K_1$ instead of $K$. Then we must replace $\mathcal{R}_{n, \alpha}$ by

$$\mathcal{R}'_{n, \alpha} := \bigcup_m A_{n, \alpha} \otimes_{\mathbb{R}} \mathbb{R}(t^{1/m}) = \bigcup_m ((A_{n, \alpha} \otimes_{\mathbb{R}} \mathbb{R}(t^{1/m})))^-.$$

(v) If $\beta \leq \alpha$ and $m \leq n$ then $\mathcal{R}_{\alpha, m} \subset \mathcal{R}_{\beta, n}$.

**Lemma 2.3.** Every nonzero $f \in \mathcal{R}_{n, \alpha}$ has a unique representation

$$f = \sum_{i \in J \subset \mathbb{N}} f_i t^{\gamma_i}$$

where the $f_i \in A_{n, \alpha}$, $f_i \neq 0$, $\gamma_i \in \mathbb{Q}$, the $\gamma_i$ are increasing, $0 \in J$, and, either $J$ is finite of the form $\{0, \ldots, n\}$, or $\gamma_i \to \infty$ as $i \to \infty$ and $J = \mathbb{N}$. The function $f$ is a unit in $\mathcal{R}_{n, \alpha}$ exactly when $f_0$ is a unit in $A_{n, \alpha}$.

**Proof.** Observe that if $a \in K, a \neq 0$ then $a$ has a unique representation $a = \sum_{i \in J \subset \mathbb{N}} a_i t^{\gamma_i}$, where the $0 \neq a_i \in \mathbb{R}, \gamma_i \in \mathbb{Q}$ and, either $J$ is finite, or $\gamma_i \to \infty$. If $f_0$ is a unit in $A_{n, \alpha}$, then $f \cdot f_0^{-1} \cdot t^{-\gamma_0} = 1 + \sum_{i=1}^{\infty} f_i \cdot f_0^{-1} \cdot t^{\gamma_i - \gamma_0}$ and $\gamma_i - \gamma_0 > 0$. □

**Definition 2.4.**

(i) In the notation of the previous lemma, $f_0$ is called the top slice of $f$.

(ii) We call $f$ regular in $\xi_n$ of degree $s$ at $a \in [I \cap \mathbb{R}]^n$ if, in the classical sense, $f_0$ is regular in $\xi_n$ of degree $s$ at $a$.

(iii) We shall abuse notation and use $\| \cdot \|$ to denote the $t$–adic norm on $K$, and the corresponding gauss–norm on $\mathcal{R}_{n, \alpha}$, so, with $f$ as in the above lemma, $\| f \| = \| t^{\gamma_0} \|$.

The standard Weierstrass Preparation and Division Theorems for $A_{n, \alpha}$ extend to corresponding theorems for $\mathcal{R}_{n, \alpha}$.

**Theorem 2.5.** (Weierstrass Preparation and Division). If $f \in \mathcal{R}_{n, \alpha}$ with $\| f \| = 1$ is regular in $\xi_n$ of degree $s$ at $0$, then there is a $\delta \in \mathbb{R}, \delta > 0$, such that there are unique $A_1, \ldots, A_s, U$ satisfying

$$f = [\xi_n^s + A_1(\xi')\xi_n^{s-1} + \ldots + A_s(\xi')]U(\xi)$$

and

$$A_1, \ldots, A_s, \in \mathcal{R}_{n-1, \delta}, \text{ and } U \in \mathcal{R}_{n, \delta} \text{ a unit.}$$

Then automatically

$$\| A_1 \|, \ldots, \| A_s \|, \| U \| \leq 1, \quad \| A_1(0) \|, \ldots, \| A_s(0) \| < 1, \text{ and } \| U(0) \| = 1.$$
Furthermore, if \( g \in \mathcal{R}_{n, \alpha} \) then there are unique \( Q \in \mathcal{R}_{n, \delta} \) and \( R_0(\xi')_0, \ldots, R_{s-1}(\xi')_0 \in \mathcal{R}_{n-1, \delta} \), satisfying
\[
\|Q\|, \|R_i\| \leq \|g\|
\]
and
\[
g = Qf + R_0(\xi')_0 + R_1(\xi')_0 \xi_n + \ldots + R_{s-1}(\xi')_0 \xi_n^{s-1}.
\]

Proof. We may assume that \( f = \sum_{\gamma \in I} f_\gamma t^{\gamma} \), where \( I \subset \mathbb{Q}^+, 0 \in I, I = I + I \) and \( I \) is well ordered. (We do not require that \( f_\alpha \neq 0 \) for all \( \alpha \in I \), but we do require \( f_0 \neq 0 \).) We prove the Preparation Theorem. The proof of the Division Theorem is similar. We shall produce, inductively on \( \gamma \), monic polynomials \( P_\gamma[\xi_n] \) with coefficients from \( \mathcal{R}_{n-1, \delta} \), and units \( U_\gamma \in \mathcal{R}_{n, \delta} \) such that, writing \( \gamma' \) for the successor of \( \gamma \) in \( I \), we have
\[
f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'}
\]
and if \( \gamma < \beta \)
\[
P_\gamma \equiv P_\beta \text{ and } U_\gamma \equiv U_\beta \mod t^{\gamma'}.
\]
Using [GR], Theorem II.D.1 (p.80), or the proof on pp. 142-144 of [ZS], we see that there is a \( 0 < \delta \leq \alpha \) such that for every \( g \in \mathcal{R}_{n, \delta} \) the Weierstrass data on dividing \( g \) by \( f_0 \) are in \( \mathcal{R}_{n, \delta} \).

\( P_0 \) and \( U_0 \) are the classical Weierstrass data for \( f_0 \), i.e. \( f_0 = U_0 P_0 \), where \( P_0 \in \mathcal{R}_{n-1, \delta}[\xi_n] \) is monic of degree \( s \), and \( U \in \mathcal{R}_{n, \delta} \) is a unit. Suppose \( P_\gamma \) and \( U_\gamma \) have been found. Then
\[
f \equiv P_\gamma \cdot U_\gamma \mod t^{\gamma'}
\]
so we have
\[
U^{-1}_\gamma \cdot f \equiv P_\gamma + g_\gamma' t^{\gamma'} + o(t^{\gamma'}),
\]
where \( g_\gamma' \in \mathcal{R}_{n, \delta} \) and we write \( o(t^{\gamma'}) \) to denote terms of order \( > \gamma' \). By classical Weierstrass division we can write
\[
g_\gamma' = P_0 \cdot Q_\gamma' + R_\gamma',
\]
where \( Q_\gamma' \in \mathcal{R}_{n, \delta} \) and \( R_\gamma' \in \mathcal{R}_{n-1, \delta}[\xi_n] \) has degree \( < s \) in \( \xi_n \). Let
\[
P_\gamma' := P_\gamma + t^{\gamma'} R_\gamma'.
\]
Then
\[
U^{-1}_\gamma \cdot f = P_\gamma + t^{\gamma'} (P_0 Q_\gamma' + R_\gamma') + o(t^{\gamma'}),
\]
\[
= (P_\gamma' + t^{\gamma'} P_\gamma Q_\gamma') + t^{\gamma'} (P_0 - P_\gamma) Q_\gamma' + o(t^{\gamma'})
\]
\[
= P_\gamma' + t^{\gamma'} (1 + t^{\gamma'} Q_\gamma') + o(t^{\gamma'}),
\]
since \( P_0 - P_\gamma = o(1) \), i.e. it has positive order. Take \( U_\gamma' := U_\gamma (1 + t^{\gamma'} Q_\gamma') \). The uniqueness of the \( A_i \) and \( U \) follows from the same induction. \( \square \)

