SOME REMARKS ON THE UNIVERSAL COEFFICIENT THEOREM IN KK-THEORY

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ABSTRACT. If a nuclear separable C*-algebra A can be approximated by C*-subalgebras satisfying the UCT, then A satisfies the UCT. It is also shown that the validity of the UCT for all separable nuclear C*-algebras is equivalent to a certain finite dimensional approximation property.

1. Introduction

Consider the category with objects separable C*-algebras and set of morphisms from A to B given by the Kasparov group $KK(A, B)$. Two C*-algebras that are isomorphic in this category are called KK-equivalent. It was shown by Rosenberg and Schochet [13] that the separable C*-algebras A KK-equivalent to abelian C*-algebras are exactly those satisfying the following universal coefficient exact sequence

$$0 \to \mathrm{Ext}(K_*(A), K_{*-1}(B)) \to KK_*(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

for any separable C*-algebra B. If A has this property we say that A satisfies the UCT.

While not all separable exact C*-algebras satisfy the UCT [15], it is an outstanding open question whether all separable nuclear C*-algebras satisfy the UCT. The class of separable nuclear C*-algebras satisfying the UCT is closed under inductive limits [13]. We use recent results in classification theory [9], [11], [5], [2] to prove the following.

Theorem 1.1. *Let A be a nuclear separable C*-algebra. Assume that for any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$ there is a C*-subalgebra B of A satisfying the UCT and such that $dist(a, B) < \epsilon$ for all $a \in \mathcal{F}$. Then A satisfies the UCT.*

Kirchberg proved that the UCT holds true for all nuclear separable C*-algebras if and only if any purely infinite simple unital separable nuclear C*-algebra with trivial K-theory is isomorphic to the Cuntz algebra $\mathcal{O}_2$ [12, Cor. 8.4.6]. We prove an analogous result for tracially AF algebras, with $\mathcal{O}_2$ replaced by the universal UHF algebra. The class of tracially AF algebras was introduced in [9] and it includes both the class of real rank zero AH algebras studied in [8] and the class of algebras constructed in [4]. A separable C*-algebra A is called *residually finite dimensional* (abbreviated *RFD*) if it has a separating sequence of finite dimensional representations. Equivalently, A embeds in a C*-algebra of

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the form $\prod_{n=1}^{\infty} M_{k(n)}$. The suspension of a C*-algebra A is denoted by $SA = C_0(0,1) \otimes A$. The C*-algebra obtained by adding a unit to A is denoted by $\widetilde{A}$.

Theorem 1.2. *The following assertions are equivalent.*

(i) Every separable nuclear C-algebra satisfies the UCT.*

(ii) Every separable nuclear simple unital tracially AF algebra A with $K_0(A) \cong \mathbb{Q}$ (as scaled ordered groups) and $K_1(A) = 0$ is isomorphic to the universal UHF algebra.

(iii) For any separable nuclear RFD C-algebra D with $K_*(D) = 0$, for any finite set $\mathcal{F} \subset D$ and any $\epsilon > 0$, there exist a representation $\pi : D \to M_k(\mathbb{C})$ and a finite dimensional C*-subalgebra B of $M_{k+1}(\widetilde{D})$ such that $dist(\begin{pmatrix} a & 0 \\ 0 & \pi(a) \end{pmatrix}, B) < \epsilon$ for all $a \in \mathcal{F}$.*

The distance in the statement is computed in $M_{k+1}(\widetilde{D})$.

2. Reduction to residually finite dimensional C*-algebras

The following proposition gathers a number of useful facts proven in [13].

Proposition 2.1. *(1) A separable C*-algebra A satisfies the UCT if and only if A is KK-equivalent to a commutative C*-algebra.*

(2) If $0 \to J \to A \to B \to 0$ is a semisplit exact sequence of separable C-algebras and if two of the C*-algebras J, A and B satisfy the UCT, then so does the third.*

(3) Let (B_i, η_i) be an inductive system of separable C-algebras and let $B = \varinjlim (B_i, \eta_i)$ be its inductive limit. If each B_i satisfies the UCT and B is nuclear, then B satisfies the UCT.*

Definition 2.2. A sequence $(A_i)_{i=1}^{\infty}$ of C*-subalgebras of A is called *exhausting* if for any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$ there is i such that $dist(a, A_i) < \epsilon$ for all $a \in \mathcal{F}$.

