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MA 266, Fall 2009, Quiz 7

(1) (10 points) Determine the inverse Laplace transform of

$$\frac{s+3}{s^2+2s+3}e^{-\pi s}.$$

Solution: Let

$$F(s) = \frac{s+3}{s^2+2s+3}e^{-\pi s};$$

then $F(s) = e^{-\pi s}G(s)$ and $f(t) = u_\pi(t)g(t-\pi)$, where

$$G(s) = \frac{s+3}{s^2+2s+3} = \frac{s+3}{(s+1)^2+2} = \frac{(s+1)+2}{(s+1)^2+2}.$$

So $G(s) = H(s+1)$ and $g(t) = e^{-t}h(t)$, where

$$H(s) = \frac{s+2}{s^2+2} = \frac{s}{s^2+2} + \frac{2}{s^2+2} = \frac{s}{s^2+2} + \sqrt{2}\frac{\sqrt{2}}{s^2+2},$$

so $a = \sqrt{2}$ in the tables and

$$h(t) = \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t.$$

Working backward, we get

$$g(t) = e^{-t}h(t) = e^{-t}\big(\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t\big)$$

and

$$f(t) = u_\pi(t)g(t-\pi) = u_\pi(t)e^{-(t-\pi)}\big(\cos\sqrt{2}(t-\pi) + \sqrt{2}\sin\sqrt{2}(t-\pi)\big).$$

(2) (10 points) Use Laplace transforms to determine the general solution of

$$y'' - 3y' + 2y = e^{-t}, \quad y(0) = 0,\ y'(0) = 1.$$

Solution: Let $Y(s) = \mathcal{L}[y(t)]$; then $\mathcal{L}[y'(t)] = sY(s) - y(0)$ and $\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$. In addition, $\mathcal{L}[e^{-t}] = 1/(s+1)$. So we have

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = (s^2 - 3s + 2)Y(s) - 1 = \frac{1}{s+1}.$$

So

$$(s-2)(s-1)Y(s) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1},$$

or

$$Y(s) = \frac{s+2}{(s+1)(s-2)(s-1)}.$$

We use partial fractions and write

$$Y(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-1}$$

or

$$s + 2 = A(s-2)(s-1) + B(s+1)(s-1) + C(s+1)(s-2).$$

Let $s = 2$ to find

$$2 + 2 = B \times 3 \times 1, \text{ so } B = 4/3;$$

let $s = 1$ to see that

$$1 + 2 = C \times 2 \times -1, \text{ so } C = -3/2;$$

let $s = -1$ to see that

$$-1 + 2 = A \times -3 \times -2, \text{ so } A = 1/6.$$

So

$$Y(s) = \frac{1}{6}\frac{1}{s+1} + \frac{4}{3}\frac{1}{s-2} - \frac{3}{2}\frac{1}{s-1},$$

and

$$y(t) = \frac{1}{6}e^{-t} + \frac{4}{3}e^{2t} - \frac{3}{2}e^{t}.$$

(3) (10 points) Determine the general solution of

$$y'' + y = g(t), \quad y(0) = 1,\ y'(0) = 1,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t. \end{cases}$$

Give a formula for $y(t)$ for $t > \pi$ that does not involve $u_\pi(t)$.

Solution: We can write $g(t)$ as $g(t) = 1 - u_\pi(t)$, so we have

$$y'' + y = 1 - u_\pi(t), \quad y(0) = 1,\ y'(0) = 1,$$

Taking Laplace transforms, we find

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = s^2Y(s) - s - 1 + Y(s) = \frac{1}{s}(1 - e^{-\pi s}),$$

so

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s(s^2+1)}(1 - e^{-\pi s}).$$

The inverse Laplace transforms of the first two terms on the right are $\cos t$ and $\sin t$. To find the inverse Laplace transform of the last term, we write

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1},$$

so

$$1 = A(s^2+1) + (Bs+C)s = s^2(A+B) + s \times C + A \times 1,$$

so $A = 1$, $C = 0$, and $B = -1$, so

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1},$$

whose inverse Laplace transform is $1 - \cos t$. We now use the formula for the inverse Laplace transform of $e^{-cs}G(s)$ to see that

$$\begin{aligned} y(t) &= \cos t + \sin t + 1 - \cos t - u_\pi(t)[1 - \cos(t - \pi)] \\ &= \sin t + 1 - u_\pi(t)[1 - \cos(t - \pi)] \\ &= \sin t + 1 - u_\pi(t)[1 + \cos t]. \end{aligned}$$

For $t > \pi$, $u_\pi(t) = 1$, so for $t > \pi$ we have

$$y(t) = \sin t + 1 - [1 + \cos t] = \sin t - \cos t.$$