Remark 2.6. We remark, for use in a subsequent paper ([CL]), that the argument of the previous proof works in the more general context that \( I \) is a well ordered subset of the value group \( \Gamma \) of a suitably complete field, for example a maximally complete field.

From the above proof or by direct calculation we have

Corollary 2.7. If \( g \in \mathcal{R}_1, \beta \in [-1, 1] \) and \( g(\beta) = 0 \) then \( \xi_1 - \beta \) divides \( g \) in \( \mathcal{R}_1 \).
Remark 2.8. Let \( f(\xi, \eta) \) be in \( R_{m+n, \alpha} \). Then there are unique \( f_\mu \) in \( R_{m, \alpha} \) such that

\[
f(\xi, \eta) = \sum_{\mu} f_\mu(\xi) \eta^\mu.
\]

The following Lemma is used to prove Theorem 2.10.

Lemma 2.9. Let \( f(\xi, \eta) = \sum_\mu f_\mu(\xi) \eta^\mu \in R_{m+n, \alpha} \). Then the \( f_\mu \in R_{m, \alpha} \) and there is a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \) and \( g_\mu \in R_{m+n, \beta} \) for \( |\mu| < d \), such that

\[
f = \sum_{|\mu| < d} \left( \sum_\nu f_\mu(\xi) g_\mu(\xi, \eta) \right) \nu^\nu,
\]
in \( R_{m+n, \beta} \).

Proof. We may assume that \( \|f\| = 1 \), and choose a \( \nu_0 \) such that \( \|f_{\nu_0}\| = 1 \). Making an \( \mathbb{R} \)-linear change of variables, and shrinking \( \alpha \) if necessary, we may assume that \( f_{\nu_0} \) is regular at 0 in \( \xi_m \) of degree \( s \), say. Write \( \xi' \) for \( (\xi_1, \ldots, \xi_{m-1}) \). By Weierstrass Division (Theorem 2.5) there is a \( \beta > 0 \) and there are \( Q(\xi, \eta) \in R_{m+n, \beta} \) and

\[
R(\xi, \eta) = R_0(\xi', \eta) + \cdots + R_{s-1}(\xi', \eta) \xi_m^{s-1} \in R_{m+n-1, \beta} \]

such that

\[
f(\xi, \eta) = f_{\nu_0}(\xi) Q(\xi, \eta) + R(\xi, \eta).
\]

By induction on \( m \), we may write

\[
R_0 = \sum_{|\mu| < d} R_{0\mu}(\xi') g_\mu(\xi', \eta),
\]

for some \( d \in \mathbb{N} \), some \( \beta > 0 \) and \( g_\mu(\xi', \eta) \in R_{m+n-1, \beta} \). Writing \( R = \sum_\nu R_\nu(\xi) \eta^\nu \), observe that each \( R_\nu \) is an \( R_{m, \beta} \)-linear combination of \( f_\nu \), since, taking the coefficient of \( \eta^\nu \) on both sides of the equation \( f(\xi, \eta) = f_{\nu_0}(\xi) Q(\xi, \eta) + R(\xi, \eta) \), we have

\[
f_\nu = f_{\nu_0} R_\nu + R_\nu.
\]

Consider

\[
f - f_{\nu_0} Q - \sum_{|\mu| < d} R_\mu(\xi) g_\mu(\xi', \eta) = S_1 \xi_m + S_2 \xi_m^2 + \cdots + S_{s-1} \xi_m^{s-1}
\]

\[
= \xi_m [S_1 + S_2 \xi_m + \cdots + S_{s-1} \xi_m^{s-2}]
\]

\[
= \xi_m \cdot S, \text{ say},
\]

where the \( S_i \in R_{m+n-1, \beta} \). Again, observe that each \( S_\nu \) is an \( R_{m, \beta} \)-linear combination of \( f_\nu \). Complete the proof by induction on \( s \), working with \( S \) instead of \( R \).

\[\square\]

Theorem 2.10. (Strong Noetherian Property). Let \( f(\xi, \eta) = \sum_\mu f_\mu(\xi) \eta^\mu \in R_{m+n, \alpha} \). Then the \( f_\mu \in R_{m, \alpha} \) and there is an integer \( d \in \mathbb{N} \), a constant \( \beta > 0 \), \( \beta \in \mathbb{R} \) and units \( U_\mu(\xi, \eta) \in R_{m+n, \beta} \) for \( |\mu| < d \), such that

\[
f = \sum_{\mu \in J} f_\mu(\xi) \eta^\mu U_\mu(\xi, \eta)
\]
in \( R_{m+n, \beta} \), where \( J \) is a subset of \( \{0, 1, \ldots, d\}^n \).
Proof. It is sufficient to show that there are an integer $d$, a set $J \subset \{0, 1, \ldots, d\}^n$, and $g_\mu \in \mathcal{R}_{m+n, \beta}^\circ$ such that
\[
(2.11) \quad f = \sum_{\mu \in J} \mathcal{F}_\mu(\xi)\eta^\mu g_\mu(\xi, \eta),
\]
since then, rearranging the sum if necessary, we may assume that each $g_\mu$ is of the form $1 + h_\mu$ where $h_\mu \in (\eta)\mathcal{R}_{m+n, \beta}^\circ$. Shrinking $\beta$ if necessary will guarantee that the $g_\mu$ are units. But then it is in fact sufficient to prove (2.11) for $f$ replaced by
\[
f_{I_i} := \sum_{\mu \in I_i} \mathcal{F}_\mu(\xi)\eta^\mu
\]
for each $I_i$ in a finite partition $\{I_i\}$ of $\mathbb{N}^n$.

