MA 442: Quiz 3

Question 1 (10 points): Assuming that we use “+∞” to mean a “a function from $(S, \rho)$ to $E^1$ that tends to $+\infty$ as $x \to p \in S$”, show that $+\infty + +\infty = +\infty$.

Answer:
The first principle is: Use the definitions.

We want to show that if $f(x) \to +\infty$ and $g(x) \to +\infty$ as $x \to p \in S$, then $f(x)g(x) \to +\infty$ as $x \to p$.

In other words, for any $K$ we want to show that there is a $\delta > 0$ such that if $x \in G_{-p}(\delta)$ then $f(x)g(x) > K$.

Well, for any $K'$ there is a $\delta > 0$ such that for any $x \in G_{-p}(\delta)$, $f(x) > K'$; similarly for $g$.

So let $K' = K$ and $K'' = 1$. Then there exists $\delta'$ and $\delta''$ such that for $x \in G_{-p}(\delta')$ $f(x) > K$ and for $x \in G_{-p}(\delta'')$, $g(x) \geq 1$. So for $\delta = \min(\delta', \delta'')$ and for all $x \in G_{-p}(\delta)$, $f(x)g(x) > K^1 = K$.

So $f(x)g(x) \to \infty$ as $x \to p$. 
Question 2 (10 points): Define $f: E^3 \to E^2$ by $f((x_1, x_2, x_3)) = (x_1, x_3)$. Prove that $f$ is a contraction on $E^3$, i.e., that $|f(x) - f(y)| \leq |x - y|$.

Answer:
We can do this fairly directly:

$$|f(x) - f(y)| = |(x_1, x_3) - (y_1, y_3)|$$
$$= \sqrt{(x_1 - y_1)^2 + (x_3 - y_3)^2}$$
$$\leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$
$$= |x - y|.$$
Question 3 (10 points):
Prove that if \( \{x_m\} \) and \( \{y_m\} \) are Cauchy sequences in \((S, \rho)\), then the sequence of distances \( \rho(x_m, y_m), m = 1, 2, 3, \ldots \), converges in \( E^1 \).

Answer: This was a homework problem.
The sequence \( \{x_m\} \) and \( \{y_m\} \) are in a general \((S, \rho)\), which may not be complete, so they may not converge.
The sequence of distances \( \{\rho(x_m, y_m)\} \) is in \( E^1 \), which is complete, so if we show the sequence \( \{\rho(x_m, y_m)\} \) is Cauchy, then it converges.

We have by the triangle inequality
\[
\rho(x_m, y_m) \leq \rho(x_m, x_n) + \rho(x_n, y_n) + \rho(y_n, y_m),
\]
so
\[
\rho(x_m, y_m) - \rho(x_n, y_n) \leq \rho(x_m, x_n) + \rho(y_n, y_m).
\]

Similarly, starting with \( \rho(x_n, y_n) \), we have
\[
\rho(x_n, y_n) - \rho(x_m, y_m) \leq \rho(x_n, x_m) + \rho(y_m, y_n).
\]

The two right-hand sides are the same, because \( \rho(a, b) = \rho(b, a) \) for any \( a, b \in S \), so combining these gives
\[
|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_m, x_n) + \rho(y_n, y_m).
\]

Since \( \{x_m\} \) and \( \{y_m\} \) are Cauchy sequences, given \( \epsilon > 0 \), there exist numbers \( K' \) and \( K'' \) such that if \( m, n > K' \) we have \( \rho(x_m, x_n) < \epsilon/2 \), and if \( m, n > K'' \) we have \( \rho(y_m, y_n) < \epsilon/2 \).

So for \( m, n > \max(K', K'') \) we have
\[
|\rho(x_n, y_n) - \rho(x_m, y_m)| \leq \rho(x_m, x_n) + \rho(y_n, y_m) < \epsilon/2 + \epsilon/2 = \epsilon.
\]

So \( \{\rho(x_n, y_n)\} \) is a Cauchy sequence in \( E^1 \), which is complete, so the sequence of distances converges.
Definitions

1. A metric space is a set $S$ with a function $\rho: S \times S \to E^1$ such that for all $x, y, z \in S$:
   1. $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$.
   2. $\rho(x, y) = \rho(y, x)$.
   3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

2. The open ball of radius $\epsilon$ about $p \in (S, \rho)$ is $G_p(\epsilon) = \{ x \in S \mid \rho(x, p) < \epsilon \}$.

3. A set $A \subseteq (S, \rho)$ is open iff $(\forall p \in A) (\exists \epsilon > 0) G_p(\epsilon) \subseteq A$.

4. A set $A \subseteq (S, \rho)$ is closed iff $S - A$ is open.

5. The usual metric on $E^n$ is given by $\rho(x, y) = |x - y|$ with the norm given by $|x| = \sqrt{x \cdot x}$ and the dot product $x \cdot y = \sum_{k=1}^{n} x_k y_k$.

6. Given any sequence $\{x_n\} \subset E^*$, we define $\limsup_{n \to \infty} x_n = \inf_{n} \sup_{k > n} x_k$ and $\liminf_{n \to \infty} x_n = \sup_{n} \inf_{k > n} x_k$.

7. A sequence in a metric space $(S, \rho)$ is Cauchy iff $(\forall \epsilon > 0) (\exists K > 0) (\forall m, n > K) \rho(x_m, x_n) < \epsilon$.

8. A metric space $(S, \rho)$ is complete iff every Cauchy sequence converges.

9. A sequence $\{x_m\} \subset (S, \rho)$ clusters at $p \in S$ iff every ball $G_p(\delta)$ contains infinitely many members of the sequence.

10. A set $A \subset (S, \rho)$ clusters at $p \in S$ iff every ball $G_p(\delta)$ contains infinitely many elements of $A$.

Corollaries

1. Every bounded sequence in $E^n$ has a cluster point.

2. $E^n$ is complete.