It is clear that any increasing sequence of C*-subalgebras of A whose union is dense in A is exhausting. Theorem 1.1 can be rephrased as follows.

Theorem 2.3. *Let A be a nuclear separable C*-algebra. Assume that there is an exhausting sequence $(A_i)_{i=1}^{\infty}$ of C*-subalgebras of A such that each A_i satisfies the UCT. Then A satisfies the UCT.*

Each A_i is exact but not necessarily nuclear.

Lemma 2.4. *It suffices to prove Theorem 2.3 under the additional assumptions that $A = \widetilde{SF}$ and $A_i = \widetilde{SF_i}$ where F is a unital RFD algebra and $(F_i)_{i=1}^{\infty}$ is an exhausting sequence of unital C*-subalgebras of F and each F_i satisfies the UCT.*

Proof. Let A and (A_i) be as in the statement of the Theorem 2.3. The C*-algebra $\widetilde{SA}$ is quasidiagonal and in fact by a result of Voiculescu [16] there is a unital completely positive map $\theta : \widetilde{SA} \to \prod M_{k(n)}$ such that the induced quotient map $\Theta : \widetilde{SA} \to \prod M_{k(n)} / \sum M_{k(n)}$ is a unital $*$-monomorphism. For each i consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & \prod M_{k(n)} & \longrightarrow & \prod M_{k(n)}/\sum M_{k(n)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \Theta & & \\
0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & F & \longrightarrow & \widetilde{SA} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & F_i & \longrightarrow & \widetilde{SA_i} & \longrightarrow & 0
\end{array}$$

where the second and third row are successive pullbacks of the first row. The C*-algebra F is nuclear as an extension of nuclear C*-algebras and F is RFD since it embeds in $\prod M_{k(n)}$. Since both $\sum M_{k(n)}$ and $\widetilde{SA_i}$ satisfy the UCT, so does F_i by Proposition 2.1(2). Here we use the fact that the third row is a semisplit extension being a pullback of a semisplit extension. The sequence $(\widetilde{SF_i})$ is exhausting in $\widetilde{SF}$ hence $\widetilde{SF}$ and also (by Proposition 2.1 (2)) F will satisfy the UCT by assumption. Finally we obtain that $\widetilde{SA}$ hence A satisfy the UCT by applying once more Proposition 2.1(2). □

3. The construction $\widetilde{SF}(\underline{\Gamma})$

In this section we revisit a construction of [4] which embeds a given RFD algebra into a simple tracially AF algebra.

Let F be a separable RFD algebra. Let

$$E = \widetilde{SF} = \{a \in C([0,1], F) : a(0) = a(1) = 0\} + \mathbb{C}1_{\widetilde{F}} \subset C([0,1], \widetilde{F})$$

and construct a sequence of unital finite dimensional representations $\sigma_n : E \to M_{r(n)}(\mathbb{C})$ such that

(i$_\sigma$) σ_n is homotopic to the evaluation map at 0, $\sigma_n^0 : E \to M_{r(n)}(\mathbb{C})$, $\sigma_n^0(a) = a(0)1_{r(n)}$.
(ii$_\sigma$) For any m the set $\{\sigma_n : n \geq m\}$ separates the elements of E.
(iii$_\sigma$) If $(h(n))$ is defined by $h(1) = 1$, $h(n+1) = r(n) + 1$, then $h(n)$ is divisible by n.

The sequence (σ_n) can be constructed by taking σ_n to be of the form

$$\sigma_n(a) = \nu_n(a(t_n)) \tag{1}$$

with ν_n a unital finite dimensional representation of $\widetilde{F}$, $t_n \in [0,1]$. By adding a suitable number of point evaluation maps to σ_n we may arrange that $h(n)$ is divisible by n. Using the inclusion $M_{r(n)}(\mathbb{C}) \subset M_{r(n)}(\mathbb{C}1_E)$ we construct a sequence of unital $*$-homomorphisms $\gamma_n : E \to M_{h(n+1)}(E)$ by

$$\gamma_n(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix} \tag{2}$$

Note that γ_n is homotopic to a unital $*$-homomorphism γ_n^0 where $\gamma_n^0(a) = \begin{pmatrix} a & 0 \\ 0 & a(0)1_{r(n)} \end{pmatrix}$.

