HIGH ORDER REGULARITY FOR CONSERVATION LAWS

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Abstract. We study the regularity of discontinuous entropy solutions to scalar hyperbolic conservation laws with uniformly convex fluxes posed as initial value problems on $\mathbb{R}$. For positive α we show that if the initial data has bounded variation and the flux is smooth enough then the solution $u(\,\cdot\,,t)$ is in the Besov space $B^\alpha_\sigma(L^\sigma)$ where $\sigma = 1/(\alpha+1)$ whenever the initial data is in this space. As a corollary, we show that discontinuous solutions of conservation laws have enough regularity to be approximated well by moving-grid finite element methods. Techniques from approximation theory are the basis for our analysis.

Key words. regularity, nonlinear approximation, Besov spaces, conservation laws

AMS(MOS) subject classifications. 35L65, 35D10, 41A25, 46E35

1. Introduction.

Discontinuities may form in the solution $u(x,t)$ of the hyperbolic conservation law

$$\text{(C)}\qquad \begin{aligned} &u_t + f(u)_x = 0, \qquad && x\in\mathbb{R}, \quad t>0,\\ &u(x,0) = u_0(x), \qquad && x\in\mathbb{R}, \end{aligned}$$

even if the flux f and the initial data u_0 are smooth. Hence, classical solutions of (C) do not generally exist. Weak solutions of (C) are not unique, but both existence and uniqueness of weak solutions that satisfy one of several auxiliary “entropy” condition were shown by Oleinik [21], Vol’pert [26], and Kružkov [14]. The regularity of these weak solutions is the topic of this paper. Namely, we are interested in smoothness spaces X that are regularity spaces for (C), i.e., $u_0 \in X$ implies that $u(\,\cdot\,,t)\in X$ for all positive t.

The Sobolev spaces $W^{\alpha,p}$ for $p\ge 1$ and $\alpha \ge 1$ contain only continuous functions, and therefore are not appropriate candidates for X. More generally, a Sobolev-type embedding theorem implies that if $\alpha p > 1$ then functions in the Besov spaces $B^\alpha_q(L^p)$ (q is a secondary index of smoothness; see §3) are again continuous. Consequently, if one desires high order smoothness ($\alpha > 1$) one must measure smoothness in L^p spaces with $0 < p \le 1/\alpha < 1$. Such Besov spaces are not locally convex topological vector spaces—they are locally quasi-convex topological vector spaces or F-spaces [22, Chapt. 11], [13]—but they are, in some sense, the right spaces in which to measure the smoothness of solutions of (C) (see [19] for a discussion).

It is well known that the space $\mathrm{BV}(\mathbb{R})$ of functions of bounded variation is a regularity space for (C). Recently, Lucier [19] has shown that if f is convex and has three bounded

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derivatives, then the spaces $\mathrm{BV}(\mathbb{R}) \cap B^\alpha_\sigma(L^\sigma)$ for $1 \le \alpha < 2$ and $\sigma = 1/(\alpha+1)$ are also regularity spaces for solutions of (C). The main result of the present paper is the following theorem, which extends results of this type to $\alpha \ge 2$. (In this paper C will denote a generic constant which may be different from one occurence to the next.)

THEOREM 1.1. *Assume that r is a positive integer and that $u_0 \in \mathrm{BV}(\mathbb{R})$ has support in $I := [0,1]$. Then there exists a constant $C_1 := C_1(r)$ such that the following statements are valid. Let $\Omega = \{y \mid |y| < C_1 \|u_0\|_{L^\infty(\mathbb{R})}\}$. Assume that there is a constant C_2 such that for all $\xi \in \Omega$, $|f^{(r+1)}(\xi)| < C_2$ and $f''(\xi) \ge 1/C_2$. Then for any positive $\alpha < r$ and time $t > 0$ there exists a constant C such that if $u_0 \in B^\alpha(I) := B^\alpha_\sigma(L^\sigma(I))$, where $\sigma = 1/(\alpha+1)$, then $u(\,\cdot\,, t)$, the solution of* (C), *has support in $I_t = [\inf_{\xi\in\Omega} f'(\xi)t, 1 + \sup_{\xi\in\Omega} f'(\xi)t]$ and $\|u(\,\cdot\,,t)\|_{B^\alpha(I_t)} \le C(\|u_0\|_{B^\alpha(I)} + 1)$.*

It may be useful to compare the case $0 < \alpha < 2$ in [19] and the case $\alpha \ge 2$ of this paper. The central idea of Lucier's theorem is to compare the error of L^1 approximation for $u(\,\cdot\,,t)$ by piecewise linear functions with N free knots with the corresponding error of approximation for u_0. A specific construction is made in which f' is approximated in $L^\infty(\mathbb{R})$ to order N^{-2} by a continuous, piecewise linear function g' and u_0 is approximated in $L^1(\mathbb{R})$ by the best discontinuous, piecewise linear function v_0 with N free knots. It is then shown that the solution $v(\,\cdot\,,t)$ of

$$\text{(P)} \qquad \begin{aligned} &v_t + g(v)_x = 0, \qquad && x \in \mathbb{R}, \quad t > 0, \\ &v(x,0) = v_0(x), \qquad && x \in \mathbb{R}, \end{aligned}$$

is piecewise linear for all time and has no more than $C(\|u_0\|_{\mathrm{BV}(\mathbb{R})})N$ pieces. The stability result (2.3) in §2 shows that $u(\,\cdot\,,t)$ can be approximated with an error not exceeding the error of approximation of u_0 plus $O(N^{-2})$. The regularity theorem is then proved by using the characterization, developed by DeVore and Popov [4] using results of Petrushev [23], [24], of the spaces $B^\alpha_\sigma(L^\sigma(I))$ of order $\alpha < r$ in terms of approximation by free knot splines of degree less than r.

This approach does not carry over directly to the case $\alpha \ge 2$ because then one would naturally approximate u_0 by piecewise polynomials v_0 of degree at least two and approximate f' by continuous, piecewise polynomials g' of the same degree. In this case $v(\,\cdot\,,t)$ is no longer a piecewise polynomial, but has pieces which are algebraic functions. This makes the proof of Theorem 1.1 much more substantial. In order to establish Theorem 1.1, we will develop in §4 various properties of algebraic curves and inverse polynomials that are analogues of properties of polynomials.

