GPOTS 2015, May 26–30

Titles and Abstracts

Department of Mathematics, Purdue University

PLENARY TALKS

KELLY BICKEL:

Operators on Matrix Weighted L^2 Spaces:

Abstract: It is well-known that every Calderón-Zygmund operator is bounded on $L^2(w)$ if and only if w is an A_2 weight and any such operator's norm depends (at most) linearly on the A_2 characteristic of w. One can similarly define matrix A_2 weights, which characterize the boundedness of Calderón-Zygmund operators on matrix weighted L^2 spaces. However, the matrix A_2 conjecture, namely whether operator norms depend linearly on the A_2 characteristic of W, is open even for simple operators.

In this talk, we will discuss several recent results related to the matrix A_2 conjecture. One set of results concern improved bounds for the norms of certain Calderón-Zygmund operators, including the Hilbert transform and sparse operators, on matrix weighted L^2 spaces. The other main result concerns a two-weight $T(1)$ theorem for band operators (operators with a finite number of diagonals) when the weights are in A_2. This is joint work with Brett Wick and Stefanie Petermichl.

KENNETH R. DAVIDSON:

Duality, convexity and peak interpolation in Drury-Arveson space

Abstract: The closed algebra $\mathcal{A}_d$ is generated by the polynomial multipliers on the Drury-Arveson space. We identify $\mathcal{A}_d^*$ as a direct sum of the preduals of the full multiplier algebra and of a commutative von Neumann algebra. This provides a natural analogue of classical results concerning the dual of the ball algebra. This is applied to questions about peak interpolation for multipliers.

This is joint work with Raphaël Clouatre.

CALEB ECKHARDT:

Representations of discrete nilpotent groups

Abstract: Most discrete groups are not Type I which produces considerable difficulties for the study of their representation theory. We discuss how recent successes of the classification program for nuclear C*-algebras show that we can characterize the C*-algebras generated by irreducible representations of nilpotent groups by their ordered K-theory. We illustrate this idea with ordered K-theory calculations for the irreducible representations of the 3-step unitriangular group. Some of this is joint work with Kleski and McKenney.

SØREN EILERS:

Classifying naturally occurring graph C^*-algebras

Abstract: Although very far from complete, the classification theory for graph C^*-algebras has come to age in recent years, and offers results which allow to decide isomorphism between many pairs of C^*-algebras that are not covered by the general classification of the Elliott program. Since graph C^*-algebras are surprisingly ubiquitous among C^*-algebras of current interest this provides a new and powerful tool for classification of non-simple C^*-algebras.

I intend to detail two successful instances of this modus operandi, concerning on one hand the *quantum lens spaces* related to Vaksman-Soibelman odd quantum spheres defined and studied by Hong and Szymański in 2003, and on the other the *C^*-algebras of right-angled Artin monoids* defined and studied by Crisp and Laca in 2002.

I am confident that there are more "cold cases" waiting to be resolved using this method, and will attempt to provide an overview of what kind of problems can and can not be expected to benefit from appealing to it. In the process I will be reporting on joint work with Xin Li, Gunnar Restorff, Efren Ruiz, and Adam Sørensen.

BEN HAYES:

A (Gentle) Introduction to Sofic Entropy

Abstract: In the last decade, several landmark results have been achieved using Popa's deformation rigidity theory reducing the isomorphism problem for certain crossed product von Neumann algebras by (necessarily) nonamenable groups to classification of isomorphism of actions of such groups on probability spaces. These results include crossed product algebras of certain Bernoulli shifts being classified by the isomorphism class of such actions. However, these results leave completely open how one classifies such actions. The answer, as I will describe in my talk, is by extending dynamical entropy for actions of amenable groups to the (significantly larger) class of sofic groups. I will also discuss other connections between sofic entropy and von Neumann algebras, including my results connecting entropy to Fuglede-Kadison determinants in essentially complete generality, generalizing previous work of Deninger, Deninger-Schmidt, Li, Kerr-Li, Bowen, Bowen-Li and Li-Thom. Time permitting, I will also discuss how one can connect the study of such actions to noncommutative harmonic analysis via the study of representations of C*-algebras. No knowledge of sofic groups or dynamical entropy will be assumed

ADRIAN IOANA:

Spectral gap and orbit equivalence rigidity for translation actions

Abstract: A dense inclusion of a countable group Γ into a locally compact group G gives rise to a left translation action $\Gamma \curvearrowright G$. In this talk, I will survey recent work motivated by the following question: how much does the equivalence relation on G of belonging to the same Γ-orbit remember about the inclusion $\Gamma < G$? When G is compact this question is related to whether the translation action has spectral gap. On the other hand, if G is locally compact but not compact, then the question is related to a certain local spectral gap

property of the translation action. I will report on recent work with R. Boutonnet and A. Salehi Golsefidy in which we establish local spectral gap, whenever Γ is a dense subgroup generated by algebraic elements of an arbitrary connected simple Lie group G. This extends to the non-compact setting recent works of Bourgain-Gamburd and Benoist-de Saxce.

MICHAEL JURY:
Dilations and Constrained Algebras

Abstract: Let $A \subset C(X)$ be a uniform algebra. The *rational dilation problem* asks whether every contractive homomorphism $\pi : A \to B(H)$ is completely contractive. (For example, when A is the disk algebra the answer is "yes," an immediate consequence of the Sz.-Nagy dilation theorem.) We investigate this question for some other uniform algebras, for example the subalgebra of the disk algebra consisting of functions with $f'(0) = 0$ (here the answer is "no"). We also obtain a simplified proof of Agler's rational dilation theorem for the annulus. (This is joint work with Scott McCullough and Michael Dritschel.)

HENRI MOSCOVICI:
Modular geometry on noncommutative tori

Abstract: The concept of intrinsic curvature, which lies at the very core of classical geometry, has only lately begun to be understood in the noncommutative framework. I will present recent results in this direction for noncommutative tori, obtained in joint works with A. Connes and with M. Lesch, which illustrate both the challenges and the rewards of doing geometry on noncommutative spaces.

