Wavelets and Image Compression

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Abstract. In this paper we present certain results about the compression of images using wavelets. We concentrate on the simplest case of the Haar decomposition and compression in L^2. Further results about compression in L^p, $p \neq 2$ are mentioned.

§1. Introduction

Transmitting digital images at video rates requires tremendous transfer of information. For example:

- If we assume that an image has 512 rows, each of which consists of 512 picture elements, or pixels, then we have $512 \times 512 = 262{,}144$ pixels.
- If the image is monochrome, and there are 256 different intensity levels with $0 \equiv$ black and $255 \equiv$ white, then each image requires $262{,}144 \times 8 = 2{,}097{,}152$ bits.
- In color, if each of the red, green, and blue components requires a monochrome image, we have 6,291,456 bits per image.
- Full motion video requires at least 24 frames a second, or 150,994,944 bits per second.
- High definition television (HDTV) may display four times as much information, requiring 603,979,776 bits/second.

For comparison, the familiar Ethernet networks commonly used to connect groups of workstations have a peak transfer rate of 10,000,000 bits/second; common fiber-optic networks have transfer rates of 140,000,000 bits/second.

Thus, algorithms that compress the information used to represent images are of great interest to engineers. One set of algorithms attempts to recreate at the receiver the precise image sent by the transmitter; these algorithms are known as lossless. Other compression algorithms are willing to lose certain features of the images that will be little noticed by the human observers

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(who, after all, are the consumers of these images) in the hope of gaining higher compression ratios. Wavelet methods of image compression lie within the class of lossy algorithms.

Rather than attempting to give a summary or an outline of the use of wavelet transforms in image compression, we will present almost completely the arguments used to analyze compression in $L^2(\Omega)$ of functions in certain smoothness classes called Besov spaces after applying the well-known Haar transform. This study discusses the basic ideas of wavelet decompositions, compression of wavelet decompositions, error, quantization, and smoothness of images. Perhaps the only thing new here, beyond the presentation, is the observation that the error of compression in $L^2(\Omega)$ does not depend on any unknown constants, mainly due to our choice of the norm used to measure the smoothness of images. Otherwise, the results here are known.

Our particular view of image compression using wavelets is taken from the paper [2]; the survey paper [4] indicates an approach to wavelets that motivates both [2] and this paper. Further information about wavelets can be found in, for example, the book by Meyer [9] and the papers by Mallat [8] and Daubechies [1]. The paper [10] surveys rather broadly the application of wavelets to signal and image processing, including image compression.

§2. Wavelet Decomposition of Images: The Haar Transform

In this section we present a particular way to view the well-known Haar transform. We hope that this simple presentation will introduce the reader to the more general wavelet transforms used in image compression.

To be specific, we consider images with 512 rows and 512 columns of pixels, each of which can take an integer grey-scale value ranging from 0 (which represents black) to 255 (which represents white). As described, it takes $2^{18} = 262144$ 8-bit bytes of information to store the image. With a natural scaling of both the domain and range variables, we consider the image to be a function f that maps the unit square $\Omega := [0,1]^2$ into $[0,1)$. We write

$$f = P_9 f = \sum_{1\times 1\, I} p_I \phi_I, \tag{1}$$

where the sum is over all (1×1 blocks of) pixels I, p_I is the pixel value at I, and ϕ_I is the characteristic function of I, i.e., $\phi_I(x) = 1$ if $x \in I$ and $\phi_I(x) = 0$ otherwise. There are $2^9 \times 2^9$ such pixels, hence the notation P_9.

One achieves a certain amount of compression simply by averaging the values of the pixels over 2×2 blocks of pixels and storing or transmitting only the block averages rather than the values of the individual pixels. We therefore introduce

$$P_8 f = \sum_{2\times 2\, I} p_I \phi_I,$$

where now p_I is the average of the pixel values on I. (This definition is consistent with what we used in (1).) There are $2^8 \times 2^8$ blocks of 2×2 pixels, which is why we used P_8. This process achieves a compression ratio of 4 to 1, i.e., a mildly distorted version of the image is specified using only one fourth as many numbers as before.