Because of the equivalence between regularity in $B^\alpha_\sigma(L^\sigma(I))$ and approximation by piecewise polynomial functions with free knots [5], Theorem 1.1 implies that $u(\,\cdot\,,t)$ can be approximated by piecewise polynomials with free knots as well as the initial data can be. Such approximations are generated by moving-grid or front-tracking finite element schemes, among others. (See, for example, [10], [18], [17], [20].) Thus, Theorem 1.1 shows that solutions of (C) have, in principle, precisely the regularity needed for good approximation by moving-grid finite element methods.

We remark that others have studied regularity for conservation laws by describing the structure of the singularity set of u or by showing that "generic" smooth initial data remains piecewise smooth for positive time [1], [2], [7], [8], [11], [16], [25].

2. Entropy solutions of hyperbolic conservation laws. In this section we recount properties of solutions of Problem (C) that we will use in the following sections. The monograph by Lax [15] and Kružkov's paper [14] are given as general references for this section.

The method of characteristics shows that C^1 solutions of (C) are constant along lines $x = x_0 + tf'(u_0(x_0))$, so near the line $t = 0$ the function $u(x,t)$ satisfies the implicit equation

$$u = u_0(x - f'(u)t). \tag{2.1}$$

Discontinuities can develop in u, and (2.1) no longer holds for all x and t; however, it is true in some sense that the solution u is *piecewise* made up of local solutions of (2.1), at least when f is convex.

This idea is made rigorous by Lax [15] who describes the solution u of (C) by means of a related minimization problem. If u_0 is continuous and f is strictly convex, he shows that

$$u(x,t) = u_0(y) \text{ where } y := y(x,t) \text{ is a solution of } \frac{x-y}{t} = f'(u_0(y)). \tag{2.2}$$

There may be many solutions to the last equation but the minimization property picks out a specific value $y(x,t)$. Lax shows that $y(x,t)$ is an increasing function of x for fixed t. Shocks occur wherever $y(x,t)$ is discontinuous in x.

We also note that entropy solutions of (C) and (P) are stable in L^1 with respect to changes in the initial data and the flux: if u and v are solutions of (C) and (P) with initial data u_0 and v_0 in $\mathrm{BV}(\mathbb{R})$ and C^1 fluxes f and g, respectively, then [18]

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1(\mathbb{R})} \le \|u_0 - v_0\|_{L^1(\mathbb{R})} + t\|f' - g'\|_{L^\infty(\mathbb{R})}\|u_0\|_{\mathrm{BV}(\mathbb{R})}. \tag{2.3}$$

3. Besov spaces and spline approximation. In this section we give the definition of Besov spaces and recall their relationship to spline approximation with free knots. This section contains a selection of relevant results from [5]. (See also [4].)

Let I be a finite interval. Fix $0 < \alpha < \infty$, $0 < q \le \infty$ and $0 < p < \infty$, and pick an integer $r > \alpha$. (Different values of r will give equivalent quasi-norms below.)

Define the $L^p(I)$ modulus of continuity $\omega_r(f,t)_p$ to be the supremum over all $0<h<t$ of $\|\Delta_h^r f\|_{L^p(I_h)}$, where $I_h = \{x \in I \mid x+rh \in I\}$, $\Delta_h^0 f(x) = f(x)$, and $\Delta_h^r f(x) = \Delta_h^{r-1} f(x+h) - \Delta_h^{r-1} f(x)$. The Besov space $B_q^\alpha(L^p(I))$ is defined to be the set of all functions $f \in L^p(I)$ for which

$$|f|_{B_q^\alpha(L^p(I))} := \left(\int_0^\infty [t^{-\alpha}\omega_r(f,t)_p]^q \, dt/t\right)^{1/q}$$

is finite. Set $\|f\|_{B_q^\alpha(L^p(I))} := \|f\|_{L^p(I)} + |f|_{B_q^\alpha(L^p(I))}$. We especially need the case when p and q are less than one.

We are particularly interested in the spaces $B^\alpha(I) := B_\sigma^\alpha(L^\sigma(I))$, $\alpha > 0$, where $\sigma := 1/(\alpha+1)$. These spaces have the property that if $\alpha' > \alpha$ then $B^{\alpha'}(I)$ is continuously embedded in $B^\alpha(I)$, which in turn is continuously embedded in $L^1(I)$. We define $B^0(I) := L^1(I)$.

The spaces $B^\alpha(I)$, $\alpha > 0$, form a real interpolation family. The real method of interpolation using K-functionals can be described as follows: For any two linear, complete, quasi-normed spaces X_0 and X_1 continuously embedded in a linear Hausdorff topological space X, define the following functional for all f in $X_0 + X_1$:

$$K(f,t,X_0,X_1) := \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\},$$

where $f_0 \in X_0$ and $f_1 \in X_1$. The new space $X_{\theta,q} := (X_0,X_1)_{\theta,q}$ $(0<\theta<1,\ 0<q\le\infty)$ consists of functions f for which

$$\|f\|_{X_{\theta,q}} := \|f\|_{X_0+X_1} + \left(\int_0^\infty [t^{-\theta}K(f,t,X_0,X_1)]^q \, dt/t\right)^{1/q} < \infty,$$

where $\|f\|_{X_0+X_1} := K(f,1,X_0,X_1)$. DeVore and Popov [4] showed that if $\beta > \gamma > \alpha \ge 0$, $q = 1/(\gamma+1)$, and θ is defined by $\gamma = (1-\theta)\alpha + \theta\beta$, then $(B^\alpha(I), B^\beta(I))_{\theta,q} = B^\gamma(I)$. In particular, $(L^1(I), B^\beta(I))_{\alpha/\beta, 1/(\alpha+1)} = B^\alpha(I)$.

