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MA 366, Spring 2010, Quiz 6

(1) (5 points) Determine the Laplace transform of $f(t) = t \times u_\pi(t)$.
Solution: To make it match line 13 in the table we write

$$f(t) = t \times u_\pi(t) = (t - \pi + \pi)u_\pi(t) = (t - \pi)u_\pi(t) + \pi u_\pi(t).$$

So we can use lines 13 and 3 (for the Laplace transforms of 1 and t) to get

$$F(s) = e^{-\pi s}\frac{1}{s^2} + \pi e^{-\pi s}\frac{1}{s}.$$

(5 points) Determine the inverse Laplace transform of $F(s) = \dfrac{s}{s^2 + 4s + 13}$.
Solution: We can't factor the denominator, so we write

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

This looks a bit like lines 9 and 10, but F doesn't have the right numerator for either of them. So we write

$$F(s) = \frac{s+2-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{3}\frac{3}{(s+2)^2+9},$$

and now the first term matches line 10 and the second matches line 9, so

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t.$$

(2) (10 points) Determine the solution $y(t)$ of the differential equation

$$y'(t) - 2y(t) = u_2(t), \quad y(0) = 1.$$

Solution: We let $Y(s) = \mathcal{L}[y(t)]$, so $\mathcal{L}[y'(t)] = sY(s) - y(0)$. Taking the Laplace transform of the equation we get from line 12

$$\frac{e^{-2s}}{s} = sY(s) - y(0) - 2Y(s) = Y(s)(s-2) - 1,$$

so

$$Y(s) = \frac{1}{s-2} + \frac{e^{-2s}}{s(s-2)}.$$

The first term matches line 2, for the second we need to use partial fractions:

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}, \text{ or } 1 = A(s-2) + Bs.$$

Set $s = 0$ to see that $A = -1/2$, set $s = 2$ to see that $B = 1/2$, so

$$Y(s) = \frac{1}{s-2} + e^{-2s}\Big(-\frac{1}{2} \times \frac{1}{s} + \frac{1}{2} \times \frac{1}{s-2}\Big).$$

We use lines 2 and 13 to get $y(t) = e^{2t} + u_2(t)[-\frac{1}{2} + \frac{1}{2}e^{2(t-2)}]$.

(3) (10 points) Determine the solution $y(t)$ of the differential equation

$$y'' + 4y' + 5y = \delta(t-2) + e^{-t}, \quad y(0) = 0,\ y'(0) = 0.$$

What is the behavior of $y(t)$ as $t \to \infty$?

Solution: Let $Y(s)$ be the Laplace transform of $y(t)$. Then using the formulas for the Laplace transforms of $y'(t)$ and $y''(t)$ we get

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - y(0) + 5Y(s) = e^{-2s} + \frac{1}{s+1}.$$

Since $y(0) = y'(0) = 0$, we get

$$(s^2 + 4s + 5)Y(s) = e^{-2s} + \frac{1}{s+1},$$

or

$$Y(s) = e^{-2s}\frac{1}{s^2+4s+5} + \frac{1}{(s+1)(s^2+4s+5)}.$$

The denominator doesn't factor further, so we complete the square:

$$Y(s) = e^{-2s}\frac{1}{(s+2)^2+1} + \frac{1}{(s+1)(s^2+4s+5)}.$$

The first term matches lines 13 and 5, we use partial fractions on the second term:

$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+5}$$

or

$$1 = A(s^2+4s+5) + (Bs+C)(s+1).$$

Setting $s = -1$, we see that the second term is zero so $1 = A((-1)^2 + 4(-1) + 5) = 2A$, so $A = 1/2$. Set $s = 0$ to get $1 = 5A + C$, so $C = 1 - 5A = 1 - 5/2 = -3/2$. Finally, set $s = 1$ (any number other than 0 and -1 will do) to get

$$1 = \frac{1}{2}(1^2 + 4 \times 1 + 5) + (B - \frac{3}{2}) \times 2 = 5 + 2B - 3,$$

so $2B = -1$ or $B = -1/2$. So

$$\frac{1}{(s+1)(s^2+4s+5)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s+3}{(s+2)^2+1} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\left[\frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}\right].$$

So we use lines 2, 9, and 10 to get

$$y(t) = u_2(t)e^{-2(t-2)}\sin(t-2) + \frac{1}{2}e^{-t} - \frac{1}{2}[e^{-2t}\cos t + e^{-2t}\sin t].$$

All terms tend to zero as $t \to \infty$, so $y(t) \to 0$ as $t \to \infty$.