There is no reason to stop this process, and we calculate $P_7 f$, $P_6 f$, all the way down to $P_0 f$, which is specified solely by the average intensity of the pixels over the entire image. We now write the image itself as

$$\begin{aligned} f = P_9 f &= (P_9 f - P_8 f) + (P_8 f - P_7 f) + \ldots (P_1 f - P_0 f) + P_0 f \\ &= \sum_{1\times 1\, I} d_I \phi_I + \sum_{2\times 2\, I} d_I \phi_I + \cdots + P_0 f \\ &= \sum_{I} d_I \phi_I + P_0 f, \end{aligned}$$

where the final sum is over all dyadic blocks of pixels. The coefficients d_I are obtained from the differences of the approximations at adjacent levels.

We began with 2^{18} pixels p_I, and we now have 4/3 as many coefficients d_I, but this redundancy is easily removed as follows. At the finest level, it is clear that we have subtracted from each pixel value the average over the 2×2 block containing it. Therefore, we organize each set of 4 coefficients d_I into blocks as

$$\boxed{\begin{matrix} a & b \\ c & d \end{matrix}} = \alpha \boxed{\begin{matrix} + & - \\ + & - \end{matrix}} + \beta \boxed{\begin{matrix} - & - \\ + & + \end{matrix}} + \gamma \boxed{\begin{matrix} - & + \\ + & - \end{matrix}}, \tag{2}$$

since $a+b+c+d=0$; here $+$ denotes $+1$ and $-$ denotes -1. By combining the coefficients d_I and their corresponding characteristic functions ϕ_I into groups of 4, we come up with a new representation using only 3 coefficients and the alternating basis functions indicated by the boxes on the right side of (2). In this way we remove 1/4 of the coefficients, i.e., we have removed the redundancy.

We let the set Ψ consist of the three functions

$$\begin{aligned} \psi^{(1)} &= \phi_{[0,1/2)\times[0,1)} - \phi_{[1/2,1)\times[0,1)}, \\ \psi^{(2)} &= \phi_{[0,1)\times[0,1/2)} - \phi_{[0,1)\times[1/2,1)}, \\ \psi^{(3)} &= \phi_{[0,1/2)\times[0,1/2)} - \phi_{[0,1/2)\times[1/2,1)} - \phi_{[1/2,1)\times[0,1/2)} + \phi_{[1/2,1)\times[1/2,1)}. \end{aligned}$$

For each $\psi \in \Psi$, $k \geq 0$, and $j = (j_1, j_2)$ with $0 \leq j_1 < 2^k$ and $0 \leq j_2 < 2^k$ (we denote this set of j by $\mathbf{Z}^2(k)$), we introduce the function $\psi_{j,k}(x) := 2^k \psi(2^k x - j) = 2^k \psi(2^k(x - j/2^k))$ for $x \in \Omega$. This is the scaled (by 2^k) dyadic dilate (by 2^k) and translate (by $j/2^k$) of ψ, and it has support on the dyadic square $2^{-k}\Omega + j/2^k$. By applying the rewrite rule (2) to the coefficients d_I, we write the image f in terms of the functions $\psi_{j,k}$ as

$$f = \sum_{k\geq 0} \sum_{j\in \mathbf{Z}^2(k)} \sum_{\psi\in\Psi} c_{j,k,\psi} \psi_{j,k} + P_0 f; \tag{3}$$

here $c_{j,k,\psi} = 0$ if $k > 8$. This is the Haar decomposition of the image f.

§3. Image Compression in $L^2(\Omega)$

Although we have argued in [2] that one achieves better compressed images when attempting to minimize the error in $L^1(\Omega)$ rather than $L^2(\Omega)$, the L^2 theory is simple enough that a complete presentation can be made in the following paragraphs.

The collection of functions $\{\psi_{j,k} \mid k \geq 0,\ j \in \mathbf{Z}^2(k),\ \psi \in \Psi\}$ is orthogonal. This is easily checked—if the supports of two such functions do not intersect, then their inner product is zero; if the support of one is strictly contained in the support of a second, then the inner product is zero because the integral of each $\psi_{j,k}$ is zero and the second function is constant on the support of the first; and one can check by hand what happens when the supports of two such functions coincide. Because of the scaling by 2^k, each $\psi_{j,k}$ has $L^2(\Omega)$-norm one, i.e., the entire set is orthonormal. Adding the characteristic function ϕ_Ω to this set preserves this orthonormality; the resulting set of functions forms a complete orthonormal basis for $L^2(\Omega)$. Thus, any f for which

$$\|f\|_{L^2(\Omega)} := \left(\int_\Omega |f(x)|^2\,dx \right)^{1/2} < \infty$$

has an expansion of the form (3), with $c_{j,k,\psi} = \int_\Omega f\psi_{j,k}$. Therefore,

