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The K-theory of abelian subalgebras of AF algebras

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1. Introduction

C^*-algebras are a special class of $\mathbb{C}$-algebras with involution. Two of the best understood classes of examples are the commutative C^*-algebras and the AF algebras. (AF stands for approximately finite-dimensional.) Typically, AF algebras are highly non-commutative. Nevertheless, the commutative subalgebras of AF algebras form a rich collection, telling us some surprising things about AF algebras.

Every finite-dimensional, unital C^*-algebra is of the form $M_{n_1} \oplus \cdots \oplus M_{n_k}$. (Here M_n means $M_n(\mathbb{C})$.) A unital AF algebra is an inductive limit $\varinjlim A_n$ of a sequence

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots$$

of finite-dimensional C^*-algebras, each unitally embedded in its successor. (The limit we have in mind is a completion of the algebraic limit $\bigcup_{n=1}^{\infty} A_n$.) Elliott [11] completely classified these C^*-algebras; the invariant is (essentially) the abelian group $K_0(A)$, sometimes called the dimension-group of A.

The typical abelian, unital C^*-algebra is $C(X)$, that is, the star-algebra of continuous, complex-valued functions on a compact Hausdorff space X, (with pointwise operations $f^*(x) = \overline{f(x)}$, $fg(x) = f(x)g(x)$, etc.). Thus, when dealing with C^*-algebras, one often turns for inspiration to topology. For example, the K-theory groups $K_0(A)$ and $K_1(A)$, defined for any C^*-algebra A, are designed so that $K_i(C(X)) \cong K^i(X)$.

Topology does not predict all phenomena in C^*-algebras. We examine one such, involving embeddings of the form $C(X) \hookrightarrow A$, where A is AF. To see what is predicted by topology, we examine the case where A is also abelian.

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Suppose that $\varinjlim A_n$ is abelian. Then each A_n is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C} \cong C(\Sigma_n)$ where Σ_n is finite. Inductive limits of C^*-algebras correspond to inverse limits of spaces. (More generally, "arrows reverse" as $X \to Y$ induces $C(Y) \to C(X)$, by composition.) Indeed, A must be isomorphic to $C(\Sigma)$, with Σ a zero-dimensional space, specifically $\varprojlim \Sigma_n$.

Thus the appropriate analog in topology is a continuous surjection $\Sigma \to X$ of a zero-dimensional, compact metrizable space Σ onto a compact Hausdorff space X. These abound, but are uninteresting. For example, if X is also a connected ANR, then one can show (for example, using some shape theory, as in [20]) that all maps from Σ to X are homotopic. If X is only connected, the induced map $\tilde{K}^0(X) \to \tilde{K}^0(\Sigma)$ is zero, basically because the positive-degree homology groups of a zero-dimensional space are zero.

By contrast, it was discovered in [18] that a unital embedding

$$\phi : C(\mathbb{T}^2) \to A$$

could be found where A was AF and the induced map

$$\phi_* : K_0(C(\mathbb{T}^2)) \to K_0(A)$$

was injective. This implied immediately the non-triviality of $[C(\mathbb{T}^2), A]_1$, (the homotopy classes of unital *-homomorphisms). The existence of such an apparently pathological embedding held consequences (cf. [18], §5) for any potential homology theory defined for C^*-algebras. Informally, it meant that at least this AF algebra could behave two-dimensionally, not like the zero-dimensional non-commutative topological space it was assumed to be.

It was later discovered, in [12], that many AF algebras exhibit this behavior. The minimal choice, in some sense, was found to be

$$A = AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}),$$

which is our notation for the unital AF algebra corresponding to the dimension group (G, G^+, u), where

$$G = \mathbb{Z}[1/2] \oplus \mathbb{Z},$$
$$G^+ = \{0\} \cup \{(r, n) \mid r > 0\}.$$
$$u = (1, 0).$$

(Of the two possible positive cones for $\mathbb{Z}[1/2] \oplus \mathbb{Z}$, this is the smaller one, and the only one we shall use in this paper.) In this case, the resulting map

$$\phi_* : \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z}[1/2] \oplus \mathbb{Z}$$

is a rational isomorphism.

Left open was the question of whether something was special about dimension two. We shall show that the answer is no. The pathology for $A = AF(\mathbb{Z}[1/2] \oplus \mathbb{Z})$ can be said to be of "infinite even degree."

In this paper, we manipulate known embeddings rather than construct embeddings directly. For a more explicit understanding of our embeddings, the reader is referred to [12], [19] and the two fine papers by Pimsner and Voiculescu that started this embedding research, [21] and [22].

The results in [7] regarding

$$\mathrm{Hom}(C(X), C(Y) \otimes M_n)$$

make it possible to manipulate an embedding of $C(\mathbb{T}^2)$ to produce AF embeddings of other commutative C^*-algebras which are also "minimal" and rational isomorphisms on K_0. The following is our main theorem. It's proof is broken up between the next two sections.