Let $H(n) = h(1)h(2)\dots h(n)$, $E_n = M_{H(N)}(E)$ and define $\Gamma_n, \Gamma_n^0 : E_n \to E_{n+1}$ by

$$\Gamma_n = id_{H(n)} \otimes \gamma_n, \quad \Gamma_n^0 = id_{H(n)} \otimes \gamma_n^0.$$

The systems of maps (γ_n) and (γ_n^0) are denoted by $\underline{\Gamma}$ and $\underline{\Gamma}^0$, respectively. Let $A = E(\underline{\Gamma}) = \varinjlim (E_n, \Gamma_n)$ and $A^0 = E(\underline{\Gamma}^0) = \varinjlim (E_n, \Gamma_n^0)$.

Assume now that $(F_i)_{i=1}^\infty$ is an exhausting sequence of unital C*-subalgebras of F. The construction which takes the pair E, $\underline{\Gamma} = (\gamma_n)$ to $E(\underline{\Gamma})$ is functorial as described in [4, Remark 4]. Thus since γ_n maps $\widetilde{SF}_i$ into $M_{H(n)}(\widetilde{SF}_i)$ we obtain unital embeddings $A_i = \widetilde{SF}_i(\underline{\Gamma}) \subset A = \widetilde{SF}(\underline{\Gamma})$. Moreover it is clear that (A_i) is an exhausting sequence of unital C*-subalgebras of A.

4. F satisfies the UCT if and only if $A = \widetilde{SF}(\underline{\Gamma})$ does so

In this section we employ the construction $\widetilde{SF}(\underline{\Gamma})$ and an argument based on shape theory to reduce the proof of Theorem 2.3 to a certain class of simple tracially AF algebras. Let F be a separable nuclear RFD C*-algebra and let $\underline{\Gamma} = (\gamma_n)$ be constructed as above with (σ_n) verifying the conditions (i_σ)–(iii_σ) of Section 3.

Proposition 4.1. *F satisfies the UCT if and only if $A = \widetilde{SF}(\underline{\Gamma})$ does so.*

Proof. The diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\Gamma_n} & E_{n+1} \\ \| & & \| \\ E_n & \xrightarrow{\Gamma_n^0} & E_{n+1} \end{array}$$

commutes up to homotopy. Therefore A is shape equivalent to $A^0 = \widetilde{SF}(\underline{\Gamma}_0)$. It follows from [3, Theorem 3.9] that A is isomorphic to A^0 in the asymptotic homotopy category of Connes and Higson. Since these algebras are nuclear, it follows that they are KK-equivalent [1]. Thus A satisfies the UCT if and only if A^0 does so. We are now going to argue that A^0 satisfies the UCT if and only if F does so. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{H(n)}(SF) & \longrightarrow & E_n & \xrightarrow{\mu_n} & M_{H(n)}(\mathbb{C}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Gamma_n^{00}} & & \downarrow{\scriptstyle \Gamma_n^0} & & \downarrow{\scriptstyle \theta_n} & & \\ 0 & \longrightarrow & M_{H(n+1)}(SF) & \longrightarrow & E_{n+1} & \xrightarrow[\mu_{n+1}]{} & M_{H(n+1)}(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

where θ_n is a unital $*$-homomorphism, $\mu_n : E_n = M_{H(n)}(\widetilde{SF}) \to M_{H(n)}(\mathbb{C})$ is the evaluation map at 0 and there is a permutation unitary $u_n \in M_{H(n+1)}(\mathbb{C})$ such that

$$\Gamma_n^{00}(a) = u_n \begin{pmatrix} a & \\ & 0 \end{pmatrix} u_n^*.$$

Passing to the inductive limit we obtain an exact sequence

$$0 \to SF \otimes \mathcal{K} \to A^0 \to \mathcal{U} \to 0 \tag{3}$$

where $\mathcal{U}$ is the universal UHF algebra.