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &= \sum_{k\geq 0} \sum_{j\in\mathbf{Z}^2(k)} \sum_{\psi\in\Psi} \|c_{j,k,\psi}\psi_{j,k}\|^2_{L^2(\Omega)} + \|p_\Omega\phi_\Omega\|^2_{L^2(\Omega)} \\ &= \sum_{k\geq 0} \sum_{j\in\mathbf{Z}^2(k)} \sum_{\psi\in\Psi} |c_{j,k,\psi}|^2 + |p_\Omega|^2.\end{aligned}$$

We wish now to find an approximation $\tilde f$ to f with the form

$$\tilde f = \sum_{(j,k,\psi)\in\Lambda} c_{j,k,\psi}\psi_{j,k} + P_0 f,$$

where Λ is a finite set, say with no more than N elements. We want to do this in a way that minimizes the $L^2(\Omega)$ error,

$$\|f - \tilde f\|^2_{L^2(\Omega)} = \sum_{(j,k,\psi)\notin\Lambda} |c_{j,k,\psi}|^2.$$

Because we have an explicit expression for the error, we see immediately that the best choice is to put the triples associated with the N largest values of $|c_{j,k,\psi}|$ into Λ. Thus, we have found an exact, explicit solution to our minimization problem.

This choice of Λ implicitly defines another parameter ϵ, which is

$$\epsilon := \inf_{(j,k,\psi)\in\Lambda} |c_{j,k,\psi}|.$$

One can start with ϵ, and put into Λ only those triples for which $|c_{j,k,\psi}| \geq \epsilon$. This is known as *threshold coding.* The number of terms N in Λ is not known until one is finished, but this method does not require one to sort the set $\{c_{j,k,\psi}\}$, which takes more time than to calculate the coefficients themselves! In fact, it is better to calculate *quantized* coefficients

$$\tilde{c}_{j,k,\psi} = \mathrm{round}(\frac{c_{j,k,\psi}}{2\epsilon})\, 2\epsilon,$$

where round(x) is the nearest integer to the real number x. In this way we store or transmit the code round$(c_{j,k,\psi}/(2\epsilon))$ and reconstruct

$$\tilde{f} = \sum_{k\geq 0} \sum_{j\in \mathbf{Z}^2(k)} \sum_{\psi\in\Psi} \tilde{c}_{j,k,\psi}\psi_{j,k} + P_0 f. \tag{4}$$

Two important properties that all these methods of calculating $\tilde{c}_{j,k,\psi}$ satisfy are

$$|c_{j,k,\psi} - \tilde{c}_{j,k,\psi}| \leq \epsilon \tag{5}$$

and

$$\text{if } |c_{j,k,\psi}| < \epsilon, \text{ then } \tilde{c}_{j,k,\psi} = 0. \tag{6}$$

§4. Error, Smoothness, and Quantization

In this section we investigate the connection between the quantization parameter ϵ, the error in our approximation (4), the number of nonzero quantized coefficients $\tilde{c}_{j,k,\psi}$, and the smoothness of the image in suitable function spaces. The results we describe are special cases of results in [5] and [3] about nonlinear approximation using wavelets.

To measure the smoothness of images, we use the Besov spaces $B^\alpha_q(L^p(\Omega))$ for $\alpha > 0$, $0 < p < \infty$, and $0 < q < \infty$. Roughly speaking, these spaces have α derivatives in $L^p(\Omega)$, with the parameter q indicating finer gradations of smoothness. A precise definition of these spaces can be found, for example, in our paper [2]. The scale of spaces $B^\alpha_\tau(L^\tau(\Omega))$, $\alpha > 0$, $1/\tau = \alpha/2 + 1/2$ is important for compression in $L^2(\Omega)$. It is a special case of results in [6] or [7] or ... that one can define the Besov space norm of a function in $B^\alpha_\tau(L^\tau(\Omega))$ in terms of the size of the Haar or wavelet coefficients of that function. In particular, if the $c_{j,k,\psi}$ are given by (3), then one can define

$$\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))} := \sum_{k\geq 0} \sum_{j\in \mathbf{Z}^2(k)} \sum_{\psi\in\Psi} |c_{j,k,\psi}|^\tau + |p_\Omega|^\tau \tag{7}$$

for $0 < \alpha < 1$, i.e., for $1 < \tau < 2$. (The equivalence holds for higher values of α when one uses smoother wavelets, e.g., the wavelets of Daubechies [1].)