Theorem 1.1. *Let X be a compact Hausdorff space with the homotopy type of a finite CW-complex*

(a) *There exists a unital embedding*

$$\phi : C(X) \to A$$

into an AF *algebra with*

$$\phi_* : K_0(C(X)) \to K_0(A)$$

rationally an isomorphism. If X is connected, then A can be the unique unital AF *algebra with*

$$K_0(A) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}^n,$$

$n = \mathrm{rank}\, \tilde{K}^0(X)$.

(b) *There exists a unital embedding*

$$\phi : C(X) \to (SA)^{\sim}$$

into the unitalized suspension of an AF *algebra with*

$$\phi_* : K_1(C(X)) \to K_1((SA)^{\sim})$$

rationally an isomorphism.

In Theorem 1.1, and throughout this paper, we regard $\mathbb{Z}[1/2] \oplus \mathbb{Z}^n$ as the dimension group G with

$$G^+ = \{0\} \cup \{(r, \mathbf{n}) \mid r > 0\},$$

$$u = (1, 0).$$

With regard to (b), we remark that if

$$A = \varinjlim A_n, \quad A_n = \bigoplus M_{[n,k]},$$

then

$$(SA)^{\sim} = \varinjlim (SA_n)^{\sim}$$

and

$$(SA_n)^{\sim} = \bigoplus \{f \in C(S^1, M_{[n,k]}) \mid f(1) \in \mathbb{C}\},$$

so the algebras used for part (b) fall into the context of limits of subhomogeneous C^*-algebras. Alternately, the theorem could have been stated using $C(S^1) \otimes A$.

Theorem 1.1 settles a question from [12]. The spherical homology [14] of

$$AF(\mathbb{Z}[1/2] \oplus \mathbb{Z})$$

is zero in all degrees except zero. In section four, we discuss how our embeddings can be used to produce new examples of almost commuting matrices and quasi-representations. In addition, Theorem 1.1 has the following corollaries involving shape theory [10], [1] and maximal abelian subalgebras (MASAs) in AF algebras.

Corollary 1.2. *If X has the homotopy type of a finite CW-complex, and $H^n(X; \mathbb{Q}) \neq 0$ for some $n \geqq 2$, then $C(X)$ is not semiprojective in the sense of Effros and Kaminker.*

Proof. First note that it suffices to prove this for X connected. By Theorem 1.1, there is a unital embedding

$$\phi : C(X) \to \varinjlim B_m = B$$

where

$$\phi_* : K_i(C(X)) \to K_i(B)$$

is rationally injective, $i \equiv n \pmod 2$ and

$$B_m = \bigoplus M_{[m,k]} \quad \text{or} \quad B_m = \bigoplus (SM_{[m,k]})^{\sim}.$$

To show that ϕ is not homotopic to any map of $C(X)$ to B_m, it suffices to show that $\psi : C(X) \to B_m$ never induces a rational injection. For n even, this is true because ψ is homotopic to a trivial (scalar-valued) homomorphism. For n odd, it follows from [7] that there exists a nonzero $x \in K_1(C(X)) \otimes \mathbb{Q}$ such that $\varrho_*(x) = 0$ whenever

$$\varrho : C(X) \to (SM_k)^{\sim}.$$

(We will explain this more fully after Proposition 2.1.)

Corollary 1.3. *For any natural numbers* $n_1, n_2, \ldots$ *with* $\sum n_k = n < \infty$, *the* AF *algebra* $AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n)$ *contains a maximal abelian subalgebra isomorphic to* $C(Y)$ *with*

$$\operatorname{rank} \check{H}^{2k}(Y; \mathbb{Q}) \geqq n_k .$$

Proof. Let $\phi : C(X) \to A$ be as in Theorem 1.1. If $C(Y)$ is any MASA containing the image of ϕ, we have a map

$$\psi : C(X) \to C(Y)$$

which is rationally injective on K_0. By the Chern character, we know that the corresponding function $f: Y \to X$ induces injections

$$f^* : H^*(X, \mathbb{Q}) \to H^*(Y; \mathbb{Q}) . \quad \square$$

Corollary 1.4. *Let* A *be a rational* AF *algebra with unit* (*i.e. the tensor product of a unital* AF *algebra with the* UHF *algebra with dimension group the rationals*) *and let* X *be a finite, connected* CW *complex. Given a group homomorphism*

$$\gamma : K_0(C(X)) \to K_0(A)$$

satisfying

$$\gamma([1]) = [1] \quad \textit{and} \quad \gamma(\tilde{K}^0(X)) \subseteqq \bigcap \ker(\tau_*) ,$$

where τ *ranges over all bounded traces on* A, *there exists a unital* $*$*-homomorphism*

$$\phi : C(X) \to A$$

with $K_0(\phi) = \gamma$.