The Besov spaces $B^\alpha(I)$ are intimately related to approximation by piecewise polynomials with free knots. For all positive integers n and r, let $\Sigma_n := \Sigma_{n,r}$ denote the collection of all piecewise polynomials on I of degree less than r with at most 2^n pieces. If f is in $L^1(I)$ and $n \ge 0$, we let

$$s_n(f)_1 := \inf_{S\in\Sigma_n} \|f - S\|_{L^1(I)}$$

be the error in approximating f in the $L^1(I)$ norm by the elements of Σ_n; $s_{-1}(f)_1 := \|f\|_{L^1(I)}$. DeVore and Popov have shown that a function f is in $B^\alpha(I)$ with $\alpha > 0$ if and only if

$$\|f\|_{\mathcal{A}_\sigma^\alpha(L^1(I))} := \left(\sum_{n=-1}^\infty (2^{n\alpha} s_n(f)_1)^\sigma\right)^{1/\sigma} < \infty, \tag{3.1}$$

and $\|f\|_{\mathcal{A}_\sigma^\alpha(L^1(I))}$ is equivalent to $\|f\|_{B^\alpha(I)}$. More generally, if $\beta > \alpha$ and $0 < q \le \infty$ then $\mathcal{A}_q^\alpha(L^1(I)) = (L^1(I), B^\beta(I))_{\alpha/\beta,q}$.

4. Properties of polynomials and algebraic functions. Whenever I is a finite interval we will use the notation

$$\|f\|_p^*(I) := \left(\frac{1}{|I|}\int_I |f(x)|^p\,dx\right)^{1/p}, \qquad 0 < p < \infty,$$

with $\|f\|_\infty^*(I) := \sup_{x\in I}|f(x)|$. The following inequalities are well known for polynomials P of degree no greater than k; see, for example DeVore and Sharpley [6].

- For each $k = 0, 1, \ldots$ and $p, q \in (0, \infty]$ there exists a C such that for all polynomials P of degree $\le k$,

$$\|P\|_p^*(I) \le C\|P\|_q^*(I). \tag{4.1}$$

- For each $k = 0, 1, \ldots$ and $p \in (0, \infty]$ there exists a C such that for all polynomials P of degree $\le k$,

$$\|P'\|_p^*(I) \le C|I|^{-1}\|P\|_p^*(I). \tag{4.2}$$

- For each $k = 0, 1, \ldots$ and $p \in (0, \infty]$ there exists a C such that for all polynomials P of degree $\le k$, and for all intervals $J \supset I$,

$$\|P\|_p^*(J) \le C\left(\frac{|J|}{|I|}\right)^k \|P\|_p^*(I). \tag{4.3}$$

The constants can be chosen to depend only on k if p and q are bounded away from 0.

An analysis of approximation by piecewise polynomials can be based on three properties: the equivalence of norms (4.1), the "inverse inequality" (4.2), and the fact that polynomials oscillate in a controlled way that depends on their degree. The rest of this section is devoted to proving similar results for certain algebraic curves.

Let P and Q be two polynomials of degree $\le d$ such that $\phi := P^{-1}$ and $\psi := Q^{-1}$ are monotone and well defined on an interval $[\alpha, \beta]$. This means that P is monotone on an interval $[a, b]$ and Q is monotone on $[a', b']$. We will consider the functions ϕ, ψ, and $A(y) := \phi(y) - \psi(y)$ for $y \in [\alpha, \beta]$.

LEMMA 4.1 (Equivalence of Norms). *Let ϕ and ψ be defined on an interval I as the functional inverses of polynomials P and Q of degree $\le d$; assume that ϕ and ψ are monotone on I. Then for all $1 \le p < d/(d-1)$*

$$\|\phi - \psi\|_p^*(I) \le C(p, d)\|\phi - \psi\|_1^*(I). \tag{4.4}$$

Proof. We can assume that P is increasing since otherwise we replace P and Q by $-P$ and $-Q$ respectively. By considering $(P-\alpha)/(\beta-\alpha)$ and $(Q-\alpha)/(\beta-\alpha)$, we can assume that $\alpha = 0$ and $\beta = 1$. Also, by a linear change of variable, we can assume that $a = 0$, $b = 1$, and $a' \geq 0$.

We assume first that Q is decreasing and consider the following cases:

Case 1: $b' \geq 1/2$. We have $P(1/8) \geq \delta > 0$ where δ depends only on d, for otherwise by (4.3) applied to P and $p = \infty$, P could not attain the value 1 at $x = 1$. Similarly, for $m = (a' + b')/2$, $Q(m) \geq \delta' > 0$ for some δ' depending only on d since otherwise Q cannot attain the value 1 at $x = a'$. Hence, for $\delta'' = \min(\delta, \delta')$, $|A(y)| \geq |m - 1/8| \geq b'/4 \geq \frac{1}{8}\max(b', 1)$ for $y \in [0, \delta'']$. On the other hand, $|A(y)| \leq \max(b', 1)$ for all $y \in [0, 1]$, so (4.4) follows for all $1 \leq p \leq \infty$.

Case 2: $b' \leq 1/2$. We have $P(3/4) \leq \delta < 1$ with δ depending only on d for otherwise (4.3) applied to $1 - P$ and $p = \infty$ would show that P could not attain the value 0 at $x = 0$. It follows that $|A(y)| \geq 3/4 - b' \geq 1/4$, $y \in [\delta, 1]$, while $|A(y)| \leq 1$ for all $y \in [0, 1]$. Hence (4.4) follows for all $1 \leq p \leq \infty$.

We consider now when Q is increasing. We can assume that Q is not a translate of P, i.e., we do not have $P(x) = Q(x + \delta)$ for some δ, for then (4.4) follows trivially. In what follows, C and δ depend on d, and C may depend on p. We consider the following cases:

Case 3: $a' \geq 1/4$ *and* $b' \leq 100$. From (4.3) for P and $p = \infty$, it follows that $P(1/8) \geq \delta$ since otherwise P cannot attain the value 1 at $x = 1$. Hence $|A(y)| \geq a' - 1/8 \geq 1/8$ on $[0, \delta]$. On the other hand $|A(y)| \leq b'$ for all $y \in [0, 1]$ and hence (4.4) follows for all $1 \leq p \leq \infty$.

Case 4: $b' \geq 100$. The value of Q at $m = (a' + b')/2$ is less than $\delta < 1$, since otherwise by (4.3) (applied to $(1 - Q)$ for $p = \infty$), Q could not attain the value 0 at $x = a'$. Hence for $y \in [\delta, 1]$, $A(y) \geq m - 1 \geq b'/4$, while for all $y \in [0, 1]$ we have $|A(y)| \leq b'$. Therefore (4.4) follows for all $1 \leq p \leq \infty$.