Several interesting things follow immediately from (7). Most importantly, it follows that $B^\alpha_\tau(L^\tau(\Omega))$ is embedded continuously into $L^2(\Omega)$, because

$$\begin{aligned}\|f\|_{L^2(\Omega)} &= \Bigl(\sum_{k\ge 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}|c_{j,k,\psi}|^2+|p_\Omega|^2\Bigr)^{1/2}\\ &\le \Bigl(\sum_{k\ge 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}|c_{j,k,\psi}|^\tau+|p_\Omega|^\tau\Bigr)^{1/\tau} = \|f\|_{B^\alpha_\tau(L^\tau(\Omega))}.\end{aligned}$$

With these preliminaries we can prove the following theorem.

Theorem 1. Assume that $f\in B^\alpha_\tau(L^\tau(\Omega))$ for some $0<\alpha<1$ and $1/\tau=\alpha/2+1/2$, and that $c_{j,k,\psi}$ are the coefficients of the Haar decomposition (3). Choose an $\epsilon>0$, and define $\tilde c_{j,k,\psi}$ by any method that satisfies (5) and (6). Let $\tilde f$ be given by (4). Then

(1) The number, N, of nonzero coefficients $\tilde c_{j,k,\psi}$ satisfies

$$N\le \epsilon^{-\tau}\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))}. \tag{8}$$

(2) The error $f-\tilde f$ satisfies

$$\|f-\tilde f\|_{L^2(\Omega)}\le 2^{1/\tau}N^{-\alpha/2}\|f\|_{B^\alpha_\tau(L^\tau(\Omega))}. \tag{9}$$

Proof: We let $\gamma_{j,k,\psi}:=c_{j,k,\psi}-\tilde c_{j,k,\psi}$ for $\psi\in\Psi$, $k\ge 0$, and $j\in\mathbf{Z}^2(k)$. Then

$$f-\tilde f=\sum_{k\ge 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}\gamma_{j,k,\psi}\psi_{j,k},$$

each $|\gamma_{j,k,\psi}|\le\epsilon$, and either $\tilde c_{j,k,\psi}=0$, in which case $|\gamma_{j,k,\psi}|=|c_{j,k,\psi}|$, or $\tilde c_{j,k,\psi}\ne 0$, in which case $|c_{j,k,\psi}|\ge\epsilon\ge|\gamma_{j,k,\psi}|$. Therefore,

$$\sum_{k\ge 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}|\gamma_{j,k,\psi}|^\tau\le\sum_{k\ge 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}|c_{j,k,\psi}|^\tau\le\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))}. \tag{10}$$

We note that for each nonzero $\tilde c_{j,k,\psi}$, $|c_{j,k,\psi}|\ge\epsilon$. Therefore, (10) implies that

$$N\epsilon^\tau\le\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))},$$

so (8) follows.

The coefficients $\gamma_{j,k,\psi}$ can be partitioned into sets $\Lambda_1,\dots,\Lambda_M$, with

$$M\le\epsilon^{-\tau}\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))},$$

such that

$$\sum_{\gamma_{j,k,\psi}\in\Lambda_n}|\gamma_{j,k,\psi}|^\tau\le 2\epsilon^\tau,\quad n=1,\dots,M. \tag{11}$$

This is accomplished simply by sorting $|\gamma_{j,k,\psi}|$ in decreasing order and adding $\gamma_{j,k,\psi}$ to Λ_1 until the sum in (11) is greater than ϵ^τ. The process is repeated with the remaining coefficients added to Λ_n, $n = 2, \dots, M$. Because each term is individually less than ϵ^τ, (11) follows. Because each sum is at least ϵ^τ, the bound on M is immediate.