Proof. Since $K_0(A) \otimes \mathbb{Q}$ is isomorphic to $K_0(A)$, γ induces

$$\gamma_1 : K_0(C(X)) \otimes \mathbb{Q} \to K_0(A) .$$

Let

$$\psi : C(X) \to AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n)$$

be any unital homomorphism inducing an isomorphism

$$\psi_* : K_0(C(X)) \otimes \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}^n .$$

Define

$$f = \gamma_1 \circ \psi_*^{-1} : \mathbb{Q} \oplus \mathbb{Q}^n \to K_0(A) .$$

We now show that f is positive. For any $r \in \mathbb{Q}$ and $\mathbf{s} \in \mathbb{Q}^n$,

$$f(r, \mathbf{s}) = r[1] + (1/q)\gamma(x)$$

for some integer q and $x \in \tilde{K}^0(X)$. If τ is any trace, then

$$\tau_*(f(r, \mathbf{s})) = r\tau(1) + (1/q)\tau_*(\gamma(x)) = r\tau(1).$$

If $(r, \mathbf{s})$ is positive, then r is positive, and so by [9], Theorem 3, $f(r, \mathbf{s})$ is positive. Since f preserves order and order unit, there exists a unital homomorphism

$$\alpha : AF(\mathbb{Q} \oplus \mathbb{Q}^n) \to A$$

with $K_0(\alpha) = f$. The required homomorphism is $\phi = \alpha \circ i \circ \psi$, where i is the obvious inclusion

$$AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n) \to AF(\mathbb{Q} \oplus \mathbb{Q}^n). \quad \square$$

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2. Applying $kk(Y, X)$

The K-theory of maps $C(X) \to C(Y) \otimes M_k$ can be effectively computed for large values of k by using the connective KK-theory of [7]. In this section, X and Y will be connected, compact spaces with the homotopy type of a finite CW-complex, with base-points x_0 and y_0. By $C_0(X)$ we mean $C_0(X \backslash \{x_0\})$, and $[A, B]$ refers to homotopy classes of (possibly non-unital) $*$-homomorphisms.

A result of G. Segal (see [24] and [23]) provides us with a representation of the reduced connective K-theory in terms of homotopy classes of $*$-homomorphisms:

$$k^q(X) = [C_0(S^{q+r}), C_0(S^r X) \otimes \mathscr{K}].$$

This formula holds for any $q \in \mathbb{Z}$ and any $r \geqq 0$ such that $q + r \geqq 1$. As a natural generalization, the formalism based on the group

$$kk(Y, X) = [C_0(X), C_0(Y) \otimes \mathscr{K}]$$

was introduced in [7]. This has proven to be a useful tool in dealing with $*$-homomorphisms of matrix algebras over continuous functions. Composition with the Bott homomorphism $C_0(S^1) \to C_0(S^3) \otimes M_2$ gives rise to an operation

$$S : k^{q+2} \to k^q.$$

A rational version of [7], Corollary 3.4.8 shows that there exist isomorphisms C_X^q so that the following diagram commutes:

$$\begin{array}{ccc} k^{q+2}(X) \otimes \mathbb{Q} & \xrightarrow{S} & k^q(X) \otimes \mathbb{Q} \\ \downarrow C_X^{q+2} & & \downarrow C_X^q \\ \bigoplus_{j \geqq 1} \tilde{H}^{q+2j}(X; \mathbb{Q}) & \hookrightarrow & \bigoplus_{j \geqq 0} \tilde{H}^{q+2j}(X; \mathbb{Q}). \end{array}$$

When $q = 0$ or 1, we may make the identification

$$k^q(X) \otimes \mathbb{Q} = \tilde{K}^q(X) \otimes \mathbb{Q}$$

and, as always,

$$\tilde{K}^q(X) \otimes \mathbb{Q} = K_q(C_0(X)) \otimes \mathbb{Q} = K_q(C_0(X) \otimes \mathcal{K}) \otimes \mathbb{Q}\,.$$

The result we need is a restatement of a universal coefficient theorem. Given

$$\phi \in \operatorname{Hom}(C_0(X), C_0(Y) \otimes \mathcal{K}),$$

representing a homotopy class $[\phi] \in kk(Y, X)$, we define

$$\Gamma_0[\phi] = (C_Y^0 \oplus C_Y^1) \circ (\phi_* \otimes \mathrm{id}_{\mathbb{Q}}) \circ (C_X^0 \oplus C_X^1)^{-1} \in \operatorname{Hom}(\tilde{H}^*(X; \mathbb{Q}), \tilde{H}^*(Y; \mathbb{Q}))\,,$$

where ϕ_* is the induced map

$$\phi_* : \tilde{K}^*(X) \to \tilde{K}^*(Y)\,.$$

We now define

$$\Gamma : kk(Y, X) \otimes \mathbb{Q} \to \operatorname{Hom}(\tilde{H}^*(X; \mathbb{Q}), \tilde{H}^*(Y; \mathbb{Q}))$$

as the unique map through which Γ_0 factors. Note that if α belongs to the image of Γ, then a suitable integer multiple of α belongs to the image of Γ_0.