Case 5: $b' \leq 1/2$. From (4.3) for $1 - P$ and $p = \infty$, $P(3/4) \leq \delta < 1$ since otherwise P could not attain the value 0 at $x = 0$. Hence, $|A(y)| \geq 3/4 - b' \geq 1/4$ for $y \in [\delta, 1]$. On the other hand, $|A(y)| \leq 1$ for all $y \in [0, 1]$, and therefore (4.4) follows for all $1 \leq p \leq \infty$.

Case 6: $a' \leq 1/4$ *and* $1/2 \leq b' \leq 100$. We let $M := \|A\|_{L^1([0,1])}$, $b_0 := \min(b', 1)$, and $b_1 := \max(b', 1)$. It follows that $\|P - Q\|_{L^1([a', b_0])} \leq \|A\|_{L^1([0,1])} = M$ and therefore by (4.1) and (4.3), $\|P - Q\|_{L^\infty([0, b_1])} \leq C_0 M$. Now let $E_k := \{y \in [0, 1] \mid |A(y)| > 2^k M\}$, $k = 0, 1, \ldots$. We fix k and show that

$$\text{meas}(E_k) \leq C 2^{-kd/(d-1)}. \tag{4.5}$$

We first observe that E_k is the union of disjoint open intervals which total at most $2d$ in number. Indeed, if y is in the boundary of E_k then $y = P(x) = Q(x + \delta)$ with

$\delta = \pm 2^k M$. For either of these choices of δ, the polynomial $P(x) - Q(x + \delta)$ has at most d zeros (unless it is identically zero) and therefore our claim follows.

Let $J := [y_0, y_0 + \mu]$, $\mu \geq \text{meas}(E_k)/(2d)$ be one of the intervals that make up E_k. We assume that $\phi(y) < \psi(y)$, $y \in J$; the other case is the same. We repeatedly move "up and to the right" by setting $x_0 = \phi(y_0)$, $x_1 = \psi(y_0)$, $y_1 = P(x_1)$, $x_2 = \psi(y_1)$, and so on. Let m be the smallest integer such that $y_m \notin J$. Then the points $x_0, \ldots, x_m$ are in $[0, b_1]$. Now $y_{j+1} - y_j = P(x_{j+1}) - Q(x_{j+1}) \leq C_0 M$, $j = 0, 1, \ldots, m - 1$. Hence

$$\text{meas}(J) \leq |y_m - y_0| \leq C_0 m M. \tag{4.6}$$

Since $|P(x) - P(x_0)| \leq C_0 m M$ for $x \in [x_0, x_m]$, and $x_m - x_0 = x_m - x_{m-1} + \cdots + x_1 - x_0 = \psi(x_{m-1}) - \phi(x_{m-1}) + \cdots + \psi(x_0) - \phi(x_0) \geq m 2^k M$, we have from (4.3) with $p = \infty$,

$$|P(x) - P(x_0)| \leq C m M (m 2^k M)^{-d}, \qquad x \in [0, b_1]. \tag{4.7}$$

For one of the values $x = 0, 1$, the left side of (4.7) is larger than $1/2$. Hence, $mM \leq C 2^{-kd/(d-1)}$. Using this in (4.6) establishes that $\text{meas}(J)$ and hence $\text{meas}(E_k)$ do not exceed $C 2^{-kd/(d-1)}$, which is (4.5).

Finally,

$$\begin{aligned} \int_0^1 |A(y)|^p \, dy &\leq M^p + \sum_{k=1}^{\infty} 2^{kp} M^p \, \text{meas}(\{y \in [0,1] \mid 2^{k-1} M \leq |A(y)| < 2^k M\}) \\ &\leq M^p (1 + \sum_{k=1}^{\infty} 2^{kp} \, \text{meas}(E_{k-1})). \end{aligned}$$

If $p < d/(d-1)$, the sum on the right side converges because of (4.5); therefore, we obtain (4.4) in this case as well. ∎

We will need to know, roughly speaking, that algebraic functions and their derivatives do not oscillate very much. This is stated more precisely in the following lemma.

Lemma 4.2 (Bounded Oscillation). *Assume that P and Q are polynomials with real coefficients in two variables of total degree less than r. Let ϕ and ψ be functions that are real analytic in the interior of an interval I and satisfy $P(x, \phi) = 0$ and $Q(x, \psi) = 0$ for $x \in I$. Let $A = \phi - \psi$. Then for $k = 0, 1, \ldots, r + 1$ either $A^{(k)}$ is identically zero on I or $A^{(k)}(x) = 0$ has finitely many solutions x in I. The number of solutions depends only on r.*

Proof. The statement that the kth derivative of ϕ is equal to the kth derivative of ψ can be written as a system of polynomial equations in $2k+3$ variables. For example, when

$k=1$, $A'(x)=0$ for a particular x if and only if there exist numbers ϕ, ψ, ϕ', and ψ' such that the following set of polynomial equations has a solution:

$$\begin{aligned} &P(x,\phi)=0, && Q(x,\psi)=0,\\ &D_1P(x,\phi)+D_2P(x,\phi)\phi'=0, && D_1Q(x,\psi)+D_2Q(x,\psi)\psi'=0,\\ &\phi'=\psi'. \end{aligned}$$

(D_1 and D_2 denote differentiation with respect to the first and second argument, respectively.) Tarski's Theorem (see Friedman [9, p. 225 ff.] or Jacobson [12, Chapt. 5] for a proof) states that such systems have solutions for x in a finite number of subintervals of I, and the number of subintervals depends only on the degree of P and Q. If $\phi'(x)=\psi'(x)$ for x in an open interval of I, then $\phi'(x)=\psi'(x)$ for all $x\in I$ because ϕ and ψ are analytic. Thus, either $A'(x)=0$ for all $x\in I$ or for a finite number, depending only on r, of values of x. This argument can be extended to higher derivatives of A in an obvious manner. $\square$

We shall also need the following "inverse inequality" for functions more general than polynomials.

