Name: ______________________________

MA 266, Fall 2009, Quiz 6

(1) (10 points) A certain vibrating system satisfies the equation

$$u'' + \gamma u' + u = 0, \quad \gamma \geq 0.$$

Find the value of the damping coefficient γ for which the quasi period of the damped motion is 50% greater than the period of the corresponding undamped motion.

Solution: The characteristic formula of the differential equation is $F(r) = r^2 + \gamma r + 1$ and the roots of $F(r) = 0$ when $0 \leq \gamma^2 < 4$ are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2} = -\frac{\gamma}{2} \pm i\sqrt{1 - \frac{\gamma^2}{4}}.$$

When $\gamma = 0$ then $r = \pm i$ and the general solution is $u = c_1 \cos t + c_2 \sin t$ and the period is 2π. When $0 < \gamma < 2$ then the roots still have nonzero imaginary part $\mu = \sqrt{1 - \gamma^2/4}$ and the solution

$$u(t) = e^{-\gamma t/2}(c_1 \cos \mu t + c_2 \sin \mu t)$$

with quasi period $2\pi/\mu$. The condition in the statement of this problem is that

$$\frac{2\pi}{\mu} = \frac{3}{2} \times 2\pi,$$

which is that $1 - \gamma^2/4 = 4/9$ or $\gamma^2/4 = 5/9$ or $\gamma = 2\sqrt{5}/3$.

(2) (10 points) Determine the general solution of

$$y^{(\text{iv})} - 2y''' + 2y'' - 2y' + y = 0.$$

Solution: The characterstic equation of the differential equation is $F(r) = r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$. By guessing one finds that $F(1) = 0$ so $r = 1$ is a root, and by long division and guessing again, one finds again that $r = 1$ is a root. So one finds that $F(r) = (r-1)^2(r^2+1)$ and the general solution is

$$y(t) = c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t.$$

(3) (10 points) Determine the general solution of

$$y'' + 2y' + 2y = e^{-t} + \sin 2t. \tag{1}$$

Solution: The characteristic equation of the associated homogeneous problem

$$y'' + 2y' + 2y = 0$$

is $F(r) = r^2 + 2r + 2 = 0$, the roots of which are

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 2}}{2} = -1 \pm i,$$

so the solution of the associated homogeneous problem is

$$y_h(t) = e^{-t}(c_1 \cos t + c_2 \sin t).$$

We'll use the method of undetermined coefficients to find a particular solution y_p of the inhomogenous problem. We have for a candidate solution

$$y_p(t) = Ae^{-t} + B\cos 2t + C\sin 2t,$$

and none of the terms on the right hand size are in the homogeneous solution, so we know that form will work. We find that

$$\begin{aligned} y_p'(t) &= -Ae^{-t} - 2B\sin 2t + 2C\cos 2t, \\ y_p''(t) &= \quad Ae^{-t} - 4B\cos 2t - 4C\sin 2t \end{aligned}$$

Plugging this into (1) gives

$$\begin{aligned} Ae^{-t} - 4B\cos 2t - 4C\sin 2t + 2(-Ae^{-t} - 2B\sin 2t + 2C\cos 2t) + 2(Ae^{-t} + B\cos 2t + C\sin 2t) \\ = e^{-t} \times A + \cos 2t \times (-2B + 4C) + \sin 2t \times (-2C - 4B) = e^{-t} + \sin 2t. \end{aligned}$$

So $A = 1$ and $-2B + 4C = 0$ and $-4B - 2C = 1$. Subtracting twice the second equation from the third gives $-4B - 2C - 2(-2B + 4C) = -10C = 1$, so $C = -1/10$ and $B = 2C = -1/5$. So the general solution is

$$y(t) = y_h(t) + y_p(t) = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t} - \frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t.$$