Proposition 2.1. *The image of the homomorphism*

$$\Gamma : kk(Y, X) \to \operatorname{Hom}(\tilde{H}^*(X; \mathbb{Q}), \tilde{H}^*(Y; \mathbb{Q}))$$

is the set of homomorphisms which send $\tilde{H}^q(X; \mathbb{Q})$ *into* $\bigoplus_{j \geqq 0} \tilde{H}^{q+2j}(Y; \mathbb{Q})$. *(That is, the homomorphisms which preserve parity and filtration by dimension.)*

Proof. The $\mathbb{Q}$-coefficient version of [7], Theorem 3.5.4 states that

$$kk(Y, X) \otimes \mathbb{Q} = [C_0(X), C_0(Y) \otimes \mathcal{K}] \otimes \mathbb{Q} \cong \operatorname{Hom}_{\mathbb{Q}[t]}(k^*(X) \otimes \mathbb{Q}, k^*(Y) \otimes \mathbb{Q})\,.$$

The isomorphism sends the homotopy class of $\phi : C_0(X) \to C_0(Y) \otimes \mathcal{K}$ to the induced map on $k^*(X) \otimes \mathbb{Q}$. The module action of t in $\mathbb{Q}[t]$ is given by S. Therefore

$$\begin{aligned} kk(Y, X) \otimes \mathbb{Q} &\cong \left\{ (\xi_q) \in \bigoplus_q \operatorname{Hom}\left(\bigoplus_{j \geqq 0} \tilde{H}^{q+2j}(X; \mathbb{Q}), \bigoplus_{j \geqq 0} \tilde{H}^{q+2j}(Y; \mathbb{Q}) \right) \middle| \, \xi_{q+2} \text{ restricts to } \xi_q \right\} \\ &\cong \left\{ \xi_0 \oplus \xi_1 \,\middle|\, \xi_q(\tilde{H}^q(X; \mathbb{Q})) \subseteq \bigoplus_{j \geqq 0} H^{q+2j}(Y; \mathbb{Q}) \right\}. \quad \square \end{aligned}$$

By the stabilization theorem [7], Theorem 6.4.2, we can replace $\mathscr{K}$ by M_k, and the image of Γ will not change, for k large. (For small k, the image can be smaller.) To finish the proof of Corollary 1.2, we note that

$$\psi : C(X) \to (SM_k)^{\sim}$$

induces the same map on K_1 as the restriction

$$\phi : C_0(X) \to C_0(S^1) \otimes M_k .$$

Let $x \in K_1(C_0(X))$ be an element that C_X^1 maps to a nonzero element of $H^n(X; \mathbb{Q})$. Since $n \geqq 3$,

$$\phi_*(x) \in (C_X^1)^{-1}(H^n(S^1; \mathbb{Q})) = \{0\} .$$

Theorem 2.2. *If X and Y are connected, compact spaces, with the homotopy type of a finite CW-complex, and*

$$\operatorname{rank} H^q(X; \mathbb{Q}) = \operatorname{rank} H^q(Y; \mathbb{Q})$$

for $q \geqq 1$ (respectively for $q \geqq 2$, even, or for $q \geqq 1$, odd) then for large k, there exists a unital embedding

$$\phi : C(X) \to C(Y) \otimes M_k$$

with $K_*(\phi)$ (respectively $K_0(\phi)$ or $K_1(\phi)$) a rational isomorphism. If $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$ (respectively $H^{\text{even}}(X; \mathbb{Z})$ and $H^{\text{even}}(Y; \mathbb{Z})$, or $H^{\text{odd}}(X; \mathbb{Z})$ and $H^{\text{odd}}(Y; \mathbb{Z})$) are torsion-free, ϕ can be chosen with $K_*(\phi)$ (respectively $K_0(\phi)$ or $K_1(\phi)$) an isomorphism.

Proof. Assume that, for all $q \geqq 1$ (the even q, odd q cases are similar),

$$\operatorname{rank} H^q(X; \mathbb{Q}) = \operatorname{rank} H^q(X; \mathbb{Q}) .$$

Then clearly there is an isomorphism

$$\eta : \tilde{H}^*(X; \mathbb{Q}) \to \tilde{H}^*(Y; \mathbb{Q})$$

preserving the grading, and thus preserving parity and filtration. By Proposition 2.1, there exists

$$\phi : C_0(X) \to C_0(Y) \otimes M_k ,$$

for any large k, with $\Gamma([\phi] \otimes (1/m)) = \eta$ for some integer m. Then $\Gamma_0([\phi]) = m\eta$, and as $m\eta$ is an isomorphism, $K_*(\phi)$ is a rational isomorphism.