LEMMA 4.3 (Inverse Inequality). *Let v be twice continuously differentiable on an open interval I and assume that v, v', and v'' each have one sign on I. If numbers p and q are given such that $0<p\le 1$ and $qp<q-p$, then there exists a constant C such that whenever $v\in L^q(I)$ then $v'\in L^p(I)$ and*

$$\|v'\|_p^*(I)\le C|I|^{-1}\|v\|_q^*(I). \tag{4.8}$$

Proof. We can assume without loss of generality that $I=(0,1)$, because the general result follows from this by scaling. We can also assume that $v\ge 0$ (consider $-v$ instead) and that $v(x)$ is increasing (otherwise, consider $v(1-x)$). If $v''\ge 0$ on I, it follows that for each x there is a $\xi\in[x,(x+1)/2]$ such that

$$v((x+1)/2)\ge v((x+1)/2)-v(x)=v'(\xi)(1-x)/2\ge v'(x)(1-x)/2.$$

Let $s=q/p>1$, and define t by $1/t+1/s=1$. Then by Hölder's inequality,

$$\begin{aligned}\int_0^1|v'(x)|^p\,dx&\le 2^p\left(\int_0^1 v((x+1)/2)^{ps}\,dx\right)^{1/s}\left(\int_0^1(1-x)^{-pt}\,dx\right)^{1/t}\\ &\le C(p,q)\left(\int_{1/2}^1 v(\tau)^{ps}\,d\tau\right)^{1/s}\end{aligned}$$

because $pt=(p^{-1}-q^{-1})^{-1}<1$. This proves (4.8).

If $v''\le 0$ on I, one proceeds similarly after noting that

$$v(x)\ge v(x)-v(x/2)=v'(\xi)(x/2)\ge v'(x)(x/2). \qquad\square$$

5. Proof of the regularity theorem. In this section we prove Theorem 1.1. The proof is divided into several steps; the first consists of constructing certain approximations to the solution $u(x,t)$ of (C). The ideas used in this construction are similar to those presented in [18] and [19].

We will assume that the initial data u_0 has support in $I := [0,1]$ and is in $\mathrm{BV}(\mathbb{R}) \cap B^\alpha(I)$ for some $\alpha > 0$. We fix an integer $r > \alpha$. For each $n > 0$, let $\tilde{S}_n$ be a best $L^1(I)$ approximation to u_0 from $\Sigma_n := \Sigma_{n,r}$, the class of discontinuous, piecewise polynomial functions of degree less than r with at most 2^n pieces, i.e., $\|u_0 - \tilde{S}_n\|_{L^1(I)} = s_n(u_0)_1$. We observe that there exists a constant C_1 that depends only on r such that

$$\|\tilde{S}_n\|_{\mathrm{BV}(\mathbb{R})} \le C_1 \|u_0\|_{\mathrm{BV}(\mathbb{R})}. \tag{5.1}$$

Indeed, let J_j be the intervals that support the polynomial pieces P_j of $\tilde{S}_n$. For each j let t_j be a point in J_j; if λ is the piecewise constant function that takes the value $c_j := u_0(t_j)$ on J_j then clearly $\mathrm{Var}_I(\lambda) \le \mathrm{Var}_I(u_0)$. (Here we assume that u_0 is taken to be right continuous, for example.) Now for each n, P_j is the best $L^1(J_j)$ approximation to u_0 on J_j among polynomials of degree less than r; in particular, it is a better approximation than the constant c_j. Hence, from (4.2) with $p = 1$ it follows that

$$\begin{aligned} \mathrm{Var}_{J_j}(P_j) &= \mathrm{Var}_{J_j}(P_j - c_j) \le C\|P_j - c_j\|_1^*(J_j) \\ &\le C[\|u_0 - P_j\|_1^*(J_j) + \|u_0 - c_j\|_1^*(J_j)] \le 2C\|u_0 - c_j\|_1^*(J_j) \\ &\le 2C\,\mathrm{Var}_{J_j}(u_0) \end{aligned}$$

Moreover, this inequality and (4.1) show that $\|P_j - c_j\|_\infty(J_j) \le C\,\mathrm{Var}_{J_j}(u_0)$ and therefore the jump in $\tilde{S}_n$ in going from J_j to J_{j+1} does not exceed $C\,\mathrm{Var}_{J_j}(u_0) + C\,\mathrm{Var}_{J_{j+1}}(u_0) + |c_{j+1} - c_j|$. This gives $\mathrm{Var}_I(\tilde{S}_n) \le C[\mathrm{Var}_I(u_0) + \mathrm{Var}_I(\lambda)]$ and (5.1) follows. In addition, one sees that

$$\begin{aligned} \|P_j - c_j\|_\infty^*(J_j) &\le C\|P_j - c_j\|_1^*(J_j) \\ &\le C[\|u_0 - P_j\|_1^*(J_j) + \|u_0 - c_j\|_1^*(J_j)] \le 2C\|u_0 - c_j\|_1^*(J_j) \\ &\le 2C\|u_0 - c_j\|_\infty^*(J_j). \end{aligned}$$

Because $|c_j| \le \|u_0\|_{L^\infty(I)}$, we have $\|\tilde{S}_n\|_{L^\infty(I)} \le C\|u_0\|_{L^\infty(I)}$.

We modify $\tilde{S}_n$ at each of its discontinuities x_i by replacing $\tilde{S}_n$ on $(x_i - \delta, x_i + \delta)$ by a linear function such that the resulting piecewise polynomial S_n is continuous on I. Clearly by choosing $\delta > 0$ sufficiently small, we will guarantee that

$$\|u_0 - S_n\|_{L^1(I)} \le 2s_n(u_0)_1,\ \|S_n\|_{\mathrm{BV}(\mathbb{R})} \le C_1\|u_0\|_{\mathrm{BV}(\mathbb{R})},$$
$$\text{and } \|S_n\|_{L^\infty(I)} \le C_1\|u_0\|_{L^\infty(I)}.$$

This C_1 is the constant of Theorem 1.1. Other properties of S_n are that it has no more than 2^{n+1} pieces and that the range of S_n is contained in $\Omega := \{y \mid |y| \le C_1\|u_0\|_{L^\infty(I)}\}$.