By Lemma 2.9 there is an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$ and $g_\mu \in \mathcal{R}_{m+n, \beta}^\circ$ for $|\mu| \leq d$, such that
\[
f = \sum_{|\mu| \leq d} \mathcal{F}_\mu(\xi)g_\mu(\xi, \eta).
\]
Rearranging, we may assume for $\nu, \mu \in \{1, \ldots, d\}^n$ that $(\bar{g}_\mu)_\nu$ equals 1 if $\mu = \nu$ and that it equals 0 otherwise.

Focus on $f_{I_i}(\xi, \eta)$, defined as above by
\[
f_{I_i}(\xi, \eta) = \sum_{\mu \in I_i} \mathcal{F}_\mu(\xi)\eta^\mu
\]
with
\[
I_i := \{0, \ldots, d\}^n \cup \{\mu: \mu_i \geq d \text{ for all } i\}
\]
and note that
\[
(2.12) \quad f_{I_i}(\xi, \eta) = \sum_{|\mu| \leq d} \mathcal{F}_\mu(\xi)g_{\mu, I_i}(\xi, \eta)
\]
with $g_{\mu, I_i}(\xi, \eta) \in \mathcal{R}_{m+n, \beta}^\circ$ defined by the corresponding sum
\[
g_{\mu, I_i}(\xi, \eta) = \sum_{\nu \in I_i} \bar{g}_{\mu, \nu}(\xi)\eta^\nu.
\]

It is now clear that $g_{\mu, I_i}$ is of the form $\eta^\mu(1 + h_\mu)$ where $h_\mu \in (\eta)\mathcal{R}_{m+n, \beta}^\circ$.

One now proceeds by noting that $f - f_{I_1}$ is a finite sum of terms of the form $f_{I_j}$ for $j > 1$ and $\{I_j\}_j$ a finite partition of $\mathbb{N}^n$ and where each $f_{I_j}$ for $j > 1$ is of the form $\eta^q(1 + h_{\ell})$ where $h_\ell$ is of the form $\eta^p(\xi, \eta')$ where $\eta'(\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_n)$ and $q$ is in $\mathcal{R}_{m+n-1, \beta}^\circ$. These terms can be handled by induction on $n$.

\[\square\]

Definition 2.13. For $\gamma \in \mathcal{K}_\text{alg}^\circ$ let $\gamma^\circ$ denote the closest element of $\mathbb{C}$, i.e. the unique element $\gamma^\circ$ of $\mathcal{K}_\text{alg}^\circ$ such that $|\gamma - \gamma^\circ| \in \mathcal{K}_\text{alg}^\circ$.

Lemma 2.14. Let $f \in \mathcal{R}_1$. If $f(\gamma) = 0$ then $f_0(\gamma^\circ) = 0$ ($f_0$ is the top slice of $f$). Conversely, if $\beta \in \mathbb{R}$ (or $\mathbb{C}$) and $f_0(\beta) = 0$ there is a $\gamma \in \mathcal{K}_\text{alg}^\circ$ with $\gamma^\circ = \beta$ and $f(\gamma) = 0$. Indeed, $f_0$ has a zero of order $n$ at $\beta \in \mathbb{C}$ if, and only if, $f$ has $n$ zeros $\gamma$ (counting multiplicity) with $\gamma^\circ = \beta$. 

Proof. Use Weierstrass Preparation and [BGR] Proposition 3.4.1.1.

Corollary 2.15. A nonzero \( f \in \mathcal{R}_{1,\alpha} \) has only finitely many zeros in the set \( \{ x \in K_{\text{alg}} : |x| \leq \alpha \} \). Indeed, there is a polynomial \( P(x) \in K[x] \) and a unit \( U(x) \in \mathcal{R}_{1,\alpha} \) such that \( f(x) = P(x) \cdot U(x) \).

Proof. Observe that \( f_0 \) has only finitely many zeros in \( \{ x \in \mathbb{C} : |x| \leq \alpha \} \), that non-real zeros occur in complex conjugate pairs, and that \( f \) is a unit exactly when \( f_0 \) has no zeros in this set, i.e. when \( f_0 \) is a unit in \( A_{1,\alpha} \), and use Lemma 2.14.

Theorem 2.16 (Quantifier Elimination Theorem). Denote by \( \mathcal{L} \) the language \( \langle +,\cdot,^{-1},0,1,<,\mathcal{R} \rangle \) where the functions in \( \mathcal{R}_n \) are interpreted to be zero outside \( I^n \). Then \( K \) admits quantifier elimination in \( \mathcal{L} \).

Proof. This is a small modification of the real quantifier elimination of [DD] as in [DMM1], using the Weierstass Preparation Theorem and the Strong Noetherian Property above. Crucial is that, in Theorems 2.5 and 2.10, as in [DD], \( \beta \) and \( \delta \) are positive real numbers so one can use the compactness of \( [-1,1]^n \) in \( \mathbb{R}^n \).

3. \( o \)-MINIMALITY

In this section we prove the \( o \)-minimality of \( K \) in the language \( \mathcal{L} \). Let \( \alpha > 1 \). As we remarked above, each \( f \in A_{n,\alpha} \) defines a function from the poly-disc \( (I_{\mathcal{C},\alpha})^n \to \mathbb{C} \), where \( I_{\mathcal{C},\alpha} := \{ x \in \mathbb{C} : |x| \leq \alpha \} \), and hence each \( f \in \mathcal{R}_{n,\alpha} \) defines a function from \( (I_{K_{\text{alg}},\alpha})^n \to K_{\text{alg}} \), where \( I_{K_{\text{alg}},\alpha} := \{ x \in K_{\text{alg}} : |x| \leq \alpha \} \). In general \( A_{n,\alpha} \) is not closed under composition. However, if \( F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha}, G_j(\xi) \in \mathcal{R}_{n,\beta} \) for \( j = 1, \ldots, m \) and \( |G_j(\xi)| \leq \alpha \) for all \( x \in (I_{K_{\text{alg}},\beta})^n \), then \( F(G_1(\xi), \ldots, G_m(\xi)) \in \mathcal{R}_{n,\beta} \). This is clear if \( F \in A_{n,\alpha} \) and the \( G_j \in A_{n,\beta} \). The general case follows easily.

For \( c, r \in K, r > 0 \), we denote the “closed interval” with center \( c \) and radius \( r \) by
\[
I(c, r) := \{ x \in K : |x - c| \leq r \}
\]
and for \( c, \delta, \varepsilon \in K, 0 < \delta < \varepsilon \), we denote the “closed annulus” with center \( c \), inner radius \( \delta \) and outer radius \( \varepsilon \) by
\[
A(c, \delta, \varepsilon) := \{ x \in K : \delta \leq |x - c| \leq \varepsilon \}.
\]

On occasion we will consider \( I(c, r) \) as a disc in \( K_{\text{alg}} \) and \( A(c, \delta, \varepsilon) \) as an annulus in \( K_{\text{alg}} \), replacing \( K \) by \( K_{\text{alg}} \) in the definitions. No confusion should result. Note that these discs and annuli are defined in terms of the real-closed order on \( K \), not the non-archimedean absolute value \( \| \cdot \| \), and hence are not discs or annuli in the sense of [BGR],[LR3] or [FP], which we will refer to as affinoid discs and affinoid annuli.