From (3) and Proposition 2.1 we see that A satisfies the UCT if and only if F does so. $\square$

Using the extension (3) and its construction we see that $K_*(A) \cong K_*(A^0) \cong K_*(SA) \oplus K_*(\mathcal{U})$. The order on $K_0(A) \cong K_0(SF) \oplus \mathbb{Q}$ is given by

$$K_0(A)^+ = K_0(A^0)^+ = \{(x,r) \in K_0(SF) \oplus \mathbb{Q} : r > 0\} \cup \{0\}. \tag{4}$$

This follows from the observation that if p, q are projections in (matrices over) E_n and the rank of $\mu_n(p)$ is strictly greater than the rank of $\mu_n(q)$, then $\Gamma^0_{n+r,n}(p) \succ \Gamma^0_{n+r,n}(q)$ for some large r. The description of positive elements is useful when we construct an AH algebra whose ordered K-theory is isomorphic to that of A.

5. $A = \widetilde{SF}(\underline{\Gamma})$ is isomorphic to an AH algebra

Let F and (F_i) be as in Lemma 2.4 and let $A = \widetilde{SF}(\underline{\Gamma})$ and $A_i = \widetilde{SF_i}(\underline{\Gamma})$ be constructed as in the previous section.

Let (a_i) be a sequence dense in A. Since the sequence (A_i) is exhausting, after passing to a subsequence, we may arrange that for each i there is a set $\{a_1^{(i)}, a_2^{(i)}, \dots, a_i^{(i)}\} \subset A_i$ such that

$$\|a_j - a_j^{(i)}\| < 1/i, \quad \text{for all } 1 \le j \le i. \tag{5}$$

Assume that we are given a sequence $\varphi_i : A_i \to B$ of unital nuclear $*$-homomorphisms. By nuclearity, for each i there are unital completely positive maps $\mu_i : A_i \to M_{k(i)}(\mathbb{C})$ and $\nu_i : M_{k(i)}(\mathbb{C}) \to B$ such that $\|\varphi_i(a_j^{(i)}) - \nu_i\mu_i(a_j^{(i)})\| < 1/i$ for all $1 \le j \le i$. By Arveson's extension theorem, μ_i extends to a unital completely positive map $\widetilde{\mu}_i : A \to M_{k(i)}(\mathbb{C})$. Define $\phi_i : A \to B$ by $\phi_i = \nu_i\widetilde{\mu}_i$. Then

$$\|\phi_i(a_j^{(i)}) - \varphi_i(a_j^{(i)})\| = \|\nu_i\mu_i(a_j^{(i)}) - \varphi_i(a_j^{(i)})\| < 1/i \tag{6}$$

for all $1 \le j \le i$.

Lemma 5.1. *(a) If $x_i \in A_i$, $x \in A$ and $\|x_i - x\| \to 0$, then $\|\phi_i(x) - \varphi_i(x_i)\| \to 0$, as $i \to \infty$.*
(b) If $x, y \in A$, then $\|\phi_i(xy) - \phi_i(x)\phi_i(y)\| \to 0$.

Proof. Since φ_i are $*$-homomorphisms, (b) follows from (a). To prove (a), for any $\epsilon > 0$ we are going to find n such that

$$\|\phi_i(x) - \varphi_i(x_i)\| < 8\epsilon \quad \text{for all} \;\; i \ge n. \tag{7}$$

Since (a_i) is dense in A, there is m such that $\|x - a_m\| < \epsilon$. Let $n \ge \max(m, 1/\epsilon)$ be such that $\|x - x_i\| < \epsilon$ for all $i \ge n$. If $i \ge n$, then from (5), we have

$$\|x_i - a_m^{(i)}\| \le \|x_i - x\| + \|x - a_m\| + \|a_m - a_m^{(i)}\| < \epsilon + \epsilon + 1/i < 3\epsilon.$$

Using (6), and assuming that $i \geq n$ we have

$$\|\phi_i(x) - \varphi_i(x_i)\| \leq \|\phi_i(x) - \phi_i(x_i)\| + \|\phi_i(x_i) - \phi_i(a_m^{(i)})\| + \\ \|\phi_i(a_m^{(i)}) - \varphi_i(a_m^{(i)})\| + \|\varphi_i(a_m^{(i)}) - \varphi_i(x_i)\| \leq \\ \|x - x_i\| + 2\|x_i - a_m^{(i)}\| + 1/i < \epsilon + 6\epsilon + 1/i < 8\epsilon.$$

□

The reader is referred to [5] for a background discussion on the total K-theory group $\underline{K}(A)$ of a C*-algebra A and the partial maps induced on $\underline{K}(A)$ by approximate morphisms. The graded group $\underline{K}(A)$ is acted un by a natural set of coefficient and Bockstein operations [14], denoted here by Λ. We need the following.