ZHUANG NIU:
The classification of unital separable simple approximately subhomogeneous C*-algebras

Abstract: A C*-algebra is locally approximately subhomogeneous (ASH) if any finite subset is approximately contained in a sub-C*-algebra which is subhomogeneous. Consider the class of Jiang-Su stable simple unital separable ASH algebras. Then this class of C*-algebras is shown to be classified by the Elliott invariant. This talk is based on the joint works with George Elliott, Guihua Gong, Huaxin Lin.

DAVE PENNEYS:
The $2D2$ subfactor

Abstract: The representation theory of subfactors generalizes the representation theory of quantum groups, and thus we think of subfactors as objects which encode quantum symmetries. In one sense, subfactors of small index are the simplest examples of subfactors, and in joint work with Afzaly and Morrison, we have a complete classification of their standard invariants to index $5+1/4$. I will discuss recent joint work with Morrison on 2-supertransitive subfactors at index $3+\sqrt{5}$. One important ingredient is a new variation of Bigelow's jellyfish algorithm which is universal for finite depth subfactors.

EMILY PETERS:

Constructing subfactors via two-strand jellyfish relations

Abstract: We are interested in constructing new examples of subfactors, and one promising way to do this is by directly constructing the subfactor's planar algebra. Recently, Bigelow-Penneys explored the connection between the shape of a subfactor's principal graph and the relations in its planar algebra, and explained that 'spoke-shaped' principal graphs give planar algebras having 'jellyfish relations'. Jellyfish relations are great because they make it easy to directly prove that a given planar algebra is a subfactor planar algebra. In joint work with Penneys, we give algorithms to construct jellyfish relations when they exist. We use this to construct subfactors, including a $3^{\mathbb{Z}/4}$ subfactor (known to Izumi but not previously appearing in the literature).

SORIN POPA:

Paving over arbitrary MASAs in von Neumann algebras

Abstract: I will present some recent work with Stefaan Vaes, where we consider a paving property for a MASA A in a von Neumann (vN) algebra M, that we call *so-paving*, involving approximation in the *so*-topology, rather than in norm (as in classical Kadison-Singer paving), but which coincides with norm-paving for the diagonal MASA in $B(H)$. We conjecture that any MASA in any von Neumann algebra satisfies *so*-paving. We use recent work of Marcus-Spielman-Srivastava to check this for all MASAs in type I vN algebras, all Cartan subalgebras in amenable vN algebras and in group measure space II_1 factors of profinite actions. The conjecture also holds true for singular MASAs in II_1 factors, where the paving size is shown to be $4\varepsilon^{-2}$.

MIKAEL RØRDAM:

Just infinite C^*-algebras

Abstract: A (discrete) group is called just infinite if it is infinite and all its non-trivial normal subgroups have finite index. There is a well-established theory for just infinite groups, and there are many interesting examples of just infinite groups (including, for example, the Grigorchuk groups). One can in a similar way define a (unital) C^*-algebra to be just infinite if it is infinite dimensional and if all its proper quotients are finite dimensional. Infinite dimensional simple C^*-algebras and essential extensions of simple C^*-algebras by finite dimensional C^*-algebras are just infinite (for trivial reasons). We show that there exist residually finite dimensional just infinite C^*-algebras (that can be chosen to be AF-algebras), and we explain some structure results for just infinite C^*-algebras. The construction of a just infinite residually finite dimensional AF-algebra can be done using an old result by Bratteli and Elliott which says that each totally disconnected spectral space arises as the primitive ideal space of an AF-algebra. We discuss possible connections to just infinite groups. This is work in progress joint with V. Grigorchuk and M. Musat.

ZHONG-JIN RUAN:

On Exotic Group C*-algebras

Abstract: Let Γ be a discrete group. We can obtain two natural group C*-algebras: the full group C*-algebra $C^*(\Gamma)$ and the reduced group C*-algebra $C_r^*(\Gamma)$. A C*-algebra A is an *exotic C*-algebra* (associatd with Γ) if there exist proper surjective quotients

$$C^*(\Gamma) \to A \to C_r^*(\Gamma).$$

In this talk, we show that for a large class of nonamenable groups, the associated exotic C^*-algebras A have poor local properties. More precisely, we demonstrate the failure of local reflexitity, exactness, and local lifting property. Additionally, A does not admit an amenable trace and, hence, is not quasidiagonal and does not have the WEP. This is a recent joint work with Matthew Wiersma.

YASUHIKO SATO:

Strict comparison and absorption of the Jiang-Su algebra

Abstract: In the recent progress of the classification theory for C* algebras, the Jiang-Su algebra $\mathcal{Z}$ is among one of the most important algebras. The algebra $\mathcal{Z}$ is constructed as a simple projectionless C* algebra with a unique tracial state. In this talk, I will begin with an overview of theory of the Jiang-Su algebra and strict comparison which is known as a kind of Murray-von Neumann equivalence in C* algebraic context. As a main result of this study, we proved that absorption of the Jiang-Su algebra $\mathcal{Z}$ and strict comparison are equivalent for nuclear C* algebras with a unique tracial state. This talk is based on a joint work with Hiroki Matui.

THOMAS SINCLAIR:

Omitting types in operator systems

Abstract: A fundamental result of Junge and Pisier states that, for $n \geq 3$, the set of n-dimensional operator spaces is not separable in the strong topology. In particular, there cannot be a universal separable operator space, refuting Kirchberg's Conjecture (A) that there is a unique C* norm on the algebraic tensor product of $B(H)$ with itself. In this talk, I will discuss some model-theoretic and descriptive-set theoretic issues concerning quantitive, finitary characterizations of various properties of operator spaces and show how they connect to the aforementioned result of Junge and Pisier and Kirchberg's embedding problem. This is based on joint work with Isaac Goldbring.

ANNA SKRIPKA:

Differentiation of operator functions.

Abstract: We will discuss modern estimates for higher order derivatives of operator functions that generalize a straightforward estimate for derivatives of scalar composite functions. These results have been obtained in response to questions of mathematical physics, which will be outlined in the talk.

AARON TIKUISIS

Regularity for C*-algebras and the Toms–Winter conjecture

Abstract: C*-algebraic regularity is a fairly recent idea which has come to occupy a central place in the Elliott classification programme for nuclear C*-algebras. It concerns roughly three properties of C*-algebras: finite noncommutative dimension (nuclear dimension/decomposition rank), tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, and strict comparison of positive elements. The Toms–Winter conjecture predicts that, among simple separable nuclear non-type I C*-algebras, these three properties coincide. In my talk, I will describe these properties, discuss the Toms-Winter conjecture and what is known, and explain the connections to Elliott's classification programme.