For \( I = I(c, r) \), \( A = A(c, \delta, \varepsilon) \) as above, we define the rings of analytic function on \( I \) and \( A \) as follows:
\[
\mathcal{O}_I := \{ f \left( \frac{x - c}{r} \right) : f \in \mathcal{R}_1 \}
\]
\[
\mathcal{O}_A := \{ g \left( \frac{\delta}{x - c} \right) + h \left( \frac{x - c}{\varepsilon} \right) : g, h \in \mathcal{R}_1, g(0) = 0 \}.
\]
The elements of \( \mathcal{O}_I \) (respectively, \( \mathcal{O}_A \)) are analytic functions on the corresponding \( K_{\text{alg}} \)-disc (respectively, annulus) as well.

**Remark 3.1.** (i) Elements of \( \mathcal{O}_A \) are multiplied using the relation 
\[
\frac{\delta}{\varepsilon} \cdot \frac{\varepsilon - c}{\varepsilon} = \frac{\delta}{\varepsilon}
\]
and the fact that \(|\frac{\delta}{\varepsilon}| < 1\). Indeed, let \( g(\xi_1) = \sum_i a_i \xi_1^i \), \( h(\xi_2) = \sum_j b_j \xi_2^j \). Then, using the relation \( \xi_1 \xi_2 = \frac{\delta}{\varepsilon} \), we have
\[
g \cdot h = \sum_{j<i} a_i b_j (\frac{\delta}{\varepsilon})^{j-i} \xi_1^i + \sum_{i<j} a_i b_j (\frac{\delta}{\varepsilon})^{j-i} \xi_2^j = f_1(\xi_1) + f_2(\xi_2).
\]
If \( g, h \in A_{1,\alpha} \) and \( \frac{\delta}{\varepsilon} \in \mathbb{R} \) then \( f_1, f_2 \in A_{1,\alpha} \), and this extends easily to the case \( g, h \in R_{1,\alpha} \) and \( \frac{\delta}{\varepsilon} \in K^\circ \). Lemma 3.6 will show that in fact the only case of an annulus that we must consider is when \( \frac{\delta}{\varepsilon} \in K^{oo} \).

(ii) We define the gauss-norm on \( \mathcal{O}_I \) by \( \|f(\frac{\varepsilon - c}{\varepsilon})\| := \|f(\xi)\| \), and on \( \mathcal{O}_A \) by \( \|g(\frac{\delta}{\varepsilon}) + h(\frac{\varepsilon - c}{\varepsilon})\| := \max\{\|g(\xi_1)\|,\|h(\xi_2)\|\} \). It is clear that the gauss-norm equals the supremum norm.

(iii) If \( f \in \mathcal{O}_I \) then \( \|\frac{\varepsilon - c}{\varepsilon} f\| = \|f\| \). If \( f \in \mathcal{O}_A \) then \( \|\frac{\varepsilon - c}{\varepsilon} f\| \leq \|f\| \) and if \( \frac{\delta}{\varepsilon} \in K^{oo} \), (i.e. is infinitesimal) then \( \|\frac{\varepsilon - c}{\varepsilon} g(\frac{\delta}{\varepsilon})\| = \|\frac{\delta}{\varepsilon}\| \cdot \|g(\frac{\delta}{\varepsilon})\| < \|g(\frac{\delta}{\varepsilon})\| \) and \( \|\frac{\varepsilon - c}{\varepsilon} h(\frac{\delta}{\varepsilon})\| = \|h(\frac{\delta}{\varepsilon})\| \).

(iv) If \( \|f\| < 1 \) then \( 1 - f \) is a unit in \( \mathcal{O}_A \). (In fact it is a strong unit - a unit \( u \) satisfying \( \|1 - u\| < 1 \).)

**Definition 3.2.** (i) We say that \( f \in \mathcal{O}_{I(0,1)} = R_1 \) has a zero close to \( a \in I(0,1) \) if \( f \), as a \( K_{\text{alg}} \)-function defined on the \( K_{\text{alg}} \)-disc \( \{|x|: |x| \leq \alpha\} \) for some \( \alpha > 1 \) has a \( K_{\text{alg}} \)-zero \( b \) with \( a - b \) infinitesimal in \( K_{\text{alg}} \). We say \( f \) has a zero close to \( I(0,1) \) if it has a zero close to \( a \) for some \( a \in I(0,1) \). For an arbitrary interval \( I = I(c,r) \) we say that \( f = F(\frac{\varepsilon - c}{\varepsilon}) \in \mathcal{O}_I \) has a zero close to \( a \in I \) if \( F \) has a zero close to \( \frac{\varepsilon - c}{\varepsilon} \in I(0,1) \), and that \( f \) has a zero close to \( I(c,r) \) if \( F \) has a zero close to \( I(0,1) \).

(ii) For \( 0 < a, b \in K^\circ \) we write \( a \sim b \) if \( \frac{a}{b}, \frac{b}{a} \in K^\circ \) and we write \( a \ll b \) if \( \frac{a}{b} \in K^{oo} \).

(iii) Let \( X \) be an interval or an annulus, and let \( f \) be defined on a superset of \( X \). We shall write \( f \in \mathcal{O}_X \) to mean that there is a function \( g \in \mathcal{O}_X \) such that \( f|_X = g \).

**Lemma 3.3.** If \( f \in \mathcal{O}_{I(c,r)} \) has no zero close to \( I(c,r) \), then there is a cover of \( I(c,r) \) by finitely many closed intervals \( I_j = I(c_j, r_j) \) such that \( \frac{1}{r} \in \mathcal{O}_{I_j} \) for each \( j \).