It follows from (11) that if we define

$$f_n := \sum_{\gamma_{j,k,\psi} \in \Lambda_n} \gamma_{j,k,\psi} \psi_{j,k},$$

then

$$\|f_n\|_{L^2(\Omega)} \le \|f_n\|_{B^\alpha_\tau(L^\tau(\Omega))} \le 2^{1/\tau}\epsilon,$$

the set $\{f_n\}$ is orthogonal, and $f - \tilde{f} = \sum_{n=1}^M f_n$. Thus

$$\begin{aligned} \|f - \tilde{f}\|^2_{L^2(\Omega)} &= \sum_{n=1}^M \|f_n\|^2_{L^2(\Omega)} \\ &\le \epsilon^{-\tau} \|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))} 2^{2/\tau}\epsilon^2 \\ &\le 2^{2/\tau}\epsilon^{\alpha\tau} \|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))} \end{aligned}$$

since $2 - \tau = \alpha\tau$. Thus, using our previous bound,

$$\epsilon^{\alpha\tau} \le N^{-\alpha} \|f\|^{\alpha\tau}_{B^\alpha_\tau(L^\tau(\Omega))},$$

and we have

$$\begin{aligned} \|f - \tilde{f}\|^2_{L^2(\Omega)} &\le 2^{2/\tau} N^{-\alpha} \|f\|^{\alpha\tau+\tau}_{B^\alpha_\tau(L^\tau(\Omega))} \\ &= 2^{2/\tau} N^{-\alpha} \|f\|^2_{B^\alpha_\tau(L^\tau(\Omega))} \end{aligned}$$

since $\alpha\tau + \tau = 2$. Taking square roots of both sides yields (9). ■

We remark that the proof given here is a variant of a proof in [2]. Note also that because of the way we expressed the $B^\alpha_\tau(L^\tau(\Omega))$ norm of f, there are no unknown (or unknowable!) constants in (8) and (9).

§5. Compression in $L^p(\Omega)$, $1 < p < \infty$

In this section we outline the results when one tries to minimize the $L^p(\Omega)$ norm of the error instead of the $L^2(\Omega)$ norm of the error.

If $f \in L^p(\Omega)$ for $1 < p < \infty$ we still obtain the Haar decomposition (3) for f. To ensure that an approximation $\tilde{f}$ of the form (4) has small error in $L^p(\Omega)$, one chooses a positive parameter ϵ and coefficients $\tilde{c}_{j,k,\psi}$ that satisfy

$$\|(c_{j,k,\psi} - \tilde{c}_{j,k,\psi})\psi_{j,k}\|_{L^p(\Omega)} \le \epsilon \tag{12}$$

and

$$\text{if } \|c_{j,k,\psi}\psi_{j,k}\|_{L^p(\Omega)} < \epsilon, \text{ then } \tilde{c}_{j,k,\psi} = 0. \tag{13}$$

It is immediately seen that properties (12) and (13) reduce to (5) and (6) when $p = 2$.

When approximating in $L^p(\Omega)$, the scale of Besov spaces $B^\alpha_\tau(L^\tau(\Omega))$, where now $\alpha > 0$ and $1/\tau = \alpha/2 + 1/p$, is most important. Again, one can show that the norm

$$\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))} = \sum_{k\geq 0}\sum_{j\in\mathbf{Z}^2(k)}\sum_{\psi\in\Psi}\|c_{j,k,\psi}\psi_{j,k}\|^\tau_{L^p(\Omega)} + |p_\Omega|^\tau$$

is equivalent to the usual norm for the space $B^\alpha_\tau(L^\tau(\Omega))$ when $0 < \alpha < \min(1, 2/p)$. It is not so obvious that $B^\alpha_\tau(L^\tau(\Omega))$ is embedded in $L^p(\Omega)$, but this is true; see [6]. One can prove the following theorem; see [2] or [3]:

Theorem 2. Assume that $f \in B^\alpha_\tau(L^\tau(\Omega))$ for some $0 < \alpha < \min(1, 2/p)$ and $1/\tau = \alpha/2 + 1/p$, and that $c_{j,k,\psi}$ are the coefficients of the Haar decomposition (3). Choose $\epsilon > 0$, and define $\tilde{c}_{j,k,\psi}$ by any method that satisfies (12) and (13). Let $\tilde{f}$ be given by (4). Then there exists a constant C, independent of f, such that

(1) The number, N, of nonzero coefficients $\tilde{c}_{j,k,\psi}$ satisfies

$$N \leq \epsilon^{-\tau}\|f\|^\tau_{B^\alpha_\tau(L^\tau(\Omega))}.$$

(2) The error $f - \tilde{f}$ satisfies

$$\|f - \tilde{f}\|_{L^p(\Omega)} \leq CN^{-\alpha/2}\|f\|_{B^\alpha_\tau(L^\tau(\Omega))}.$$

