Name: ______________________________

MA 266, Fall 2009, Quiz 3

(1) (10 points) Determine the solution of the following initial value problem. Determine the interval in which the solution is defined.

$$y' = (1 - 2x)/y, \quad y(1) = -2.$$

Solution: This equation is separable, since it can be written as $yy' = 1 - 2x$. Integrating we get

$$\frac{1}{2}y^2 = x - x^2 + C.$$

Setting $y(1) = -2$, we see that $C = 2$, so

$$y^2 = 2x - 2x^2 + 4 \text{ or } y = -\sqrt{2x - 2x^2 + 4}.$$

(You get the minus sign before the square root because $y(1) < 0$.)

The solution exists where $2x - 2x^2 + 4 \geq 0$, or $x^2 - x - 2 = (x - 2)(x + 1) \leq 0$ or for x between -1 and 2.

(2) (10 points) Determine the solution of the following initial value problem. Determine the interval in which the solution is defined.

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2, \quad y(1) = 1.$$

Solution: You can write the equation as $y' = 1 + (y/x) + (y/x)^2$, so it is homogeneous. So we set $v = y/x$ or $y = xv$ and $y' = xv' + v$. So

$$xv' + v = 1 + v + v^2 \text{ or } \frac{1}{1 + v^2}v' = \frac{1}{x}.$$

Integrating gives

$$\ln|x| + C = \arctan v = \arctan(y/x).$$

Using $y(1) = 1$ shows that $C = \arctan 1 = \pi/4$, so the solution is $\ln|x| + \pi/4 = \arctan(y/x)$. Solving for y gives

$$y = x \tan\big(\ln x + \pi/4\big) \quad (|x| = x \text{ because } x_0 = 1 > 0).$$

Now, $\tan t$ blows up when t is an odd multiple of $\pi/2$; $\ln x_0 + \pi/4 = \ln 1 + \pi/4 = \pi/4$, which is between $-\pi/2$ and $\pi/2$, so we need $-\pi/2 < \ln x + \pi/4 < \pi/2$. Solving for x gives $e^{-3\pi/4} < x < e^{\pi/4}$. With a calculator, one finds

$$.09478022484215486 < x < 2.1932800507380152.$$

(3) (10 points) Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/m, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

Solution: The volume of liquid in the tank is always 200 L. Let $Q(t)$ be the amount of die in the tank, measured in grams. The concentration of die in the tank at any given time is given by $Q(t)/200$.

We have $Q(0) = 200 \text{ L} \times 1 \text{ g/L} = 200$ g. The rate of change of $Q(t)$ is given by

$$\begin{aligned}\frac{dQ(t)}{dt} &= \text{rate of die going into tank} - \text{rate of dye leaving tank}\\ &= 0 \text{ g/L} \times 2 \text{ L/m} - \frac{Q(t)}{200} \text{ g/L} \times 2 \text{ L/min}\\ &= -\frac{1}{100}Q(t) \text{ g/m}\end{aligned}$$

Solving for $Q(t)$ gives

$$Q(t) = 200e^{-t/100} \text{ g.}$$

There will be 2 g of die in the tank when it is 1% of the original value. So we want to find the time $\bar{t}$ when

$$2 \text{ g} = 200\, e^{-\bar{t}/100} \text{ g,}$$

or

$$\bar{t} = -100 \ln(1/100) \text{ m} = 100 \ln 100 \text{ m.}$$

If you use a calculator, you see that is is about 460.51701859880916 m, or a bit less than 8 hours.