**Proof.** It is sufficient to consider the case \( I(c,r) = I(0,1) \). Cover \( I(0,1) \) by finitely many intervals \( I(c_j, r_j) \) and use Corollary 2.15. \( \Box \)

**Remark 3.4.** The function \( f(x) = 1 + x^2 \) has no zeros close to \( I(0,1) \). It is not a unit in \( \mathcal{O}_{I(0,1)} \), but it is a unit in both \( \mathcal{O}_{I(-\frac{1}{2}, \frac{1}{2})} \) and \( \mathcal{O}_{I(\frac{1}{2}, \frac{3}{2})} \). The function \( g(x) = x \) is not a unit in \( \mathcal{O}_{I(\delta, \epsilon)} \) for any \( 0 < \delta \in K^{oo} \) and \( 0 < \epsilon \in K^\circ \setminus K^{oo} \). It is of course a unit in \( \mathcal{O}_{A(0,\delta,\epsilon)} \). The function \( \frac{1}{g} = \frac{1}{x} \in A(0,\delta,1) \) for all \( \delta > 0 \) but is not in \( \mathcal{O}_I \).
for $I = I(\frac{1+\delta}{2}, \frac{1-\delta}{2})$, for any $\delta \in K^{\infty}$. Thus we see that if $X_1 \subset X_2$ are annuli or intervals it does not necessarily follow that $O_{X_2} \subset O_{X_1}$. However the following are clear. If $I_1 \subset I_2$ are intervals, then $O_{I_1} \subset O_{I_2}$. If $0 < \delta \in K^{\infty}$, $0 < r \in K^{\infty}$, $r < 1$, and $A = A(0, \delta, 1)$, $I = I(\frac{1+r}{2}, \frac{1-r}{2})$ then $O_A \subset O_I$. If $A_1 \subset A_2$ are annuli that have the same center, then $O_{A_2} \subset O_{A_1}$. If $0 < \delta < c$ and $0 < r < \frac{c}{\delta}$ for some $1 < \alpha \in \mathbb{R}$, and $I = I(c, r) \subset A(0, \delta, 1) = A$, then $O_A \subset O_I$. (Writing $x = c - y$, $|y| \leq r$ we see that $\frac{y}{x} = \frac{\delta}{c-y} = \frac{\delta}{x} + \frac{\delta}{c} \sum (\frac{y}{x})^k = \frac{\delta}{x} \sum (\frac{y}{x})^k$.)

Restating Corollary 2.15 we have

**Corollary 3.5.** If $f \in O_I(c, r)$ there is a polynomial $P \in K[\xi]$ and a unit $U \in O_I(c, r)$ such that $f(\xi) = P(\xi) \cdot U(\xi)$.

**Lemma 3.6.** If $\varepsilon < N\delta$ for some $N \in \mathbb{N}$, then there is a covering of $A(c, \delta, \varepsilon)$ by finitely many intervals $I_j$ such that for every $f \in O_{A(c, \delta, \varepsilon)}$ and each $j, f \in O_{I_j}$.

**Proof.** Use Lemma 3.3 or reduce directly to the case $\varepsilon = 1, 0 < \delta = r \in \mathbb{R}$ and the two intervals $[-1, -r] = I(-\frac{1+r}{2}, \frac{1-r}{2})$ and $[r, 1] = I(\frac{1+r}{2}, \frac{1-r}{2})$. □

The following Lemma is key for proving $o$-minimality.

**Lemma 3.7.** Let $f \in A(c, \delta, \varepsilon)$. There are finitely many intervals and annuli $X_j$ that cover $A(c, \delta, \varepsilon)$, polynomials $P_j$ and units $U_j \in O_{X_j}$ such that for each $j$ we have $f_{|X_j} = (P_j \cdot U_j)_{|X_j}$.

**Proof.** By the previous lemma, we may assume that $c = 0$, $\varepsilon = 1$ and $\delta \in K^{\infty}$ (i.e. $\delta$ is infinitesimal, say $\delta = \gamma$ for some $\gamma > 0$). Let

$$f(x) = g(\frac{\delta}{x}) + h(x) \quad \text{with} \quad g(\xi), h(\xi) \in \mathcal{R}_1, \ g(0) = 0,$$

and

$$g(\xi) = \sum_{i \in I \subset \mathbb{N}} t^{\alpha_i} \xi^{n_i} g_i(\xi),$$

with $n_i > 0, g_i(0) \neq 0, g_i \in A_{1, \alpha}$ for some $\alpha > 1$. Observe that

$$xg(\frac{\delta}{x}) = \sum_{i \in I} (t^{\alpha_i} \delta)(\frac{\delta}{x})^{n_i-1} g_i(\frac{\delta}{x}).$$

Hence (see Remark 3.1) for suitable $n \in \mathbb{N}$, absorbing the constant terms into $h$, we have

$$x^nf(x) = \overline{g}(\frac{\delta}{x}) + \overline{h}(x)$$

where $\overline{g}(0) = 0$ and $||\overline{g}|| < ||\overline{h}||$. (For use in Section 4, below, note that this argument does not use that $K$ is complete or of rank 1.) Multiplying by a constant, we may assume that $||\overline{h}|| = 1$. Let

$$\overline{g}(\xi) = \sum_{i \in I} t^{\beta_i} \xi^{m_i} \overline{g}_i(\xi), \ \text{with} \ m_i > 0 \ \text{and} \ \beta_i > 0 \ \text{for each} \ i,$$

and

$$\overline{h}(\xi) = \xi^{\gamma_0} \overline{h}_0(\xi) + \sum_{i \in J \subset \mathbb{N} \setminus 0} t^{\gamma_i} \overline{h}_i(\xi), \ \text{with} \ \overline{h}_0(0) \neq 0 \ \text{and the} \ \gamma_i > 0 \ \text{increasing}.$$
Since $\|g\| = \|t^{\delta_0}\| < 1$ there is a $\delta'$ with $\delta \leq \delta' \in K^\infty$ (i.e. $\|\delta'\| < 1$) such that $\|t^{\delta'}\| < \|t^{\delta_0}\|$ and $\|g\| < \|t^{\delta_0}\|$. Splitting off some intervals of the form $I(\frac{1}{\pi}, \frac{1}{\pi}) = [-1, -r]$ or $I(\frac{1}{\pi}, \frac{1}{\pi}) = [r, 1]$ for $r > 0$, $r \in \mathbb{R}$ (on which the result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume that $\overline{K}$, is a unit in $A_{1, \alpha}$, for some $\alpha > 1$. So

$$x^n f(x) = \overline{g}(\frac{\delta}{x}) + \overline{h}(x)$$

$$= \overline{h_0}(x)x^{k_0} \left[1 + \sum_{i=1}^{\infty} \left(\frac{\delta'}{x}\right)^{k_0} t^{\gamma_i} \frac{\overline{h_i}(x)}{\overline{h_0}(x)} + \frac{1}{\overline{h_0}(x)} \left(\frac{\delta}{x}\right)^{k_0} \left(\frac{\delta'}{x}\right)^{k_0} g\left(\frac{\delta}{\delta'}\frac{\delta'}{x}\right)\right].$$

By our choice of $\delta'$ and Remark 3.1 the quantity in square brackets is a (strong) unit. Hence we have taken care of an annulus of the form $A(0, \delta', 1)$ for some $\delta'$ with $|\delta| \leq |\delta'|$ and $|\delta'| < 1$.