One can ask whether these methods, and these estimates, are optimal. If one asks the question in an asymptotic sense, then the answer is yes. In other words, if for a particular f there is a constant C and best approximations $\tilde{f}_N$, $N > 0$, with $\leq N$ nonzero Haar coefficients such that

$$\|f - \tilde{f}_N\|_{L^p(\Omega)} \leq CN^{-\alpha/2}$$

for some $0 < \alpha < \min(1, 2/p)$, then f is in the Besov space $B^\beta_\sigma(L^\sigma(\Omega))$ for all $0 < \beta < \alpha$ and $1/\sigma = \beta/2 + 1/p$. In fact, one has the equivalence

$$\left(\sum_{N=1}^\infty [N^{\alpha/2}\|f - \tilde{f}_N\|_{L^p(\Omega)}]^\tau \frac{1}{N}\right)^{1/\tau} < \infty \Longleftrightarrow \|f\|_{B^\alpha_\tau(L^\tau(\Omega))} < \infty \qquad (14)$$

and the left side of (14) is an equivalent norm for $\|f\|_{B^\alpha_\tau(L^\tau(\Omega))}$ (modulo constants); see [3]. Thus, our approximation algorithms are as good as *any* algorithms that use wavelets.

By calculating explicitly $\|\psi_{j,k}\|_{L^p(\Omega)}$, one sees that (12) is equivalent to

$$|c_{j,k,\psi} - \tilde{c}_{j,k,\psi}|2^{2k(\frac{1}{2}-\frac{1}{p})} < \epsilon.$$

The parameter p determines the relative importance of errors in intensity (given by $2^k|c_{j,k,\psi} - \tilde{c}_{j,k,\psi}|$) and the spatial frequency of the functions $\psi_{j,k}$

(this influence is measured by the term $2^{2k/p}$). Thus, varying p allows one to vary the relative importance of *contrast* and *frequency* in the introduction of errors to f to obtain $\tilde{f}$. One can use this flexibility to attempt to match the high-frequency response of the human visual system itself. We claim in [2] that when one follows this approach, one obtains that $p = 1$ roughly matches the high-frequency response of the human visual system.

The attentive reader will note that the above theorems were stated for $1 < p$, and, in fact, the theorems are *false* when using Haar wavelets, or any other orthogonal wavelets, when $p \leq 1$. These algorithms can be modified, however, to use redundant, non-orthogonal representations (using, e.g., the ϕ_I) to work in $L^p(\Omega)$ for $0 < p \leq 1$; see [2] and [3]. Alternately, one can work in the Hardy spaces $H^p(\Omega)$.

§6. Examples

Here we present two figures to illustrate the use of the compression algorithms. Figure 1 is the green component of the commonly used color image lena. Figure 2 is lena compressed using the $L^1(\Omega)$ compression algorithm and a modified Haar transform. The compressed image uses 12,068 nonzero coefficients, which occupy 8,925 bytes; see [2] for complete details. Thus, we have achieved a compression ratio of 29.37 to one.

Figure 1. The green component of the color image lena.

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References

1. I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. on Pure and Appl. Math. **XLI** (1988), 909–996.

Figure 2. Lena compressed in $L^1(\Omega)$ at 29.37 to one.

2. R. DeVore, B. Jawerth, and B. Lucier, Image compression through wavelet transform coding, IEEE Trans. Information Theory, to appear.
3. R. DeVore, B. Jawerth, and V. Popov, Compression of wavelet decompositions, Amer. J. Math., to appear.
4. R. DeVore and B. Lucier, Wavelets, Acta Numerica, to appear.
5. R. DeVore and V. Popov, Free multivariate splines, Constr. Approx. **3** (1987), 239–248.
6. R. DeVore and V. Popov, Interpolation of Besov spaces, Trans. Amer. Math. Soc. **305** (1988), 397–414.
7. M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, J. of Functional Analysis **93** (1990), 34–170.
8. S. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$, Trans. Amer. Math. Soc. **315** (1989), 69–87.
9. Y. Meyer, *Ondelettes et Opérateurs I: Ondelettes*, Hermann, 1990.
10. O. Rioul and M. Vetterli, Wavelets and signal processing, IEEE Signal Processing Magazine **8** (1991), issue 4, 14–38.

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