Theorem 5.2. [5] *Let A be a unital separable C*-algebra. Suppose that there is an exhausting sequence $(A_i)_{i=1}^{\infty}$ of unital C*-subalgebras of A such that each A_i is simple, exact, tracially AF, and satisfies the UCT. Then for any finite subset $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exists a $\underline{K}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any unital simple infinite-dimensional tracially AF algebra B, and any two unital nuclear completely positive contractions $\varphi, \psi : A \longrightarrow B$ which are δ-multiplicative on $\mathcal{G}$, with $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}$, there exists a unitary $u \in U(B)$ such that $\|u\varphi(a)u^* - \psi(a)\| < \epsilon$ for all $a \in \mathcal{F}$.*

Proof. If $A = A_i$ for some i the result was proved in [5]. To derive Theorem 5.2, it suffices to apply [5, Thm. 6.7] for a finite set $\mathcal{F}'$ that approximates $\mathcal{F}$ and is contained in some A_i. □

The classification of separable simple unital nuclear tracially AF algebras satisfying the UCT was completed by Lin [11], who succeeded in proving a key lifting result. Lin's lifting result extends to certain exact C*-algebras as follows (the case when A is nuclear is due to Lin).

Theorem 5.3. [2] *Let A, B be infinite dimensional separable simple unital tracially AF C*-algebras. Suppose that A is exact and satisfies the UCT. Then for any $\alpha \in \mathrm{KK}(A, B)$ such that the induced map $\alpha_* : K_0(A) \to K_0(B)$ is order preserving and $\alpha_*[1_A] = [1_B]$ there is a nuclear unital $*$-homomorphism $\varphi : A \to B$ such that $\varphi_*(x) = \alpha_*(x)$ for all $x \in \underline{\mathrm{K}}(A)$. If $\psi : A \to B$ is another nuclear $*$-homomorphism with $\psi_* = \varphi_* : \underline{K}(A) \to \underline{K}(B)$, then there is a sequence of unitaries $u_n \in B$ such that $\|\varphi(a) - u_n\psi(a)u_n^*\| \to 0$ for all $a \in A$.*

By Theorem 5.2, if $\psi : A \to B$ is another $*$-homomorphism with $\psi_* = \varphi_* : \underline{K}(A) \to \underline{K}(B)$, then there is a sequence of unitaries $u_n \in B$ such that $\|\varphi(a) - u_n\psi(a)u_n^*\| \to 0$ for all $a \in A$.

Let F' be an abelian C*-algebra that has the same K-theory groups as F and construct as in the previous section a C*-algebra $B = \widetilde{SF'}(\underline{\Gamma}')$ with exactly the same integers $r(n)$. This is certainly possible since all the irreducible representations of an abelian C*-algebra are one-dimensional. Reasoning as above $K_0(B) \cong K_0(SF') \oplus \mathbb{Q}$ with $K_0(SF) \cong K_0(SF')$ and

$$K_0(B)^+ = \{(x, r) \in K_0(SF') \oplus \mathbb{Q} : r > 0\} \cup \{0\}. \tag{8}$$

Therefore there is an isomorphism of groups $\beta_* : K_*(B) \to K_*(A)$ whose K_0-component is an isomorphism of scaled ordered groups.

Proposition 5.4. *The C^*-algebra $A = \widetilde{SF}(\underline{\Gamma})$ is isomorphic to the AH algebra $B = \widetilde{SF'}(\underline{\Gamma}')$; hence it satisfies the UCT.*