STUART WHITE:

Covering dimension and classification of maps by traces

Abstract: Given a finite injective separably acting von Neumann algebra M and a II_1 factor N, it is well known that *-homomorphisms from M into the ultrapower N^ω are completely determined upto unitary equivalence by their behaviour on traces. In this talk I will discuss C*-analogues of this fact, allowing for a weakening of unitary equivalence in order to circumvent K-theoretic obstructions, and give applications to the covering dimension of simple nuclear C*-algebras. This is joint work with Bosa, Brown, Sato, Tikusis and Winter.

CONTRIBUTED TALKS

SCOTT ATKINSON:

Convex Sets Associated to C*-Algebras

Abstract: Given a separable C*-algebra $\mathfrak{A}$, we can associate to it an invariant given by the weak approximate unitary equivalence classes of $*$-homomorphisms from $\mathfrak{A}$ to a chosen McDuff II_1-factor M. We will see that this object takes the form of a convex set. This invariant is closely related (and sometimes affinely homeomorphic) to the trace space of $\mathfrak{A}$, but it is in general a finer invariant than the trace space. In addition to discussing its basic properties, we will also discuss some interesting questions arising from this new invariant.

STEPHEN AVSEC:

Symmetries of Noncommutative Brownian Motion

Abstract: Classically, Freedman's theorem states that a sequence of random variables $(X_1, X_2, \dots)$ is rotatable if and only if $X_j = g_j\sigma$ where $(g_1, g_2, \dots)$ are iid Gaussian variables and σ is independent. A sequence is rotatable if the joint distribution of every subsequence is invariant under orthogonal transformations. Freedman's theorem along with the classical Gaussian functor implies that the only random process with rotatable increments is (conditionally) a Brownian motion. In this talk, we will discuss noncommutative random processes with rotatable increments as well as those with quantum rotatable increments.

MARCEL BISCHOFF:

A planar algebraic description of defect lines in conformal field theory

Abstract: Let $\mathcal{A}$ be a completely rational conformal net (of type III factors) on the circle. Recently, together with Y. Kawahigashi, R. Longo and K.-H. Rehren, we obtained a classification of certain boundary conditions, so-called $\mathcal{A}$-topological defect lines, between local conformal nets on Minkowski space based on $\mathcal{A}$. This classification is given in terms of data (essentially subfactors) related to representation theory of $\mathcal{A}$.

We introduce a notion of fusion, which correspond to putting defect lines next to each other and intertwiners, which are bounded operators intertwining actions between a configuration of incoming and outgoing defect lines. The spaces of these intertwiners form a kind of a subfactor planar algebra describing the behaviour of the defect lines.

ARNAUD BROTHIER:

Approximation properties for subfactors

Abstract: The study of analytical properties for subfactors has been initiated by Popa when he defined in this context the notions of amenability, property (T), and the Haagerup property. He showed that those properties only depend on the standard invariant associated to the subfactor. Popa and Vaes recently developed a representation theory for subfactors, standard invariants, and rigid C*-tensor categories. They proved that the Temperley-Lieb-Jones standard invariants have the Haagerup property by using a previous result on quantum groups and their new framework.

I will present a new proof of this theorem by using exclusively planar algebra technology and the original definition of the Haagerup property for subfactors. This is a joint work with Vaughan Jones.

JULIAN BUCK:

Large Subalgebras of C^*-Algebras

Abstract: The concept of a large subalgebra of a C^*-algebra was introduced by N.C. Phillips as an abstract characterization for the approximating subalgebras which appear in the study of transformation group C^*-algebras and various other crossed products. We discuss results by Archey, Phillips, and the author on what properties of large subalgebras imply about the structure of the full agebra of interest. Applications to crossed products provide the primary examples of current interest.

BRANIMIR ĆAĆIĆ: **Strict deformation quantisation via crossed products**

Abstract: It is well known that an irrational noncommutative 2-torus can be equivalently constructed either as the crossed product of the circle by an irrational rotation or as a strict deformation quantisation of the 2-torus along its translation action on itself. We give an alternative construction of the strict deformation quantisation of an algebra A along the action of a compact Abelian Lie group G as a fixed-point subalgebra of a crossed product of A, which not only generalises the alternative construction of the irrational noncommutative 2-tori as irrational rotation algebras, but also gives rise to generalisations of Brain, Mesland,

and Van Suijlekom's unbounded KK-theoretic factorisations of irrational noncommutative 2-tori over the circle and of irrational θ-deformed 3-spheres over the 2-sphere. This is joint work with Bram Mesland.

JOSÉ R. CARRIÓN:

Almost flat K-theory of classifying spaces

Abstract: For a discrete countable group Γ, Dadarlat has shown that the quasidiagonality of $C^*(\Gamma)$ imposes K-theoretic restrictions on the classifying space $B\Gamma$: its K-theory classes must be almost-flat. (We assume $B\Gamma$ has a finite model.) We discuss a correspondence between between almost flat K-theory classes on $B\Gamma$ and group homomorphisms $K_0(C^*(\Gamma)) \to \mathbb{Z}$ that are implemented by discrete asymptotic homomorphisms from $C^*(\Gamma)$ to matrix algebras. This correspondence allows us to better understand just when all the elements of $K^0(B\Gamma)$ are almost-flat. This is joint work with Marius Dadarlat.

IONUT CHIFAN:

Product rigidity for von Neumann algebras arising from non-elementary hyperbolic groups

Abstract: Two groups Γ and Λ are called W^*-equivalent if they give rise to isomorphic von Neumann algebras. I will show that whenever $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are icc hyperbolic groups and Λ is an arbitrary group such that $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ is W^*-equivalent to Λ it follows that $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ and, up to amplifications, Γ_i is W^*-equivalent to Λ_i for all i. This strengthens some results of N. Ozawa and S. Popa from 2003. The talk based on a joint work with Rolando de Santiago and Thomas Sinclair.

ALEXANDRU CHIRVASITU:

Finiteness and simplicity for discrete quantum group algebras

Abstract: The C^*-algebras of free groups exhibit very different behavior depending on which completion one considers: The reduced algebra is simple while the full C^*-algebra is residually finite-dimensional, in the sense that it admits enough finite-dimensional representations.