Observe that the change of variables $y = \frac{\delta}{\pi}$ interchanges the sets $\{x : \|y\| = |\delta|\}$ and $\{x : \|y\| = 1\}$ and $\{y : \|y\| = 1, |y| \leq 1\}$. Hence, as above, there is a $\delta'' \in K^\infty$ with $|\delta| < |\delta''|$ and a covering of the annulus $\delta \leq |x| \leq \delta''$ by finitely many intervals and annuli with the required property.

It remains to treat the annulus $\delta'' \leq |x| \leq \delta'$. Using the terminology of [LR3], observe that on the much bigger affineoid annulus $|\delta''| \leq |x| \leq |\delta'|$ the function $f$ is strictly convergent, indeed even overconvergent. Hence, as in [LR3] Lemma 3.6, on this affinoid annulus we can write

$$f = \frac{P(x)}{x^\ell} \cdot U(x)$$

where $P(x)$ is a polynomial and $U(x)$ is a strong unit (i.e. $\|U(x)-1\| < 1$).

**Corollary 3.8.** If $X$ is an interval or an annulus and $f \in \mathcal{O}_X$, then the set $\{x \in X : f(x) \geq 0\}$ is semialgebraic (i.e. a finite union of (closed) intervals).

**Proof.** This is an immediate corollary of 3.5 and 3.7 since units don’t change sign on intervals and since an annulus has two intervals as connected components. □

**Definition 3.9.** For $c = (c_1, \ldots, c_n)$, $r = (r_1, \ldots, r_n)$ we define the poly–interval

$I(c, r) := \{x \in K^n : |x_i - c_i| \leq r_i, i = 1, \ldots, n\}$. This also defines the corresponding polydisc in $(K_{alg})^n$. The ring of analytic functions on this poly–interval (or polydisc) is

$$\mathcal{O}_{I(c, r)} := \left\{ f \left(\frac{x_1 - c_1}{r_1}, \ldots, \frac{x_n - c_n}{r_n}\right) : f \in \mathcal{R}_r \right\}.$$ 

**Lemma 3.10.** Let $\alpha, \beta > 1, F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m, \alpha}$ and $G_j(\xi_1, \ldots, \xi_n) \in \mathcal{R}_{n, \beta}$ with $\|G_j\| \leq 1$ for $j = 1, \ldots, m$. Let $X = \{x \in [-1, 1]^n : |G_j(x)| \leq 1\}$ for $j = 1, \ldots, m$. There are (finitely many) $c_i = (c_{i1}, \ldots, c_{in}) \in \mathbb{R}^n$, $\varepsilon_i \in \mathbb{R}$, $\varepsilon_i > 0$ with $\varepsilon_i < |c_{ij}|$ if $c_{ij} \neq 0$ such that the (poly) intervals $I_i = \{x \in K : |x_i - c_{ij}| < \varepsilon_i$ for $j = 1, \ldots, n\}$ cover $X$, $|G_j(x)| < \alpha$ for all $x \in I_i$, $j = 1, \ldots, m$, and there are $H_i \in \mathcal{O}_{I_i}$ such that

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}.$$ 

**Proof.** Use the compactness of $[-1, 1]^n \cap \mathbb{R}^n$ and the following facts. If $\|G_j\| = 1$ then $|G_j(x) - G_{j0}(x)|$ is infinitesimal for all $x \in K^\infty$, where $G_{j0}$ is the top slice of $G_j$. If $\|G_j\| < 1$ then $|G_j(x)| \in K^\infty$ for all $x \in [-1, 1]^n$. □
Corollary 3.11. (i) Let $I$ be an interval, $F(\eta_1, \ldots, \eta_m) \in R_m$, and $G_j \in O_1$ for $j = 1, \ldots, m$. Then there are finitely many intervals $I_i$ covering $I$ and functions $H_i \in O_1$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}$$

(ii) Let $A$ be an annulus, $F(\eta_1, \ldots, \eta_m) \in R_m$, and $G_j \in O_A$ for $j = 1, \ldots, m$. Then there are finitely many $X_i$, each an interval or an annulus, covering $A$ and $H_i \in O_{X_i}$ such that for each $i$

$$F(G_1, \ldots, G_m)|_{X_i} = H_i|_{X_i}.$$  

Proof. Part (i) reduces to Lemma 3.10 once we see that if $|G_j| > 1$ we can use Corollary 2.15 to restrict to the subintervals of $I$ around the zeros of $G_j$ on which $|G_j| \leq C$ for some $1 < C \in R$. On the rest of $I$, $F(G_1, \ldots, G_m)$ is zero.

For (ii), we may assume that $A = A(0, \delta, 1)$ with $\delta$ infinitesimal and that $G_j(x) = G_{j_1}(\frac{x}{\delta}) + G_{j_2}(x)$. As in (i), we may reduce to the case that $|G_j| \leq 1$, using Lemma 3.7 instead of Corollary 2.15, and using Lemma 3.6 and Remark 3.4. Apply Lemma 3.10 to the functions $F$ and $G_j(\xi_1, \xi_2) = G_{j_1}(\xi_1) + G_{j_2}(\xi_2)$. The case $c = (0, 0)$ gives us the annulus $|\frac{\delta}{x}| \leq \varepsilon$, $|x| \leq \varepsilon$ i.e. $\frac{\delta}{x} \leq |x| \leq \varepsilon$. The case $c = (0, c_2)$ with $c_2 \neq 0$ gives us $\frac{\delta}{x} \leq \varepsilon$ and $|x - c_2| \leq \varepsilon$ with $\varepsilon < |c_2|$. This is equivalent to $|x - c_2| \leq \varepsilon$ since $c, c_1 \in R$, and hence equivalent to $-\varepsilon \leq x - c_2 \leq \varepsilon$ or $c_2 - \varepsilon \leq x \leq c_2 + \varepsilon$ which is an interval bounded away from 0. The case $c = (c_1, 0)$ gives us $|\frac{\delta}{x} - c_1| \leq \varepsilon$, $|x| \leq \varepsilon$ (since $\varepsilon < |c_1|$) which is equivalent to $|\frac{\delta}{x} - c_1| \leq \varepsilon$ or $c_1 - \varepsilon \leq \frac{\delta}{x} \leq c_1 + \varepsilon$ or (considering the case $c_1 > 0$, the case $c_1 < 0$ is similar) $\frac{\delta}{c_1 + x} \leq x \leq \frac{\delta}{c_1 - x}$ which is part of an annulus that can be reduced to intervals using Lemma 3.6. The case $c = (c_1, c_2)$ with both $c_1, c_2 \neq 0$ is vacuous since either $x$ or $\frac{\delta}{x}$ is infinitesimal on $A$ and $\varepsilon < |c_1|, |c_2|$.