Since X and Y are connected, we can extend ϕ so that

$$\phi : C(X) \to C(Y) \otimes M_k$$

is unital and still have $K_*(\phi)$ a rational isomorphism. However, ϕ may not be injective.

If ϕ is not injective, choose null-homotopic maps $g_j: Y \to X$ such that

$$X = g_1(Y) \cup g_2(Y) \cup \cdots \cup g_r(Y).$$

Then

$$\tilde{\phi}: C(X) \to C(Y) \otimes M_{k+r},$$

defined by

$$\tilde{\phi}(f) = \phi(f) \oplus (f \circ g_1) \oplus \cdots \oplus (f \circ g_r)$$

is injective, and has the same *K*-theory as ϕ.

The same proof works in the torsion-free case, except using Corollary 3.4.8, Theorem 3.5.5 and Corollary 6.4.4 of [7]. □

3. Constructing the embeddings

Our starting point is the unital embedding

$$\phi_2: C(S^2) \to AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}),$$

from [12], which induces the natural inclusion

$$K_0(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z}[1/2] \oplus \mathbb{Z}.$$

Composing the obvious inclusion

$$C(S^{2n}) \hookrightarrow C(S^2 \times \ldots \times S^2)$$

with $\phi \otimes \cdots \otimes \phi$ leads to an embedding which is only injective on K_0.

Lemma 3.1. *The homomorphism*

$$(\mathbb{Z}[1/2] \oplus \mathbb{Z}) \otimes (\mathbb{Z}[1/2] \oplus \mathbb{Z}) \to \mathbb{Z}[1/2] \oplus \mathbb{Z},$$
$$(r, m) \otimes (s, n) \mapsto (rs, mn)$$

preserves order and order unit.

Proof. The order and order unit that we put on $G_1 \otimes G_2$, is that derived from the isomorphism

$$G_1 \otimes G_2 \cong K_0(AF(G_1) \otimes AF(G_2)).$$

If

$$AF(G_1) = \varinjlim A_n,$$
$$AF(G_2) = \varinjlim B_n$$

for finite-dimensional algebras A_n and B_n,

$$AF(G_1 \otimes G_2) = \varinjlim A_n \otimes B_n .$$

The positive cone of $G_1 \otimes G_2$ is generated by the minimal projections of $A_n \otimes B_n$. Each minimal projection of $A_n \otimes B_n$ is equivalent to a simple tensor of minimal projections. Therefore, $(G_1 \otimes G_2)^+$ is generated by

$$\{g_1 \otimes g_2 \mid g_i \in G_i^+\} .$$

Of course, the order unit is the tensor product of order units.

In the case at hand, we need only check that (rs, mn) is positive when (r, m) and (s, n) are positive. Since (r, m) and (s, n) nonzero and positive means $r > 0$ and $s > 0$, this is obvious. □

Lemma 3.2. *If X is a finite wedge of compact oriented manifolds, then a $*$-homomorphism to any C^*-algebra, $\phi : C(X) \to A$, which is rationally injective on $K_* = K_0 \oplus K_1$, is itself injective.*

Proof. A non-injective ϕ will factor through $C(Y)$, where Y is X with some small open disk D removed. In the case that X is a compact oriented n-dimensional manifold, it is easy to use de Rahm cohomology to show that

$$H^n(X; \mathbb{R}) \to H^n(X \setminus D; \mathbb{R})$$

sends the fundamental class to zero. Therefore, in the general case,

$$H^n(X; \mathbb{R}) \to H^n(X \setminus D; \mathbb{R})$$

is not injective. The result now follows via the Chern character. □

Remark. The proof of Lemma 3.2 shows that we need only assume

$$\phi_*(\mathrm{ch}^{-1}([X])) \neq 0$$

with $[X]$ denoting the fundamental class, to conclude that ϕ is injective.