The main result will be that a similar dichotomy occurs for group algebras of discrete quantum groups arising as Pontryagin duals of the so-called free unitary and orthogonal groups. The latter are non-commutative geometric analogues of the ordinary unitary and orthogonal groups respectively, and their definition will be recalled in the course of the talk.

RAPHAËL CLOUÂTRE:

Completely contractive representations of multiplier algebras of Nevanlinna-Pick spaces

Abstract: In the spirit of previous work of Agler and Ambrozie-Engliš-Müller, we investigate completely contractive representations of multiplier algebras of Hilbert spaces of analytic functions with a complete Nevanlinna-Pick reproducing kernel. We extract dilation theoretic and function theoretic information from a certain positivity property of the reproducing kernels. We also exhibit representations of the associated Toeplitz algebra that have

the unique extension property, and gain some insight into the structure of the C^*-envelope of these multiplier algebras. This is joint work with Michael Hartz.

VALENTIN DEACONU:
Symmetries of graph C^*-algebras

Abstract: Given a discrete locally finite graph $E = (E^0, E^1, r, s)$, we consider symmetries of the associated C^*-correspondence $\mathcal{H}_E$ and of the graph algebra $C^*(E)$, defined using actions and representations of a group G. Examples include self-similar actions of groups on trees.

We study the fixed point algebra $C^*(E)^G$ and the crossed product $C^*(E) \rtimes G$. The group G acts also on the AF-core $C^*(E)^{\mathbb{T}}$ and $C^*(E)^{\mathbb{T}} \rtimes G \cong (C^*(E) \rtimes G)^{\mathbb{T}}$. If G is finite, then $C^*(E) \rtimes G$ is isomorphic to the C^*-algebra of a graph of C^*-correspondences, constructed using orbits and characters of the stabilizer groups. Some of the results can be extended to actions and representations of groupoids.

SLAVISA DJORDJEVIĆ: **Operator matrices and MP-inverse with presribed domain**

Abstract: In this talk we will present ideas to find the best approximate solution of an equation the type $Ax = b$, where A is the linear operator between two Hilbert spaces X and Y, with limitation that x belong to prescribed subspace K of X.

ADAM DOR ON:
Isomorphisms of tensor algebras arising from weighted partial dynamical systems

Abstract: Given a compact space X, a (clopen) *weighted partial dynamical system* is a double (σ, w) comprised of a d-tuple $\sigma = (\sigma_1, ..., \sigma_d)$ of continuous maps $\sigma_i : X_i \to X$, and a d-tuple $w = (w_1, ..., w_d)$ of continuous weights $w_i : X_i \to (0, \infty)$ where each X_i is clopen in X. Weighted partial dynamical systems encompass many different classical objects such as non-negative matrices, multivariable dynamical systems and distributed function systems.

We associate certain canonically constructed C*-correspondences to weighted partial dynamical systems, and classify their non-self-adjoint tensor algebras up to bounded / isometric isomorphism in terms of the graphs and weighted transition functions of the systems. This will provide us with generalizations and analogues of results by Davidson, Katsoulis, Kribs, Solel and others, in the classification of tensor algebras via classical dynamical objects.

CATALIN DRAGAN:
Sums of equivalent sequences of positive operators in von Neumann factors

Abstract: Let A be a positive operator in an infinite σ-finite von Neumann factor $\mathcal{M}$ and let $\{B_j\}_{j=1}^{\infty} \subset \mathcal{M}^+$. We give sufficient conditions for the decomposition $A = \sum_{j=1}^{\infty} C_j$ to hold when $C_j \sim B_j$ for all j (the equivalence $C \sim B$ means $C = XX^*$ and $B = X^*X$ for some $X \in \mathcal{M}$) and when C_j are unitarily equivalent to B_j for all j. This extends recent work of Bourin and Lee for the case of $B_j = B$ and $\mathcal{M} = B(\mathcal{H})$ and answers affirmatively their conjecture. For the case when $B_j = B$ for all j we provide necessary conditions, which in the type III case are also sufficient.

EUSEBIO GARDELLA

Group representations on L^p spaces.

Abstract: We study group representations on L^p spaces, for $p \in [1, \infty)$, and the associated Banach algebras $F^p_\lambda(G)$ [Herz, 1973] and $F^p(G)$ [Phillips, 2012]. These are, respectively, generalizations of the reduced group C^*-algebra, and the full group C^*-algebra, which are the cases $p = 2$ of this construction. While some classical results from group C^*-algebras carry over to the L^p analogs (for instance, a characterization of amenability of G in terms of the canonical map from full to reduced), the algebras $F^p_\lambda(G)$ and $F^p(G)$ tend to be very rigid objects when p is not equal to 2. The main result of the talk is that for p and q not equal to 2, and for locally compact groups G and H, there is a contractive isomorphism $F^p_\lambda(G) \cong F^q_\lambda(H)$ if and only if G and H are isomorphic, and p and q are either equal or Hölder conjugate. When $p = q = 1$, this was first obtained by Wendel in the 60's. More generally, we are able to describe all contractive, nondegenerate maps $F^p_\lambda(G) \to F^p_\lambda(H)$ for locally compact groups G and H, and for $p \neq 2$. As an application, we show that if G is torsion free and $p \neq 2$, then $F^p_\lambda(G)$ does not contain any idempotent e such that $\|e\| = \|1 - e\| = 1$. Whether this also holds for $p = 2$ is a very relevant open problem.

Similar conclusions hold for the L^p analogs of the von Neumann algebra of a group, and it is conceivable that our techniques constitute a step forward proving or disproving a famous conjecture of Herz (which essentially states that the two 'natural' analogs of the von Neumann algebra of the group are always the same).

This is based on joint work with Hannes Thiel (University of Münster).

ELIZABETH GILLASPY:

Separable representations of k-graph C^*-algebras, and some applications

Abstract: A higher-rank graph, or k-graph, is a small category which gives rise to a C^*-algebra (modeled after the Cuntz-Krieger algebras). The resulting C^*-algebra is consequently separable, but the representations of k-graph C^*-algebras in the literature are almost always on an unseparable ℓ^2-space. In recent work with Farsi, Kang, and Packer, we have found a way to construct separable representations of the C^*-algebras of finite k-graphs. Our construction also gives other useful information, in particular about the KMS states of the C^*-algebra, and a wavelet-type decomposition of the separable Hilbert space.