We now construct an approximation g_n to f on Ω. There exists an $r-1$ times continuously differentiable, piecewise polynomial function g_n of degree at most r with knots at the points $j/2^n \in \Omega$ that satisfies

$$\|f^{(k)} - g_n^{(k)}\|_{L^\infty(\Omega)} \le C\|f^{(r+1)}\|_{L^\infty(\Omega)}2^{-n(r+1-k)}, \quad k = 0, \ldots, r; \tag{5.2}$$

see, for example, [3]. It follows therefore that for n sufficiently large $\inf_{\xi\in\Omega} g_n''(\xi) \ge \frac{1}{2}\inf_{\xi\in\Omega} f''(\xi) > 1/(2C) > 0$. Clearly we can require this last property for small n and retain (5.2) as well. Hence g_n' is increasing on Ω.

Our interest now is to describe the solution $v(x,t)$ to problem (P) when $v_0 = S_n$ and $g = g_n$. We fix t and introduce three special types of points in I. The first are the knots of S_n, that is, points where S_n changes from one polynomial piece to another. By construction there are at most 2^{n+1} such points.

The second type of special points are isolated points x where $S_n(x) = j/2^n \in \Omega$ for some j. If S_n is polynomial of degree less than r on an interval $J_j \subset I$ then between any two consecutive points x_i and x_{i+1} of type two either $S_n(x_i) = S_n(x_{i+1})$, in which case $S_n'(\xi) = 0$ for some ξ in $[x_i, x_{i+1}]$, or $|S_n(x_i) - S_n(x_{i+1})| = 1/2^n$. Because S_n' has no more than $r-2$ zeros in J_j, if $k > r-1$ points of type two are in J_j, then the variation of S_n on J_j must be at least $(k-r+1)/2^n$. Because $\mathrm{Var}_I(S_n)$ is bounded, there are no more than $\left((r-1) + \|S_n\|_{\mathrm{BV}(\mathbb{R})}\right)2^{n+1} \le \left((r-1) + C_1\|u_0\|_{\mathrm{BV}(\mathbb{R})}\right)2^{n+1}$ points of the second type.

Let J be a maximal open interval which contains no points of the two types already described. Since the range of S_n on J is contained in an interval $[i/2^n, (i+1)/2^n]$, $P(s) := s + tg_n'(S_n(s))$ is a polynomial on J. A point of the third type is a point where P changes monotonicity. Because there are at most $C2^n$ intervals J and at most r^2 points of type three in each interval, there will be at most $C2^n$ points of type three with C depending only on r and $\|u_0\|_{\mathrm{BV}(\mathbb{R})}$.

We denote by x_j the points of any of the three types described above. According to (2.2), the solution $v(x,t)$ of (P) satisfies $v(x,t) = v(y_n)$ for some solution $y_n := y_n(x,t)$ to the equation

$$\frac{x - y_n}{t} = g_n'(S_n(y_n)). \tag{5.3}$$

Consider now a maximal interval I_0 of x values on which y_n takes values in an interval J which contains no points of the three types described above. Since $s + tg_n'(S_n(s))$ is by definition of the points of type three a monotone function of u for $u \in J$, there is at most one solution y_n to (5.3) in J. Thus, because y_n increases as x increases and there are only $C2^n$ intervals I_0, the solution $y_n(x,t)$, and hence $v(x,t)$, has at most $C2^n$ points of transition. Between these points of transition the solution $v(x,t)$ is a solution to the algebraic equation

$$v = P(x - tQ'(v)),$$

where P is the polynomial piece for S_n on I_0, and Q is the polynomial piece for g_n' on J. Thus v is a piecewise algebraic function of degree less than r^2. In what follows we will denote v by $S_n(x,t)$.

That the support of $S_n(\,\cdot\,,t) \subset I_t$ is well known [14]. The stability result (2.3) implies that

$$\begin{aligned}\|u(\,\cdot\,,t) - S_n(\,\cdot\,,t)\|_{L^1(\mathbb{R})} &\le \|u_0 - S_n(\,\cdot\,,0)\|_{L^1(\mathbb{R})} + t\|f' - g_n'\|_{L^\infty(\mathbb{R})}\|u_0\|_{\mathrm{BV}(\mathbb{R})} \\ &\le Cs_n(u_0)_1 + Ct\|u_0\|_{\mathrm{BV}(\mathbb{R})}2^{-rn}\end{aligned} \tag{5.4}$$

Proof of Theorem 1.1. Assume first that α is close to r and $u_0 \in B^\alpha(I)$. Then by the characterization (3.1), $\sum[2^{n\alpha}s_n(u_0)_1]^\sigma < \infty$. From (5.4) we obtain that $S_n(\,\cdot\,,t)$ converges to $u(\,\cdot\,,t)$ in $L^1(I_t)$ and therefore

$$u = S_0 + \sum_{n=0}^{\infty}(S_{n+1} - S_n) = \sum_{n=-1}^{\infty} T_n,$$

where $T_{-1} := S_0$ and for later use we define $S_{-1} := 0$.

From the form of the function $S_n(x,t)$ discussed above, we can write for $n = -1, 0, \ldots$

$$T_n = \sum_{j=1}^{N} A_j, \qquad N \le C2^n,$$

where C depends on r, t, and $\|u_0\|_{\mathrm{BV}(\mathbb{R})}$. (All further constants will depend on at most these three quantities and $\|f^{(r+1)}\|_{L^\infty(\Omega)}$.) Here $A_j = (\phi_j - \psi_j)\chi_j$ with ϕ_j and ψ_j algebraic functions, and χ_j the characteristic function of an interval I_j. We can further assume by Lemma 4.2 that $A_j^{(k)}$ has one sign on I_j for $k = 0, \ldots, r+1$ and $1 \le j \le N$.

We fix j and measure the smoothness of $A := A_j$. For this, fix h and consider the sets Γ of all x such that $\{x, x+h, \ldots, x+rh\} \subset I := I_j$, Γ' of all $x \notin \Gamma$ for which $\{x, x+h, \ldots, x+rh\} \cap I \ne \phi$, and Γ'' of all remaining $x \in \mathbb{R}$.