Proof. By construction of B, there is an isomorphism of groups $\beta_* : K_*(B) \to K_*(A)$ whose K_0-component is an isomorphism of scaled ordered groups. Since B satisfies the UCT, β_* lifts to an element $\beta \in KK(B, A)$. Since B is also nuclear, by Theorem 5.3 there is a unital $*$-homomorphism $\psi : B \to A$ with $\psi_* = \beta_* : \underline{K}(B) \to \underline{K}(A)$. Let $\alpha_* \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ be defined by $\alpha_* = \psi_*^{-1}$. Let $\alpha_*^{(i)} \in \mathrm{Hom}_\Lambda(\underline{K}(A_i), \underline{K}(B))$ be obtained by composing α_* with the map $\underline{K}(A_i) \to \underline{K}(A)$ induced by the inclusion $A_i \hookrightarrow A$. We have that $\alpha_*^{(i)}[1] = [1]$ and $\alpha_*^{(i)} : K_0(A_i) \to K_0(B)$ is positive since it is given by the composition of $\psi_*^{-1} : K_0(A) \to K_0(B)$ with $K_0(A_i) \to K_0(A)$. Since A_i is exact and satisfies the UCT, by Theorem 5.3 there is a unital $*$-homomorphism $\varphi_i : A_i \to B$ with $\varphi_{i\,*} = \alpha_*^{(i)} : \underline{K}(A_i) \to \underline{K}(B)$. Let (ϕ_i) be constructed as discussed before Lemma 5.1. We claim that

$$\phi_{i,\sharp}(p) \to \alpha_*[p] \quad \text{for all} \quad p \in Proj_\infty(A \otimes C), \tag{9}$$

where C is a fixed unital abelian C*-algebra with $\underline{\mathrm{K}}(A) = K_0(A \otimes C)$. For simplicity we are going to prove (9) only for $p \in A$ a projection. The general case is similar. Let $p_i \in A_i$ be a sequence of projections with $\|p_i - p\| \to 0$. Then $\|\phi_i(p) - \varphi_i(p_i)\| \to 0$ by Lemma 5.1. Therefore there is n such that $\|p_i - p\| < 1$ and $\|\chi(\phi_i(p)) - \varphi_i(p_i)\| < 1$, where χ is the characteristic map of the interval $[2/3, 1]$, and $i \geq n$. Consequently, if $i \geq n$, then

$$\phi_{i,\sharp}(p) = [\chi(\phi_i(p))] = [\varphi_i(p_i)] = \varphi_{i\,*}[p_i] = \alpha_*^{(i)}[p_i] = \alpha_*[p_i] = \alpha_*[p]. \tag{10}$$

Using Theorems 5.3 and 5.2 we construct a unital $*$-homomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha_*$. This goes as follows. Let $\mathcal{F}_n = \{a_1, \dots, a_n\}$ and let $\epsilon_n = 2^{-n}$. Let $(\mathcal{P}_n, \mathcal{G}_n, \delta_n)$ be a $\underline{K}(A)$-triple given by Theorem 5.2 for the input $\mathcal{F}_n$, ϵ_n. One may assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and $\delta_n > \delta_{n+1}$. Passing to a subsequence of (ϕ_i) we may further assume that $\phi_{n\,\sharp}(p) = \alpha_*[p]$ for all $p \in \mathcal{P}_n$ and all n. By Theorem 5.2 there is a sequence of unitaries (u_n) in B such that $\|u_n^* \phi_n(a) u_n - \phi_{n+1}(a)\| < \epsilon_n$ for $a \in \mathcal{F}_n$. It follows that $\Phi_n = u_1 u_2 \dots u_{n-1} \phi_n u_{n-1}^* \dots u_2^* u_1^*$ is a sequence of unital completely positive and asymptotically multiplicative maps such that $(\Phi_n(a))$ is a Cauchy sequence for all $a \in \{a_1, a_2, \dots\}$, a dense set in A. Thus Φ_n converges to a unital $*$-homomorphism $\varphi : A \to B$ with $\varphi_* = \alpha_* : \underline{K}(A) \to \underline{K}(B)$. Now φ and ψ are $*$-homomorphisms such that $\psi \circ \varphi$ is approximately unitarily equivalent to id_A and $\varphi \circ \psi$ is approximately unitarily equivalent to id_B by Theorem 5.2. One concludes that $A \cong B$ by Elliott's intertwining argument [7]. □

6. Conclusion of proofs

Proof of Theorems 1.1, 2.3.

The result follows by putting together Lemma 2.4, Proposition 4.1 and Proposition 5.4.

Proof of Theorem 1.2.

(i) ⇒ (ii) Let A be as in (ii). By assumption A satisfies the UCT and it has the same ordered K-theory as the universal UHF algebra $\mathcal{U}$. By the isomorphism theorem of [9] or [5], A is isomorphic to $\mathcal{U}$.