ILJA GOGIĆ

CB-norm approximation of derivations by elementary operators

Abstract: We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps. On unital C^*-algebras A, however, it is natural to regard two-sided multiplication maps $x \mapsto axb$ $(a, b \in A)$ as basic building blocks (instead of rank one operators). We can therefore try to approximate a more general map on A, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by *elementary operators*. In this talk I will consider the problem of description of those derivations of unital C^*-algebras that can be approximated by elementary operators in the completely bounded norm. This is a joint work in progress with Richard Timoney (Trinity

College Dublin).

ERIN GRIESENAUER:

Algebras of Cross Sections and Azumaya Algebras

Abstract: We study algebras of cross sections of holomorphic matrix bundles that arise naturally in Geometric Invariant Theory and Noncommutative Function Theory. If the sections are based on a Stein compact set, then our algebras may be viewed as noncommutative analogues of function algebras. We describe in detail how our algebras arise and show that they are Azumaya algebras.

STEPHEN HARDY:

Pseudocompact C*-algebras

Abstract: We are interested in C*-algebras which are elementarily equivalent to ultraproducts of finite-dimensional C*-algebras in the continuous first-order logic of metric structures. We will discuss results due to Henson and Moore in the Banach space case, some of the finiteness properties pseudocompact C*-algebras enjoy, and ongoing research questions.

MICHAEL HARTZ:

Classification of multiplier algebras of Nevanlinna-Pick spaces

Abstract: Nevanlinna-Pick spaces are Hilbert function spaces for which an analogue of the Nevanlinna-Pick interpolation theorem from complex analysis holds. Their multiplier algebras, which are non-selfadjoint operator algebras, have attracted considerable attention in recent years. The investigation of the classification problem for these algebras was initiated by Davidson, Ramsey and Shalit.

I will report on the current state of this problem and talk about recent work which uses a somewhat different perspective on these algebras.

LEONARD HUANG

Generalized Fixed-Point Algebras and Square-Integrable Representations of Twisted C^*-Dynamical Systems

Abstract: In his paper *Generalized Fixed Point Algebras and Square-Integrable Group Representations*, Ralf Meyer extended Marc Rieffel's work on proper group actions and square-integrable group representations. He observed that Rieffel's notion of a 'proper group action on a C^*-algebra' could be captured by the theory of square-integrable group representations on Hilbert C^*-modules. Not only did he reproduce Rieffel's fixed-point algebras in this general setting, he also obtained a complete classification of isomorphism classes of Hilbert modules over the reduced crossed-product C^*-algebra corresponding to a given C^*-dynamical system.

In this talk, we show how to extend Meyer's theory itself to handle square-integrable representations of Busby-Smith twisted C^*-dynamical systems. For concreteness, we focus on continuous twisted C^*-dynamical systems and show that almost all of Meyer's results continue to hold in this setting. We then explain the measure-theoretic difficulties involved

when working with twisted C^*-dynamical systems that are merely measurable. We also mention how recent work by Alcides Buss and Sergey Neshveyev might lead to further development of the theory in the case of a locally compact quantum group twisted by a unitary cocycle.

MARIUS IONESCU:

A Stabilization Theorem for Fell bundles over Groupoids

Abstract: Given a second countable saturated upper-semicontinuous Fell bundle $p : \mathcal{B} \to G$ over a second countable Hausdorff locally compact groupoid we build a groupoid dynamical system $(\mathcal{K}(V), G, \alpha)$ such that $C^*(G, \mathcal{B})$ and $\mathcal{K}(V) \rtimes_\alpha G$ are Morita equivalent. We prove the Morita equivalence by showing that $p : \mathcal{B} \to G$ and the Fell bundle associated with $(\mathcal{K}(V), G, \alpha)$ are equivalent Fell bundles. Our results generalize previous work of Alex Kumjian, Paul S. Muhly, and of Alcides Buss, Ralf Meyer, and Chenchang Zhu. As an application of our results, we describe the lattice of ideals of the C^*-algebra of a continuous Fell bundle using the corresponding results that Renault proved for groupoid dynamical systems. We also characterize the simplicity of C^*-algebras of continuous Fell bundles in terms of the minimality of the action of G on the primitive ideal space of the C^*-algebra A over $G^{(0)}$. This talk is based on joint work with Alex Kumjian and Dana P. Williams.

BHISHAN JACELON:

One dimensional models for well behaved C^*-algebras

Abstract: Well behaved C^*-algebras, for example those that are simple and $\mathcal{Z}$-stable, often exhibit properties analogous to those of von Neumann factors. One technique for establishing certain such behaviour is to transfer the problem at hand to a suitable subalgebra of nuclear dimension one (the existence of which is guaranteed by classification results). For example, in recent joint work with Karen Strung and Andrew Toms, we employ this strategy to study the distance between unitary orbits of self adjoint elements. I will discuss this and other potential applications.

VICTOR KAFTAL:

Ideals of the multiplier algebras of simple finite C*-algebras

Abstract: Let $\mathcal{A}$ be a simple stable C*-algebra that has strict comparison of positive elements by traces (and satisfies additional conditions that will be discussed). Then TFAE

(i) The tracial simplex $T(\mathcal{A})$ has only finitely many extremal traces;

(ii) The multiplier algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has strict comparison of positive elements by traces.

(iii) The multiplier algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has finitely many ideals.

(iv) The corona algebra $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})/\mathcal{A} \otimes \mathcal{K}$ is purely infinite.

This result can be extended to corners $P\mathcal{M}(\mathcal{A} \otimes \mathcal{K})P$ of the multiplier algebra in the case that the scale, i.e., the evaluation map $\hat{P}$, is quasicontinuous.

This is a report on joint work with Ping Ng, and Shuang Zhang.