Lemma 3.3. *For all $n \geqq 1$, there is an embedding*

$$\phi_{2n} : C(S^{2n}) \to AF(\mathbb{Z}[1/2] \oplus \mathbb{Z})$$

for which $K_0(\phi_{2n})$ is the canonical embedding

$$K_0(C(S^{2n})) = \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z}[1/2] \oplus \mathbb{Z} .$$

Proof. By Lemma 3.2, it suffices to find any homomorphism with the correct K-theory. The base-case, ϕ_2, is provided by [12]. Given ϕ_{2n}, we let ϕ_{2n+2} be the composition

$$\begin{array}{ccccc} C(S^{2n+2}) & \hookrightarrow & C(S^{2n})\otimes C(S^2) & \xrightarrow{\phi_{2n}\otimes\phi_2} & AF(\mathbb{Z}[1/2]\oplus\mathbb{Z})^{\otimes 2} \\ & & & & \downarrow \\ & & & & AF(\mathbb{Z}[1/2]\oplus\mathbb{Z}). \end{array}$$

The vertical map is any unital homomorphism inducing the map of Lemma 3.1 (which exists by [11]). The first map is induced by any degree-one continuous function from $S^{2n}\times S^2$ onto S^{2n+2}. □

Lemma 3.4. *For any natural numbers k and l such that $k+l$ is a power of two, the homomorphism*

$$\begin{array}{ccc} \mathbb{Z}^k\oplus(\mathbb{Z}[1/2]\oplus\mathbb{Z})^{\oplus l} & \to & \mathbb{Z}[1/2]\oplus\mathbb{Z}^l, \\ (m_1,\ldots,m_k)\oplus(r_1,s_1)\oplus\cdots\oplus(r_l,s_l) & \mapsto & \left(\dfrac{m_1+\cdots+m_k+r_1+\cdots+r_l}{k+l}, s_1,\ldots,s_l\right) \end{array}$$

preserves order, and order unit.

Proposition 3.5. *If X is a finite-wedge of even-spheres $X=\bigvee_{j=1}^{l} S^{2n_j}$, with $n_j\geq 1$, then there exists a unital embedding*

$$\phi: C(X)\to AF(\mathbb{Z}[1/2]\oplus\mathbb{Z}^l)$$

such that $K_0(\phi)$ is the natural inclusion

$$K_0(C(X))=\mathbb{Z}\oplus\mathbb{Z}^l\hookrightarrow\mathbb{Z}[1/2]\oplus\mathbb{Z}^l.$$

Proof. $C(X)$ sits inside $\bigoplus C(S^{2n_j})$ in an obvious fashion. Choose k such that $k+l$ is a power of two. Adding on k one-dimensional representations of $C(X)$ produces an embedding

$$C(X)\hookrightarrow \mathbb{C}^k\oplus\bigoplus C(S^{2n_j})$$

which is an isomorphism on $\tilde{K}^0$. The required embedding is the above followed by

$$\begin{array}{ccc} \mathbb{C}^k\oplus\bigoplus C(S^{2n_j}) & \xrightarrow{\mathrm{id}\oplus\phi_{2n_1}\oplus\cdots\oplus\phi_{2n_l}} & AF(\mathbb{Z}^k\oplus(\mathbb{Z}[1/2]\oplus\mathbb{Z})^l) \\ & & \downarrow \\ & & AF(\mathbb{Z}[1/2]\oplus\mathbb{Z}^l). \end{array}$$

The vertical map is provided by Lemma 3.4 and [1]. □

We now turn to the proof of Theorem 1.1. First we prove (a) for X a connected, finite CW-complex. Let Y be a wedge of even spheres, with rank $\tilde{H}^{2q}(X; \mathbb{Q})$ copies of S^{2q}, for $q \geqq 1$. Theorem 2.2 provides an embedding

$$C(X) \hookrightarrow C(Y) \otimes M_{2^k}$$

that is a rational isomorphism on K_0. Proposition 3.5 now gives an embedding

$$C(X) \hookrightarrow M_{2^k}(AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n))$$

which is a rational isomorphism on K_0. This looks like the wrong AF algebra, but

$$M_{2^k}(AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n)) \cong AF(\mathbb{Z}[1/2] \oplus \mathbb{Z}^n),$$

the isomorphism given, at the dimension group level, by

$$(r, \mathbf{n}) \mapsto (r 2^{-k}, \mathbf{n}).$$

For X not connected, just express $C(X)$ as the direct sum of $C(X_j)$ and embed each $C(X_j)$ separately.

We need only prove (b) for X a connected, finite CW-complex, by the same argument. Let Y be a finite CW-complex with

$$\operatorname{rank} H^{2q}(Y; \mathbb{Q}) = \operatorname{rank} H^{2q+1}(X; \mathbb{Q})$$

for $q \geqq 0$. In general, Y will not be connected. A wedge of even-spheres, including zero-spheres, will do. By part (a),

$$\phi : C(Y) \to B,$$

a unital AF embedding, exists, with $K_0(\phi)$ a rational isomorphism. Suspending, we get an embedding

$$SC(Y) \to SB,$$

a rational isomorphism on K_1. Since

$$(SC(Y))^{\sim} \cong C(Z)$$

with

$$Z = (\mathbb{R} \times Y) \cup \{\mathrm{pt}\},$$

we have, for all q,

$$\operatorname{rank} H^{2q+1}(Z; \mathbb{Q}) = \operatorname{rank} H^{2q}(Y; \mathbb{Q}) = \operatorname{rank} H^{2q+1}(X; \mathbb{Q}).$$

Theorem 2.2 provides an embedding

$$C_0(X) \to C_0(Z) \otimes M_k$$

which is rationally isomorphic on K_1. By composition, we have an embedding, rationally isomorphic on K_1,

$$C_0(X) \to M_k \otimes SB = SA,$$

where A is the AF algebra $M_k \otimes A$. Adding units, we have proven part (b).