For $x \in \Gamma''$, $\Delta_h^r(A,x) = 0$, so

$$\int_{\Gamma''} |\Delta_h^r(A,x)|^\sigma\,dx = 0. \tag{5.5}$$

For $x \in \Gamma'$, $\Delta_h^r(A,x) \le 2^r(|A(x)| + \cdots + |A(x+rh)|)$. Since Γ' has measure no greater than $2r\min(h,|I|)$, we have for a fixed $p > 1$ with $p < (2r)/(2r-1)$, by Hölder's inequality

$$\int_{\Gamma'} |\Delta_h^r(A,x)|^\sigma\,dx \le C[\min(h,|I|)]^{1-\sigma/p}\left(\int_I |A(x)|^p\right)^{\sigma/p}. \tag{5.6}$$

We can write $A = \phi - \psi$ where ϕ is a piece of S_{n+1} and ψ is a piece of S_n. From (2.2), we can write ϕ as the solution of

$$g'_{n+1}(\phi) = \frac{x - (I + tg'_{n+1} \circ P_1)^{-1}(x)}{t}, \tag{5.7}$$

where P_1 is one of the polynomial pieces in the definition of $S_{n+1}(0)$; similarly for ψ. (We recall that on each piece of ϕ, the function g'_{n+1} can be taken as a polynomial, by our construction.) Because there exist constants C_1 and C_2 such that for all n and for all $\xi \in \Omega$, $0 < C_1 < g''_n(\xi) < C_2$, one knows that g'_n and $(g'_n)^{-1}$ are uniformly Lipschitz continuous for all n. Therefore,

$$\begin{aligned}
\|\phi - \psi\|_p^*(I) &\le C\|g'_n(\phi) - g'_n(\psi)\|_p^*(I) \\
&\le C\|g'_{n+1}(\phi) - g'_n(\psi)\|_p^*(I) + C\|g'_{n+1}(\phi) - g'_n(\phi)\|_p^*(I) \\
&\le C\|g'_{n+1}(\phi) - g'_n(\psi)\|_p^*(I) + C2^{-rn} \\
&= \frac{C}{t}\|(I + tg'_{n+1} \circ P_1)^{-1} - (I + tg'_n \circ P_2)^{-1}\|_p^*(I) + C2^{-rn} \\
&\le \frac{C}{t}\|(I + tg'_{n+1} \circ P_1)^{-1} - (I + tg'_n \circ P_2)^{-1}\|_1^*(I) + C2^{-rn} \\
&= C\|g'_{n+1}(\phi) - g'_n(\psi)\|_1^*(I) + C2^{-rn} \\
&\le C\|g'_n(\phi) - g'_n(\psi)\|_1^*(I) + C2^{-rn} \\
&\le C\|\phi - \psi\|_1^*(I) + C2^{-rn}.
\end{aligned} \tag{5.8}$$

Here the third inequality is because $|g'_{n+1} - g'_n| \le |f' - g'_{n+1}| + |f' - g'_n| \le C2^{-rn}$; the first equality is (5.7) and the inequality that follows is by Lemma 4.1. Therefore, from (5.6) and (5.8) we can conclude that

$$\int_{\Gamma'} |\Delta_h^r(A, x)|^\sigma \, dx \le C[\min(h, |I|)]^{1-\sigma/p}|I|^{-\sigma+\sigma/p}\left(\int_I |A(x)|\, dx + |I|2^{-rn}\right)^\sigma. \tag{5.9}$$

We next consider $x \in \Gamma$. Because $A^{(r)}$ is monotone on I, we know that for each x there is a ξ such that

$$|\Delta_h^r(A, x)| = C(r)h^r|A^{(r)}(\xi)| \le Ch^r \max(|A^{(r)}(x)|, |A^{(r)}(x + rh)|).$$

Without loss of generality assume that the maximum is attained by the first term. For a number $\epsilon > 0$ to be specified in a moment, let $\alpha_r := \alpha$ and $\alpha_k := \alpha_{k+1} - 1 - \epsilon$, $k = r-1, \dots, 0$, and let $\sigma_k := 1/(\alpha_k + 1)$. Then by choosing ϵ appropriately, we will have $\sigma_0 = p$, where p is as in (5.8). (Here we must assume that α is close enough to r.) We also have that $0 < \sigma_k \le 1$ for $k = r, \dots, 1$, and that $\sigma_k\sigma_{k-1} < \sigma_{k-1} - \sigma_k$; therefore, Lemma 4.3 implies that

$$\|A^{(r)}\|_{\sigma_r}^*(I) \le C|I|^{-1}\|A^{(r-1)}\|_{\sigma_{r-1}}^*(I) \le \cdots \le C|I|^{-r}\|A\|_{\sigma_0}^*(I).$$

We then apply (5.8) to find that

$$
\begin{aligned}
\int_\Gamma |\Delta_h^r(A,x)|^\sigma\,dx &\le Ch^{r\sigma}\int_I |A^{(r)}(x)|^\sigma\,dx\\
&\le Ch^{r\sigma}|I|^{-r\sigma+1}\left(\frac{1}{|I|}\int_I |A(x)|^p\,dx\right)^{\sigma/p}\\
&\le Ch^{r\sigma}|I|^{-r\sigma-\sigma+1}\left(\int_I |A(x)|\,dx+2^{-rn}|I|\right)^\sigma.
\end{aligned}
\tag{5.10}
$$

Because $\Gamma=\phi$ if $h>|I|/r$, (5.10), (5.9), and (5.5) imply that

$$
\begin{aligned}
\int_{\mathbb{R}} |\Delta_h^r(A,x)|^\sigma\,dx \le C\left([\min(h,|I|)]^{1-\sigma/p}|I|^{-\sigma+\sigma/p}+|I|^{-r\sigma-\sigma+1}h^{r\sigma}\chi(h)\right)&\\
\times\left(\int_I |A(x)|\,dx+2^{-rn}|I|\right)^\sigma&,
\end{aligned}
$$

where χ is the characteristic function of $[0,|I|/r]$. It follows that $\omega_r(A,h)_\sigma^\sigma$ is also less than the right hand side of our latest inequality. Therefore,

$$
\begin{aligned}
&\int_0^\infty h^{-\alpha\sigma}\omega_r(A,h)_\sigma^\sigma\,dh/h\\
&\qquad\le C\Bigg(|I|^{-\sigma+\sigma/p}\int_0^{|I|} h^{-\alpha\sigma-\sigma/p}\,dh+|I|^{1-\sigma}\int_{|I|}^\infty h^{-\alpha\sigma-1}\,dh\\
&\qquad\qquad+|I|^{-r\sigma-\sigma+1}\int_0^{|I|} h^{(r-\alpha)\sigma-1}\,dh\Bigg)\left(\int_I |A(x)|\,dx+2^{-rn}|I|\right)^\sigma\\
&\qquad\le C|I|^{-\alpha\sigma-\sigma+1}\left(\int_I |A(x)|\,dx+2^{-rn}|I|\right)^\sigma\\
&\qquad\le C\left(\int_I |A(x)|\,dx+2^{-rn}|I|\right)^\sigma,
\end{aligned}
\tag{5.11}
$$

because $-\alpha\sigma-\sigma+1=0$.