STEVE KALISZEWSKI:

On Nondegeneracy of Coactions of Groups on C^*-Algebras

Abstract: One of the motivating principles behind the theory of C^*-coactions is to extend results on crossed products by abelian groups to the non-abelian case: Coactions of G take over the role played by actions of $\widehat{G}$. Notable examples of this include Imai's generalization of Takai-Takesaki duality for crossed products by actions, and Katayama duality for crossed products by coactions. But — along with other important results — Katayama's theorem requires *nondegenerate* coactions. While many classes of coactions are "automatically" nondegenerate, it is not known whether all coactions are nondegenerate. In this talk, we do not present a solution to the general problem, but we make an effort to reduce the considerable confusion it has caused. *This is joint work with John Quigg.*

GABRIELl KANTÚN-MONTIEL: **Invertibility along an operator**

Abstract: We study the inverse along an element in the case of the algebra of bounded linear operators on a Banach space and characterize it as an outer inverse with prescribed range and nullspace. This inverse generalizes the group, Drazin and Koliha-Drazin inverses.

A. SAMIL KAVRUK:

Relative Weak Injectivity for Operator Systems

Abstract: Extending C. Lance's weak expectation property, E. Kirchberg considered the notion of relative weak injectivity for arbitrary pairs of C*-algebras. In this presentation we focus in this context on non-commutative versions of several order theoretic concepts, such as Tight Riesz Interpolation and Riesz decomposition properties. We will show that Arveson's extension theory, with some order preservation assumptions, is tightly connected with relative weak injectivity. We discuss several formulations of Connes' embedding problem. For general operator systems, we will extend some of the result of E. Effros and formulate function theoretic characterizations of (c,max), (min,c), and (er,max)-nuclearities. Time permitting, we will discuss weak rigidity for operator systems, a joint work with C. Kleski.

CRAIG KLESKI:

Rigidity of completely positive maps on operator systems

Abstract: Let S be an operator system in $B(H)$ generating a C*-algebra A. We explore the extent to which completely positve maps on A are determined by their behavior on S under various natural assumptions. For example, if A is nuclear and if every factor representation of A has Arveson's unique extension property, then any unital completely positive map on A with codomain A'' extending the identity map on S must be multiplicative. This is a noncommutative variant of Saskin's classical result on the connection between Choquet boundaries of function spaces and Korovkin sets. We also discuss applications to complete isometries of C*-algebras and other sorts of rigidity having connections to the weak expectation property.

DANIEL MARKIEWICZ:

C*-envelopes of tensor algebras arising from stochastic matrices

Abstract: Given a subproduct system X in the sense of Shalit and Solel, one can define a (non-selfadjoint) *tensor algebra* $\mathcal{T}_+(X)$ associated to X. There has been important and interesting work on the characterization of the C*-envelope $C^*_{\text{env}}(\mathcal{T}_+(X))$. The special case when the subproduct system is actually a *product* system reduces to the study of a single C*-correspondence E. Katsoulis and Kribs have shown that $C^*_{\text{env}}(\mathcal{T}_+(E))$ is the Cuntz-Pimsner algebra $\mathcal{O}(E)$. On the other hand, Davidson, Ramsey and Shalit have shown that when X is a commutative subproduct system of finite dimensional Hilbert spaces, then $C^*_{\text{env}}(\mathcal{T}_+(X))$ is the Toeplitz algebra $\mathcal{T}(X)$. More recently, Kakariadis and Shalit proved that if X is a subproduct system of finite dimensional Hilbert spaces associated to a two sided subshift, then either $C^*_{\text{env}}(\mathcal{T}_+(X)) = \mathcal{T}(X)$ or $C^*_{\text{env}}(\mathcal{T}_+(X)) = \mathcal{O}(X)$, depending on a combinatorial condition on the subshift.

We prove that in contrast to the plausible dichotomy suggested above, when X is the subproduct system associated to a finite and irreducible stochastic matrix, the C*-envelope $C^*_{\text{env}}(\mathcal{T}_+(X))$ need not be $*$-isomorphic to either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

This talk is based on joint work with Adam Dor-On.

ADAM MORGAN

Cuntz-Pimsner Algebras Associated to Tensor Products of C^*-Correspondences

Abstract: Given two C^*-correspondences X and Y over C^*-algebras A and B, there is a standard construction which gives an $A \otimes B$ correspondence $X \otimes Y$ called the external tensor product of X and Y. Our main result shows that, under mild hypotheses, the Cuntz-Pimsner algebra $\mathcal{O}_{X\otimes Y}$ embeds as a certain subalgebra of $\mathcal{O}_X \otimes \mathcal{O}_Y$ and that this subalgebra can be described in a natural way in terms of the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$.

SOUMYASHANT NAYAK:

The Schur-Horn theorem, numerical ranges and C*-convexity

Abstract: The Schur-Horn theorem is a classical theorem in the theory of matrices which characterizes the diagonals of Hermitian matrices with a given set of eigenvalues. At its heart, its a convexity result and has inspired a lot of work in symplectic geometry in the form of AGS theorem , Kirwan convexity theorem, etc. In recent years a number of attempts to generalize the theorem in the context of operators on separable infinite-dimensional Hilbert spaces have been made. We will discuss some results due to R. Kadison and W. Arveson. Time permitting, we will also discuss some basic results on C*-convexity and matricial ranges in this context.

RICKY NG:

The operator system SOH and a new tensor product of operator systems

Abstract:Consider a complete order isomorphism ϕ from an operator system S onto its matrix-order dual S^d. It is natural to ask if ϕ could be a complete norm isomorphism; i.e. can a self-dual operator system simultaneously be isomorphic, as operator spaces, to

its matrix order dual ? The answer turns out to be negative since its cb-condition number is bounded below by 2. Based on Pisiers operator Hilbert space OH, we create a self-dual operator system SOH and show that this isomorphism attains the minimum ch-condition number. We examine further properties of SOH and use it to create a new tensor product on operator systems by approximate CP-factorization.

This is joint work with Vern Paulsen.

RACHAEL NORTON:

Displacement Theory in the Context of W^*-correspondences

Abstract: We show how the so-called displacement equation can be formulated in the context of W^*-correspondences to prove a generalization of the Nevanlinna-Pick theorem. The proof of the theorem we obtain follows the line of reasoning used by Constantinescu and Johnson in 2003 to prove a noncommutative extension of the Nevanlinna-Pick theorem and our result generalizes their theorem. Our result is also similar in appearance to a generalization of the Nevanlinna-Pick theorem proved by Muhly and Solel in 2004. However, there are differences. We present an outline of our proof and, time permitting, we discuss the differences we have detected.