4. Quasi-representations and asymptotically commuting matrices

We now consider the implications of our embeddings with respect to finite matrices. Connes, Gromov and Moscovici [5] define a quasi-representation of a unital C^*-algebra A to be a positive, unital map $\sigma: A \to M_n$. To measure lack of multiplicativity, they introduce

$$\|\sigma\|_F = \sup\{\|\sigma(ab) - \sigma(a)\sigma(b)\| \,|\, a, b \in F\}$$

(F a finite subset of A). Given $x_1, \ldots, x_r \in K_0(A)$, choose representatives

$$[p_i] - [q_i] = x_i$$

where p_i and q_i are projections in $M_{n(i)}(A)$. For some finite F and positive ε,

$$\|\sigma\|_F < \varepsilon \Rightarrow \|\sigma(p_i)^2 - \sigma(p_i)\| < 1/4$$

(cf. [5], Proposition 5). We define a "push-forward" of x_i by σ to $\mathbb{Z} \cong K_0(M_n)$ by

$$\sigma_*(x_i) = \operatorname{Trace}(P_i) - \operatorname{Trace}(Q_i)$$

where P_i, respectively Q_i, is the spectral projection for $\sigma(p_i)$, respectively $\sigma(q_i)$, corresponding to the set $\{z \,|\, |z-1| \leqq 1/2\}$. Different representatives for the x_i lead to the same values for $\sigma_*(x_i)$ when $\|\sigma\|_{F'} < \varepsilon'$ for some finite $F' \supseteqq F$ and $\varepsilon \geqq \varepsilon' > 0$. See [17] for a specific example of pushing a projection over $C(\mathbb{T}^2)$ forward to a matrix that is approximately a projection.

Theorem 4.1. *Suppose X has the homotopy type of a finite CW-complex, and $e \in \tilde{K}^0(X)$ is non-torsion. Then there is a sequence of positive, unital, linear maps*

$$\psi_n: C(X) \to M_n$$

such that

$$\|\psi_n(fg) - \psi_n(f)\psi_n(g)\| \to 0,$$

for all $f, g \in C(X)$ and, for large n, $\psi_{n*}(e) \neq 0$.

Proof. This follows from Theorem 1.1 and the following lemma. □

Remark. These quasi-representations cannot be approximated by representations, since $\phi_{n*}(e) = 0$ for any representation $\phi_n: C(X) \to M_n$. By approximate, we mean $\|\phi_n(f) - \psi_n(f)\| \to 0$ for all $f \in C(X)$.

Remark. By [5], Proposition 8, each ψ_n induces an associated element of cyclic cohomology, which must be nontrivial by [5], Proposition 9. The existence of quasi-representations that extract non-trivial topological information is to be expected in view of the E-theory of Connes and Higson [6] whose cocycles are based on asymptotic homomorphisms.

Lemma 4.2. *Let A be a nuclear, unital C^*-algebra and let $\varinjlim B_n$ be an inductive limit of a system of C^*-algebras with unital, injective connecting maps. Any completely positive, unital map $\psi : A \to \varinjlim B_n$ can be approximated by completely positive, unital maps $\psi_n : A \to B_n$ with $\psi_n(a) \to \psi(a)$ for all $a \in A$.*

Proof. Consider the C^*-algebras

$$l^\infty(B_n) = \{(b_n)_{n=1}^\infty \mid b_n \in B_n \quad \text{and} \quad \sup \|b_n\| \leqq \infty\},$$

$$c_0(B_n) = \{(b_n)_{n=1}^\infty \mid b_n \in B_n \quad \text{and} \quad \lim \|b_n\| = 0\}.$$

If $b = \lim_{n\to\infty} b_n$, for $b_n \in B_n$, then

$$\Lambda(a) = (b_n)_{n=1}^\infty + c_0(B_n) \in l^\infty(B_n)/c_0(B_n)$$

gives a well-defined $*$-homomorphism. Now apply the Choi-Effros [3] lifting theorem to produce a completely positive, unital map Ψ with the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\Psi} & l^\infty(B_n) \\ \downarrow & & \downarrow \\ \varinjlim B_n & \xrightarrow{\Lambda} & l^\infty(B_n)/c_0(B_n). \end{array}$$

It is clear that composing Ψ with the projection onto B_n produces the desired ψ_n. □

Remark. One may check that

$$K_*(l^\infty(B_n)) \subseteqq \prod_1^\infty K_*(B_n)$$

is a subgroup of the infinite product,

$$K_*(l^\infty(B_n)/c_0(B_n)) = K_*(l^\infty(B_n))/\bigoplus K_*(B_n)$$

and Λ induces injections on K-theory.