We can now estimate the smoothness of $T_n=T_n(\,\cdot\,,t)$. Because $\sigma<1$, we know that

$$
\omega_r(T_n,h)_\sigma^\sigma\le\sum_{j=1}^N \omega_r(A_j,h)_\sigma^\sigma. \tag{5.12}
$$

Hence, (5.11) and Hölder's inequality imply that

$$
\begin{aligned}
\int_0^\infty h^{-\alpha\sigma}\omega_r(T_n,h)_\sigma^\sigma\,dh/h &\le C\sum_{j=1}^N\left(\int_{I_j} |A(x)|\,dx+2^{-rn}|I_j|\right)^\sigma\\
&\le CN^{1-\sigma}\left(\|T_n\|_{L^1(I_t)}+2^{-rn}|I_t|\right)^\sigma\\
&\le CN^{\alpha\sigma}\left(\|T_n\|_{L^1(I_t)}^\sigma+2^{-rn\sigma}|I_t|^\sigma\right).
\end{aligned}
\tag{5.13}
$$

Consider now the expression for u, $u(\,\cdot\,,t)=\sum_{n=-1}^{\infty}T_n$. Using (5.12) and the continuous embedding of $B^\alpha([0,1])$ into $L^1([0,1])$, we obtain

$$
\begin{aligned}
\int_0^\infty \omega_r(u,h)_\sigma^\sigma h^{-\alpha\sigma-1}\,dh &\le \sum_{n=-1}^{\infty}\int_0^\infty \omega_r(T_n,h)_\sigma^\sigma h^{-\alpha\sigma-1}\,dh\\
&\le C\sum_{n=-1}^{\infty}2^{n\alpha\sigma}\left(\|T_n\|_{L^1(I_t)}^\sigma+2^{-rn\sigma}\right)\\
&\le C\sum_{n=-1}^{\infty}2^{n\alpha\sigma}\left(s_n(u_0)_1^\sigma+2^{-rn\sigma}\right)\\
&\le C\|u_0\|_{B^\alpha([0,1])}^\sigma+C\|u_0\|_{L^1([0,1])}^\sigma+C\\
&\le C\|u_0\|_{B^\alpha([0,1])}^\sigma+C,
\end{aligned}
\tag{5.14}
$$

because from (2.3), for $n=-1,0\ldots,$

$$
\begin{aligned}
\|T_n(t)\|_{L^1(I_t)}&=\|S_{n+1}(t)-S_n(t)\|_{L^1(I_t)}\\
&\le\|S_{n+1}(t)-u(t)\|_{L^1(I_t)}+\|u(t)-S_n(t)\|_{L^1(I_t)}\\
&\le\|S_{n+1}(0)-u_0\|_{L^1(I_t)}+\|u_0-S_n(0)\|_{L^1(I_t)}+C2^{-rn}\\
&\le 4s_n(u_0)_1+C2^{-rn}.
\end{aligned}
$$

By (5.14), $\|u(\,\cdot\,,t)\|_{B^\alpha(I_t)}\le C\|u_0\|_{B^\alpha([0,1])}+C$. This proves the theorem for α close to r.

We shall now complete the proof by using interpolation. Fix a value of $\beta<r$ with β close to r so that the above analysis holds for β. We can estimate the K-functional $K(u,h):=K(u,h,L^1,B^\beta)$. Let $u_0\in L^1([0,1])$, let v_0 be any function in $B^\beta([0,1])$, and let $u(x,t)$ and $v(x,t)$ be the solutions to (C) corresponding to these initial conditions. Then, from the stability estimate (2.3) and (5.14), we see that

$$\|u(\,\cdot\,,t)-v(\,\cdot\,,t)\|_{L^1(I_t)}+h\|v(\,\cdot\,,t)\|_{B^\beta(I_t)}\le C[\|u_0-v_0\|_{L^1([0,1])}+h\|v_0\|_{B^\beta([0,1])}+h].$$

We recall that $\|u(\,\cdot\,,t)\|_{L^1(I_t)}=\|u_0\|_{L^1([0,1])}$ and $v(\,\cdot\,,t)\equiv 0$ when v_0 is chosen to be zero. Therefore, we can take an infimum over all $v_0\in B^\beta([0,1])$ to see that

$$K(u(\,\cdot\,,t),h)\le\begin{cases}\|u_0\|_{L^1([0,1])}, & \text{for } h>1,\\ CK(u_0,h)+Ch, & \text{for } h\le 1.\end{cases}$$

Apply the (θ,q) norm to $K(u(\,\cdot\,,t),h)$ with $\theta=\alpha/\beta$ and $q=1/(\alpha+1)$:

$$
\begin{aligned}
\|u(\,\cdot\,,t)\|_{B^\alpha(I_t)}&\le\|u(\,\cdot\,,t)\|_{L^1(I_t)}+\left(\int_0^\infty[h^{-\theta}K(u(\,\cdot\,,t),h)]^q\,dh/h\right)^{1/q}\\
&\le\|u_0\|_{L^1([0,1])}+C\left(\int_0^1[h^{-\theta}(K(u_0,h)+h)]^q\,dh/h\right)^{1/q}\\
&\qquad+C\left(\int_1^\infty[h^{-\theta}\|u_0\|_{L^1([0,1])}]^q\,dh/h\right)^{1/q}\\
&\le C\|u_0\|_{B^\alpha([0,1])}+C,
\end{aligned}
$$

by the equivalence in §3. This holds for all $\alpha<\beta$, and hence for all $\alpha<r$. $\square$

Acknowledgments. We thank Leonard Lipshitz for several helpful discussions.

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