TRON OMLAND:

On the K-theory of certain Cuntz-Li algebras

Abstract: In this talk, we consider a family of Cuntz-Li algebras $\mathcal{Q}_S$ arising from a set S of mutually relatively prime numbers. Two algebras of this type have previously been studied: $\mathcal{Q}_{\mathbb{N}}$ by Cuntz and $\mathcal{Q}_2$ by Larsen and Li, corresponding to the sets $S = \{\text{all primes}\}$ and $S = \{2\}$, respectively.

For every such S, the algebra $\mathcal{Q}_S$ is a untial Kirchberg algebra in the UCT class, and our goal is to compute the K-theory of $\mathcal{Q}_S$.

SATISH PANDEY: **A Spectral Characterization of $\mathcal{AN}$ Operators**

Abstract:We establish a spectral characterization theorem for the operators on complex Hilbert spaces of arbitrary dimensions that attain their norm on every closed subspace. We construct example to show that the class of these operators is not closed under addition. Nevertheless, we prove that the intersection of these operators with the positive operators form a proper cone in the real Banach space of hermitian operators.

SUJAN PANT:

Primeness results for von Neumann algebras associated with surface braid groups

Abstract: In this talk I introduce a new class $\mathcal{P}$ of non-amenable groups which give rise to prime von Neumann algebras. This means that for every $\Gamma \in \mathcal{P}$ its group von Neumann algebra $L(\Gamma)$ cannot be decomposed as a tensor product of diffuse von Neumann algebras. I show that $\mathcal{P}$ is fairly large as it contains many examples of groups intensively studied in various areas of mathematics, notably: all infinite central quotients of pure surface braid groups—in particular, most pure braid groups on punctured surfaces of genus at least 1; all mapping class groups of (punctured) surfaces of genus 0,1,2; most Torelli groups and

Johnson kernels of (punctured) surfaces of genus 0,1,2; and, all groups hyperbolic relative to finite families of residually finite, exact, infinite, proper subgroups.

BENJAMIN PASSER:

A Noncommutative Borsuk-Ulam Theorem

Abstract: The Borsuk-Ulam theorem in algebraic topology states that any odd, continuous map from the sphere $\mathbb{S}^k$ to Euclidean space $\mathbb{R}^k$ of the same dimension must have a zero, and the theorem has many alternative forms. Each of these leads to an algebraic translation in terms of the C^*-algebra $C(\mathbb{S}^k)$. When we replace $C(\mathbb{S}^k)$ with a noncommutative sphere, we reach a list of conjectures, some more plausible than others. For a particular family of noncommutative spheres $C(\mathbb{S}^k_\rho)$, I will discuss these so-called noncommutative Borsuk-Ulam theorems (and non-theorems) and some key elements of their proofs.

GELU POPESCU:

Euler characteristic on noncommutative polyballs

Abstract: We introduce and study the Euler characteristic associated with algebraic modules generated by arbitrary elements of certain noncommutative polyballs. We provide several asymptotic formulas and prove some of its basic properties. We show that the Euler characteristic is a complete unitary invariant for the finite rank Beurling type invariant subspaces of the tensor product of full Fock spaces $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$, and prove that its range coincides with the interval $[0, 1)$. We obtain an analogue of Arveson's version of the Gauss-Bonnet-Chern theorem from Riemannian geometry, which connects the curvature to the Euler characteristic. In particular, we prove that if M is an invariant subspace of $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$, $n_i \geq 2$, which is graded (generated by multi-homogeneous polynomials), then the curvature and the Euler characteristic of the orthocomplement of M coincide.

JOHN QUIGG:

Landstad duality and a theorem of Pedersen

Abstract: We show how a theorem of Pedersen characterizing exterior equivalent actions on a C^*-algebra can be parlayed into an equivalence between two equivariant categories of C^*-algebras. In one category, isomorphisms correspond to outer conjugacies of actions, while isomorphisms in the other category are equivariant isomorphisms of the crossed products that respect the generalized fixed point algebras. This category equivalence is a variation of Landstad's original characterization of actions up to equivariant isomorphism, where we now allow more morphisms. Time permitting, we will compare our "outer duality" with Landstad duality and also with Imai-Takai crossed-product duality. This is joint work with Steve Kaliszewski and Trom Omland.

TRAVIS RUSSELL:

Characterizations of Ordered Self-Adjoint Operator Spaces

Abstract: In this talk, I will present two characterizations of self-adjoint operator spaces which take into account both the norm and the positive cone induced at each matrix level.

I will demonstrate a few applications, including a generalization of the Arveson extension theorem.

BENJAMIN RUSSO:

The Equivalence of Lifting and Factorization for Tuples of 3-Isometries.

Abstract: An operator T is called a 3-isometry if there exists a $B_1(T^*, T)$ and $B_2(T^*, T)$ such that

$$Q(n) = T^{*n}T^n = 1 + nB_1(T^*, T) + n^2B_2(T^*, T)$$

for all natural numbers n. These operators are related to the 3-symmetric operators introduced by Helton, which have a connection to disconjugacy theory from the theory of Sturm-Liouville equations. Of particular importance in the study of 3-isometries is the associated operator polynomial $Q(s)$ where s is a real number. We call an operator J a Jordan operator of order 2 if $J = U + N$ where U is unitary, N is nilpotent order 2, and U and N commute. Jordan operators are a canonical example of 3-isometries. By use of the theory of completely positive maps, it can be shown that a class of 3-isometries satisfying a particular positivity condition on the operator pencil $Q(s)$ can be lifted to Jordan operators. In the multi-variable setting we can define an analogous class for tuples of commuting 3-isometries. It is a natural question to ask whether tuples of commuting 3-isometries satisfying an analogous positivity condition always lift to tuples of commuting Jordan operators. In this talk, we will discuss our exploration into the equivalence of lifting to the factorization of the associated operator polynomial $Q(s)$. We will also show that this factorization is not always possible by modifying an example of Choi.