Lemma 3.12. Let $X$ be an interval or an annulus and let $f, g \in O_X$. There are finitely many subintervals and subannuli $X_i \subset X$, $i = 1, \ldots, \ell$ such that

$$\{ x \in X : |f(x)| \leq |g(x)| \} \subset \bigcup X_i \subset X,$$

and, except at finitely many points,

$$\frac{f}{g}|_{X_i} \in O_{X_i}.$$  

Proof. We consider the case that $X$ is an interval, $I$, and may take $I = I(0, 1) = [-1, 1]$. We may assume by Corollary 3.5 that $f$ and $g$ have no common zero. If $g$ has no zeros close to $I(0, 1)$, we are done by Lemma 3.3. Let $\alpha_1, \ldots, \alpha_n$ be the distinct elements of $[-1, 1] \cap R$ such that $g$ has at least one zero close to (i.e. within an infinitesimal of) $\alpha$. Breaking into subintervals and making changes of variables we may assume that $n = 1$ and that $\alpha_1 = 0$. Again making a change of variables (over $K$) we may assume that $g$ has at least one zero with zero “real” part, i.e. of the form $a = \sqrt{-1} \alpha$ for some $\alpha \in K$. Let $N$ denote the number of zeros of $f \cdot g$ close to 0 in $I(0, 1)$. Let $\delta = 3 \cdot \max\{|x| : x \in K_{alg} \text{ close to 0 and } f(x) \cdot g(x) = 0\}$. If $\delta = 0$, then $a = \alpha = 0$ and there is no other zero of $f \cdot g$ close to zero in $I(0, 1)$. Then there is a $\delta' > 0$ such that for $|x| < \delta'$ we have $|g(x)| < |f(x)|$. Then the interval $I(0, \delta')$ drops away, and on the annulus $A(0, \delta', 1)$ the function $g$ is a unit. If $\delta > 0$, we consider the interval $I(0, \delta)$ and the annulus $A(0, \delta, 1)$ separately, and
proceed by induction on \( N \). So suppose \( \delta > 0 \) and \( f \cdot g \) has \( N \) zeros close to 0 in \( I(0,1) \). Let \( \alpha \) be as above. If \( \alpha \sim \delta \) then the zero \( a = \sqrt{-1} \alpha \) is not close to \( I(0,\delta) \) and by restricting to \( I(0,\delta) \) we have reduced \( N \). If \( |\alpha| < \delta \) then this zero is close to 0 in \( I(0,\delta) \), but the largest zeros (those of size \( \delta/3 \)) are not close to 0 in \( I(0,\delta) \), and hence restricting to \( I(0,\delta) \) again reduces \( N \).

It remains to consider the case of the annulus \( A(0,\delta,1) = \{ x : \delta \leq |x| \leq 1 \} \) where all the zeros of \( g \) are within \( \delta/3 \) of 0. By Lemma 3.7 we may assume that \( g(x) = P(x) \cdot U(x) \) where \( U \) is a unit and all the zeros of \( P \) are within \( \delta/3 \) of 0. Let \( \alpha_i, i = 1, \ldots, \ell \) be these zeros. For \( |x| \geq \delta \) we may write \( \frac{1}{x-\alpha_i} = \frac{1}{x} \frac{1}{1-\alpha_i/x} = \frac{1}{x} \sum_{j=0}^{\infty} \left( \frac{\alpha_i}{x} \right)^j = \frac{1}{x} \sum_{j=0}^{\infty} \left( \frac{\alpha_i}{x} \right)^j \left( \frac{1}{x} \right)^j \) and \( |\alpha_i| \leq \frac{1}{4} \). Hence \( \frac{1}{g} \in O_{A(0,\delta,1)} \). This completes the case that \( X \) is an interval.

The case that \( X \) is an annulus is similar – one can cover \( X \) with finitely many subannuli \( X_i \) and subintervals \( Y_j \) so that for each \( i \) \( g|_{X_i} \) is a unit in \( O_{X_i} \), and for each \( j \) \( f|_{Y_j} \cdot g|_{Y_j} \in O_{Y_j} \).

From Corollary 3.11 and Lemma 3.12 we now have by induction on terms:

**Proposition 3.13.** Let \( f_1, \ldots, f_\ell \) be \( L \)-terms in one variable, \( x \). There is a covering of \([-1,1]\) by finitely many intervals and annuli \( X_i \) such that except for finitely many values of \( x \), we have for each \( i \) and \( j \) that \( f_i|_{X_i} \in O_{X_i} \) (i.e. \( f_i|_{X_i} \) agrees with an element of \( O_{X_i} \), except at finitely many points of \( X_i \)).

This, together with Corollary 3.8 gives

**Theorem 3.14.** \( K \) is \( o \)-minimal in \( L \).

4. FURTHER EXTENSIONS

In this section we give extensions of the results of Sections 2 and 3 and the results of [LR3].

Let \( G \) be an (additive) ordered abelian group. Let \( t \) be a symbol. Then \( t^G \) is a (multiplicative) ordered abelian group. Following the notation of [DMM1] and [DMM2] (but not [DMM3] or [LR2]) we define \( \mathbb{R}((t^G)) \) to be the maximally-complete valued field with additive value group \( G \) (or multiplicative value group \( t^G \)) and residue field \( \mathbb{R} \). So

\[
\mathbb{R}((t^G)) := \left\{ \sum_{g \in I} a_g t^g : a_g \in \mathbb{R} \text{ and } I \subset G \text{ well-ordered} \right\}.
\]

We shall be a bit sloppy about mixing the additive and multiplicative valuations. \( I \subset G \) is well-ordered exactly when \( t^I \subset t^G \) is reverse well-ordered. The field \( K \) of Puiseux series, or its completion, is a proper subfield of \( \mathbb{R}_1 := \mathbb{R}((t^G)) \). Considering \( G = \mathbb{Q}^m \) with the lexicographic ordering, we define

\[
\mathbb{R}_m := \mathbb{R}((t^{\mathbb{Q}^m})).
\]

It is clear that if \( G_1 \subset G_2 \) as ordered groups, then \( \mathbb{R}((t^{G_1})) \subset \mathbb{R}((t^{G_2})) \) as valued fields. Also, \( \mathbb{R}((t^{G})) \) is Henselian and, if \( G \) is divisible, then \( \mathbb{R}((t^{G})) \) is real-closed. We shall continue to use \( < \) for the corresponding order on \( \mathbb{R}((t^G)) \).
In analogy with Section 2, we define