Our embeddings also produce examples of asymptotically commuting matrices that are fundamentally different from any discovered previously. Given two sequences of matrices S_n, $T_n \in M_{N(n)}$, we call (S_n) and (T_n) asymptotically commuting matrices if

$$\lim_{n\to\infty} \|[S_n, T_n]\| = 0.$$

If $(T_n^{(1)}), \ldots, (T_n^{(k)})$ are pairwise asymptotically commuting and self-adjoint (by which we mean $(T_n^{(j)})^* = T_n^{(j)}$ for all n and j), then we define their joint-spectrum $\sigma(T_n^{(1)}, \ldots, T_n^{(k)})$ as the spectrum of the unital C^*-algebra generated by

$$(T_n^{(1)}) + \mathscr{I}, \ldots, (T_n^{(k)}) + \mathscr{I}$$

in $\mathscr{A}/\mathscr{I}$, where

$$\mathscr{A} = \{(R_n)_{n=1}^{\infty} \mid R_n \in M_{N(n)} \text{ and } \sup \|R_n\| \leqq \infty\},$$

$$\mathscr{I} = \{(R_n)_{n=1}^{\infty} \mid R_n \in M_{N(n)} \text{ and } \lim \|R_n\| = 0\}.$$

Examples of asymptotically commuting matrices have been found which cannot be approximated by commuting matrices. The joint spectrum for Voiculescu's example [25] is a torus, while for Davidson's [8] it is a sphere. Many other examples are possible. To be specific, we have the following result.

Corollary 4.3. *For any g, there exist self-adjoint, asymptotically commuting matrices*

$$A_n, B_n, C_n \in M_{2^n},$$

with joint-spectrum equal to a g-hole torus, such that there exists no sequences A_n', B_n', C_n' *of self-adjoint matrices with*

$$[A_n', B_n'] = [A_n', C_n'] = [B_n', C_n'] = 0$$

and

$$\lim \|A_n - A_n'\| = \lim \|B_n - B_n'\| = \lim \|C_n - C_n'\| = 0.$$

Proof. The AF algebra $A = AF(\mathbb{Z}[1/2] \oplus \mathbb{Z})$ can be written as

$$A \cong \varinjlim M_{2^n} \oplus M_{2^n} \oplus M_{2^n} \oplus M_{2^n}.$$

(See [12].) As in the proof of Lemma 4.2, define

$$\Lambda : A \to \mathscr{A}/\mathscr{I} \quad (N(n) = 2^{[n/4]}).$$

By the remark above, Λ is injective on $K_0(A)$. Passing to a subsequence, and applying Theorem 1.1, we have a unital star-homomorphism

$$\phi : C(X) \to \mathscr{A}/\mathscr{I} \quad (N(n) = 2^n),$$

which is injective on $K_0(C(X))$, where X is the g-hole torus. Any homomorphism

$$C(X) \to M_{2^n}$$

must send $\tilde{K}^0(X)$ to zero. The same must hold for any homomorphism

$$C(X) \to \mathscr{A}.$$

Therefore ϕ does not lift to $\mathscr{A}$.

Since X can be embedded in $\mathbb{R}^3$, there are self-adjoint elements a, b and c which generate $C(X)$. Let A_n, B_n and C_n be self-adjoint matrices such that

$$\phi(a) = (A_n) + \mathscr{I} ,$$
$$\phi(b) = (B_n) + \mathscr{I} ,$$
$$\phi(c) = (C_n) + \mathscr{I} .$$

These are asymptotically commuting. By Lemma 3.2, ϕ must be injective, so the joint-spectrum of (A_n), (B_n) and (C_n) must be homeomorphic to X, and we now identify these two spaces. If A_n', B_n', C_n' are commuting self-adjoint matrices close to A_n, B_n, C_n for some n, then $\sigma(A_n', B_n', C_n')$ is close to X. A small perturbation, found using the joint functional calculus, produces matrices A_n'', B_n'', C_n'' with joint spectrums contained in X. Therefore, the existence of commuting, self-adjoint approximating sequences to (A_n), (B_n) and (C_n) would imply that ϕ can be lifted, a contradiction. □

Remark. The g-hole torus in Corollary 4.1 can be replaced by any compact, oriented, even-dimensional manifold, but more sequences of matrices are needed.

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