CHRISTOPHER SCHAFHAUSER:

Brown's Embedding Theorem for Cuntz-Pimsner Algebras

Abstract: Suppose A is an AF-algebra and $\alpha : A \to A$ is an automorphism. N. Brown has characterized when $A \rtimes_\alpha \mathbb{Z}$ embeds into an AF-algebra in terms of the ordered K-theory of A and α. We extend this result to Cuntz-Pimsner algebras $\mathcal{O}_A(H)$ where A is an AF-algebra and H is a "regular" C^*-correspondence over A. In the process we compute the ordered K-theory of the crossed product $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$ for any "regular" C^*-correspondence H over any C^*-algebra A, where γ is the usual gauge action.

PAUL SKOUFRANIS:

Problems involving Majorization in II_1 Factors:

Abstract: A notion of majorization for n-tuples of complex numbers plays an interesting role in a diverse collection of problems in linear algebra. When applied to the n-tuple eigenvalue lists of self-adjoint matrices, several fascinating operator theoretic results can be obtain. One example of this that has received much attention in recent years is the Schur-Horn Theorem, which classifies the possible diagonal n-tuples of a self-adjoint matrix based on its eigenvalues. In this talk, we will discuss the notion of majorization of self-adjoint operators in II_1 factors and its application. In particular, we will discuss the Schur-Horn Theorem in II_1 factors (due to Ravichandran), a classification of possible diagonals

of operators in II_1 factors based on singular values (joint work with Matt Kennedy), and other applications of majorization in II_1 factors (joint work with Ken Dykema). We will also discuss a generalization to tracial C*-algebras of real rank zero.

JACK SPIELBERG:

The C^*-algebras of a left-cancellative small category

Abstract: The construction of C^*-algebras from symbolic dynamics was initiated by Cuntz and Krieger in 1980. Many mathematicians participated in the subsequent generalizations to directed graphs, higher-rank graphs, quasi-lattice-ordered groups, and beyond. *Categories of paths* were introduced as a common generalization of all of these. The notion of "pathiness" was given there by three assumptions on the morphisms in a small category: left-cancellation, right-cancellation, and non-existence of inverses. It was shown, for example, that the theory of higher-rank graphs depends only on these assumptions, and the presentation by generators and relations can be derived from them. In the paper introducing categories of paths, it was indicated that right-cancellation is not necessary for the general development. In this talk we show that the same is true for non-existence of inverses. In fact, only a minor adjustment to the definitions in that paper are necessary for the theory to apply to arbitrary left-cancellative small categories. We also describe an example involving subsemigroups of the Baumslag-Solitar groups.

RICHARD M. TIMONEY:

Operators approximable by hypercyclic operators

Abstract: We report on work of James Boland discussing the operator norm closure of the class of hypercyclic operators on a (separable infinite dimensional) Banach space X. The results make a step towards a spectral characterization due to Herrero (1991) in the Hilbert space case. In particular operators of the form $T = I + N$ with N nilpotent are all limits of hypercyclic operators.

QINGYUN WANG:

Tracial Rokhlin property for actions of amenable groups on C*-algebras

Abstract The tracial Rokhlin property for finite group actions on simple C*-algebras was introduced by Chris Phillips in to study the structure of the crossed product. It should be viewed as the C*-version of outness that has the closest relationship with outness of actions on von Neumann algebras, which is the non-commutative version of freeness of actions on measure spaces. Later it was introduced for actions of $\mathbb{Z}$ and then generalized by Matui and Sato for actions of discrete amenable groups.
In this talk, we shall present a definition of (weak) tracial Rokhlin property for actions of countable discrete amenable groups that further genralize Matui and Sato's definition. We show that generic examples, like Bernoulli shift on the tensor product of copies of the Jiang-Su algebra, has the weak tracial Rokhlin property. We show that forming crossed product from actions with the tracial Rokhlin property preserves the class of C*-algebras with real rank zero, stable rank one and has strict comparison for projections, generalizing

the structural results obtained by Phillips and Osaka. In joint work with Chris Phillips and Joav Oravitz, we also show that pureness and $\mathcal{Z}$-stability could be preserved by forming crossed product by actions with the weak tracial Rokhlin property. As a consequence, we show that certain class of classifiable C*-algebras are preserved.

CHENXU WEN: **Unique maximal amenable extension for the radial masa**

Abstract: J. Peterson conjectured that for the free group factor, any diffuse amenable subalgebra has a unique maximal amenable extension. It is shown by Houdayer that the generator masa is the unique maximal amenable extension of any diffuse subalgebra of itself. We show that the same conclusion holds for the radial masa.

VREJ ZARIKIAN:

Unique Pseudo-Expectations for C^*-Inclusions

Abstract: A pseudo-expectation for a C^*-inclusion $D \subset C$ is a generalization of a conditional expectation. Precisely it is a ucp map $\Phi : C \to I(D)$ such that $\Phi|_D = id$. (Here $I(D)$ is the injective envelope of D.) Whereas a C^*-inclusion may not admit any conditional expectations, it must have at least one pseudo-expectation, by injectivity. In this talk, based on recent joint work with David Pitts, we investigate the relationship between a C^*-inclusion and its pseudo-expectation space, in particular how the existence of a unique pseudo-expectation relates to structural properties of the inclusion. First we consider examples, emphasizing the cases of abelian inclusions and W^*-inclusions. Then we state some general results, the strongest of which hold when D is abelian. In that case there is a Krein-Milman theorem for the pseudo-expectation space, and an order-theoretic characterization of the unique pseudo-expectation property. Finally, as applications, we explore the connection between the unique pseudo-expectation property and norming (in the sense of Pop, Sinclair, and Smith), as well as its impact on C^*-envelope calculations.

DA ZHENG:

The Operator System Generated by Cuntz Isometries

Abstract: In this talk, we consider the operator system $\mathcal{S}_n$ $(2 \leq n < \infty)$ generated by Cuntz isometries $S_1, \ldots, S_n$, that is, n isometries $S_1, \ldots, S_n$ with $\sum_{i=1}^n S_i S_i^* = I$. We define an operator system $\mathcal{E}_n \subseteq M_n$ and prove that $\mathcal{S}_n$ is completely order isomorphic to a quotient of $\mathcal{E}_n$. We also study tensor products of $\mathcal{S}_n$ and examine certain types of nuclearity of $\mathcal{S}_n$ in the operator system category. Finally, some properties of the dual operator system $\mathcal{S}_n^d$ and its connection with the joint numerical radius are discussed.