**Definition 4.1.**

\[
\mathcal{R}_{n,\alpha}(G) := A_{n,\alpha} \hat{\otimes} \mathbb{R}((t^G)) \\
:= \{ \sum_{g \in I} f_g t^g : f_g \in A_{n,\alpha} \text{ and } I \subset G \text{ well ordered} \}
\]

\[
\mathcal{R}_n(G) := \bigcup_{\alpha>1} \mathcal{R}_{n,\alpha}(G)
\]

\[
\mathcal{R}(G) := \bigcup_{n} \mathcal{R}_n(G).
\]

As in Section 2, the elements of \( \mathcal{R}_n(G) \) define functions from \( I^n \subset \mathbb{R}((t^G)) \) to \( \mathbb{R}((t^G)) \). Indeed this interpretation is a ring endomorphism. In other words, the field \( \mathbb{R}((t^G)) \) has analytic \( \mathcal{R}(G) \)-structure. (See [CLR] and especially [CL] for more about fields with analytic structure.) The elements of \( \mathcal{R}_n(G) \) are interpreted as zero on \( \mathbb{R}((t^G)) \setminus I^n \).

**Theorem 4.2.** The Weierstrass Preparation Theorem (Theorem 2.5) and the Strong Noetherian Property (Theorem 2.10) hold with \( \mathcal{R} \) replaced by \( \mathcal{R}(G) \).

**Proof.** Only minor modifications to the proofs of Theorems 2.5 and 2.10 are needed.

The arguments of Sections 2 and 3 show

**Theorem 4.3.** If \( G \) is divisible then \( \mathbb{R}((t^G)) \) admits quantifier elimination and is \( o \)-minimal in \( \mathcal{L}_G \).

**Corollary 4.4.** If \( G_1 \subset G_2 \) are divisible, then \( \mathbb{R}((t^{G_1})) \prec \mathbb{R}((t^{G_2})) \) in \( \mathcal{L}_{G_1} \).

We shall show in Section 5 that, though for \( m < n \) we have \( \mathbb{R}_m \not\prec \mathbb{R}_n \) in \( \mathcal{L}_{\mathbb{Q}^m} \), there is a sentence of \( \mathcal{L}_{\mathbb{Q}^n} \) that is true in \( \mathbb{R}_n \) but is not true in any \( o \)-minimal expansion of \( \mathbb{R}_m \).

The results of [LR3] also extend to this more general setting.

**Definition 4.5.** We define the ring of *strictly convergent power series over \( \mathbb{R}((t^G)) \) as

\[
\mathbb{R}((t^G))^* := \{ \sum_{g \in I} a_g(\xi) t^g : a_g(\xi) \in \mathbb{R}[\xi] \text{ and } I \text{ well ordered} \},
\]

and the subring of *overconvergent power series over \( \mathbb{R}((t^G)) \) as

\[
\mathbb{R}((t^G))^*(\mathbb{Q}) := \{ f : f(\gamma \xi) \in \mathbb{R}((t^G))^* (\xi) \text{ for some } \gamma \in \mathbb{R}((t^G)), \| \gamma \| > 1 \},
\]

\[
\mathcal{R}(G)_{over} := \bigcup_n \mathbb{R}((t^G))^*(\{\xi_1, \ldots, \xi_n\}),
\]
and the corresponding overconvergent language as
\[ \mathcal{L}_{G, over} := (+, \cdot, ^{-1}, 0, 1, <, R(G)_{over}). \]

As in [LR3] we have

**Theorem 4.6.** If \( G \) is divisible then \( \mathbb{R}((t^G)) \) admits quantifier elimination and is \( \sigma \)-minimal in \( \mathcal{L}_{G, over} \).

Of course the \( \sigma \)-minimality follows immediately from Theorem 4.3.

5. **Extensions of the example of Hrushovski and Peterzil [HP]**

In this section we show that with minor modifications, the idea of the example of [HP] can be iterated to give a nested family of examples. This relates to a question of Hrushovski and Peterzil whether there exists a small class of \( \sigma \)-minimal structures such that any sentence, true in some \( \sigma \)-minimal structure, can be satisfied in an expansion of a model in the class. Combining with expansions with the exponential function, one perhaps can elaborate the tower of examples further.

Consider the functional equation

\[ (* \star) \quad F(\beta z) = \alpha z F(z) + 1, \]

and suppose that \( F \) is a “complex analytic” solution for \( |z| \leq 1 \). By this we mean that, writing \( z = x + \sqrt{-1}y \), \( F(z) = f(x, y) + \sqrt{-1}g(x, y) \), \( F(z) \) is differentiable as a function of \( z \). This is a definable condition on the two “real” functions, \( f, g \) of the two “real” variables \( x, y \). Then

\[ F(z) = \sum_{k=0}^{\infty} a_k z^k \]

where

\[ a_k = \frac{\alpha^k}{\beta^{k(k+1)}}. \]

(\( \alpha, \beta \) are parameters).

By this we mean that for each \( n \in \mathbb{N} \) there is a constant \( A_n \) such that

\[ (** \star) \quad |F(z) - \sum_{k=0}^{n} a_k z^k| \leq A_n |z|^{n+1} \]

is true for all \( z \) with \( |z| \leq 1 \). Indeed, by [PS] Theorem 2.50, one can take \( A_n = C \cdot 2^{n+1} \), for \( C \) a constant independent of \( n \).

Consider the following statement: \( F(z) \) is a complex analytic function (in the above sense) on \( |z| \leq 1 \) that satisfies \( (*) \); the number \( \beta > 0 \) is within the radius of convergence of the function \( f(z) = \sum_{n=1}^{\infty} (n-1)!z^n \) and \( \alpha > 0 \).

This statement is not satisfiable by any functions in any \( \sigma \)-minimal expansion of the field of Puiseux series \( K_1 \), or the maximally complete field \( \mathbb{R}_1 = \mathbb{R}((t^2)) \), because, if it were, we would have \( \|\beta\| = \|t^\gamma\|, \|\alpha\| = \|t^\delta\| \), for some \( 0 < \gamma, \delta \in \mathbb{Q} \), and for suitable choice of \( n \) the condition \( (** \star) \) would be violated. On the other
hand, if we choose $\alpha, \beta \in \mathbb{R}_2$ with $\text{ord}(\alpha) = (1, 0)$ and $\text{ord}(\beta) = (0, 1)$ then $\Sigma a_k z^k \in \mathbb{R}_2 \langle \langle z \rangle \rangle^*$ satisfies the statement on $\mathbb{R}_2$.

This process can clearly be iterated to give, in the notation of Section 4,

**Proposition 5.1.** For each $m$ there is a sentence of $\mathcal{L}_{Q_m}$ true in $\mathbb{R}_m$ but not satisfiable in any $o$–minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$.

**References**


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