Name: ______________________________

MA 266, Fall 2009, Quiz 4

(1) (10 points) Use Euler's method with stepsize $h = \frac{1}{2}$ to approximate the value of $y(1)$ if $y(t)$ satisfies

$$y'(t) = \frac{1}{2}y(t) + 1 - t, \quad y(0) = 1.$$

You may wish to use the following table to organize your computations.

k	t_k	y_k	$f(t_k, y_k)$	$y_{k+1} = y_k + h\,f(t_k, y_k)$
0	0	1	3/2	7/4
1	1/2	7/4	11/8	39/16
2	1	39/16		

Solution: We have $y' = f(t, y)$ with $f(t, y) = \frac{1}{2}y + 1 - t$. The stepsize h is 1/2, so $t_k = kh$ in general, and $t_0 = 0$, $t_1 = 1/2$, and $t_2 = 1$ in particular. Thus $y(1) = y(t_2)$, so we want to approximate $y(1)$ by y_2.

So we fill in the table. We already said what the t_ks are, and we have y_0=1.

The $k = 0$ line: $t_0 = 0$ and $y_0 = 1$ and

$$f(t_0, y_0) = \frac{1}{2}y_0 + 1 - t_0 = \frac{1}{2} \times 1 + 1 - 0 = \frac{3}{2};$$
$$y_1 = y_0 + h\,f(t_0, y_0) = 1 + \frac{1}{2} \times \frac{3}{2} = \frac{7}{4}.$$

The $k = 1$ line: $t_1 = 1/2$ and

$$y_1 = \frac{7}{4} \quad \text{(copied from the } k = 0 \text{ line);}$$
$$f(t_1, y_1) = \frac{1}{2}y_1 + 1 - t_1 = \frac{1}{2} \times \frac{7}{4} + 1 - \frac{1}{2} = \frac{11}{8};$$
$$y_2 = y_1 + h\,f(t_1, y_1) = \frac{7}{4} + \frac{1}{2} \times \frac{11}{8} = \frac{39}{16}.$$

And finally, on the $k = 2$ line: $t_2 = 1$, and

$$y_2 = \frac{39}{16} \quad \text{(copied from the } k = 1 \text{ line)}$$

So we have $y(1) = y(t_2) \approx y_2 = 39/16$.

(2) (10 points) What are the equilibrium solutions of

$$y'(t) = y(t)^3 - y(t)?$$

Classify each equilibrium solution as stable, unstable, or semi-stable.

Solution: We have $y' = f(y) = y^3 - y = y(y^2 - 1) = y(y-1)(y+1)$ so the equilibrium solutions are $y = 0$, $y = 1$, and $y = -1$. We have

$$y' = f(y) \text{ is } \begin{cases} > 0 & \text{for } 1 < x, \\ < 0 & \text{for } 0 < x < 1, \\ > 0 & \text{for } -1 < x < 0, \\ < 0 & \text{for } x < -1. \end{cases}$$

Therefore, the solutions $y = 1$ and $y = -1$ are unstable, and the solution $y = 0$ is stable.

(3) (10 points) Determine the general solution of the differential equation

$$\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)\, dy = 0.$$

Solution: The equation can be written

$$M(x,y)\, dx + N(x,y)\, dy = 0,$$

with

$$M(x,y) = \left(\frac{y}{x} + 6x\right) \text{ and } N(x,y) = (\ln x - 2),$$

so we have to assume that $x > 0$ for $N(x,y)$ to be defined. We see that

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x},$$

so the equation is exact. So there exists a function $\psi(x,y)$ such that $\partial\psi/\partial x = M$ and $\partial\psi/\partial y = N$. We calculate

$$\psi(x,y) = \int M(x,y)\, dx + h(y) = \int \left(\frac{y}{x} + 6x\right) dx + h(y) = y \ln x + 3x^2 + h(y).$$

Using $N(x,y) = \partial\psi/\partial y$ gives

$$(\ln x - 2) = \frac{\partial \psi}{\partial y} = \ln x + h'(y),$$

so $h'(y) = -2$ and $h(y) = -2y$. So the general solution is

$$\psi(x,y) = y \ln x + 3x^2 - 2y = C.$$