(ii) ⇒ (iii) Assume that D is a nuclear separable RFD C*-algebra and $K_*(D) = 0$. Construct a simple tracially AF algebra $\widetilde{D}(\underline{\Gamma})$ as in Section 3 where $\Gamma = (\gamma_n)$, $\gamma_n : \widetilde{D} \to M_{r(n)+1}(\widetilde{D})$, $\gamma_n(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix}$ and (σ_n) is a sequence of unital finite dimensional representations of $\widetilde{D}$ satisfying the conditions (ii$_\sigma$)–(iii$_\sigma$) of Section 3. By construction $\widetilde{D}(\underline{\Gamma})$ has the same ordered K-theory as the universal UHF algebra $\mathcal{U}$. By (ii) $\widetilde{D}(\underline{\Gamma})$ is isomorphic to $\mathcal{U}$, hence D has the desired approximation property.

(iii) ⇒ (i) By [13] it suffices to prove that all nuclear separable C*-algebras A with $K_*(A) = 0$ satisfy the UCT. Fix such an A and argue as in the proof of Lemma 2.4 to produce an extension

$$0 \to \sum M_{k(n)} \xrightarrow{\theta} F \to SA \to 0 \tag{11}$$

with F RFD. Since $K_*(A) = 0$, θ induces an isomorphism of groups $\theta_* : K_*(\sum M_{k(n)}) \to K_*(F)$. The mapping cone C*-algebra $C_\theta = \{(f, x) \in C_0[0,1)\otimes F \oplus \sum M_{k(n)} \ : \ f(0) = \theta(x)\}$ is RFD since both $\sum M_{k(n)}$ and F are so. The boundary map $\delta : K_*(\sum M_{k(n)}) \to K_{*-1}(SF)$ associated with the exact sequence

$$0 \to SF \to C_\theta \to \sum M_{k(n)} \to 0 \tag{12}$$

is an isomorphism since it identifies with θ_*, modulo the isomorphism $K_{*-1}(SF) \cong K_*(F)$. This shows that $K_*(C_\theta) = 0$. Set $D = SC_\theta$ and construct a C*-algebra $\widetilde{D}(\underline{\Gamma})$ as in Section 3 by using a sequence of unital finite dimensional representations $\sigma_n : \widetilde{D} \to M_{r(n)}(\mathbb{C}1_{\widetilde{D}})$ satisfying the conditions (i$_\sigma$)–(iii$_\sigma$) of Section 3 and the following

(iv$_\sigma$) There is a sequence of finite dimensional C*-algebras $B_n \subset M_{r(n)+1}(\widetilde{D})$ such that

$$\lim_{n\to\infty} dist(\begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix}, B_n) = 0, \quad \text{for all } a \in \widetilde{D}.$$

The existence of the sequence (σ_n) follows from our assumptions in condition (iii) of Theorem 1.2 and the following discussion. Let $\mathcal{F}$, ϵ, π and B be as in condition (iii). Since D is RFD, after adding to π a finite dimensional representation we may assume that $\|\pi(a)\| \geq \|a\|(1-\epsilon)$ for all $a \in \mathcal{F}$. By replacing π by $\pi \oplus \pi \circ \iota$ where $\iota : D = SC_\theta \to D$ is given be $\iota(a)(t) = a(1-t)$, $t \in [0,1]$, we may arrange that π is null-homotopic. If $h > k$ and $\widetilde{\pi} : \widetilde{D} \to M_h(\mathbb{C}_{\widetilde{D}})$ is the unitalization of π, then $\widetilde{\pi}$ is homotopic to the evaluation map $\sigma^0(a + \lambda 1) = \lambda 1_h$, $a \in D$ and

$$dist(\begin{pmatrix} \widetilde{a} & 0 \\ 0 & \widetilde{\pi}(\widetilde{a}) \end{pmatrix}, B + \mathbb{C}1) < \epsilon, \quad \text{for all } \widetilde{a} \in \mathcal{F} + \mathbb{C}1.$$

By applying this construction for an increasing sequence of finite sets $(\mathcal{F}_n)$ whose union is dense in D and $\epsilon_n = 1/n$, we obtain a sequence $\sigma_n = \widetilde{\pi}_n$ with the desired properties.

Using the approximation property (iv$_\sigma$) it is not hard to see that $\widetilde{D}(\underline{\Gamma})$ is isomorphic to an AF algebra (see the discussion below), and so it satisfies the UCT. By Proposition 4.1 (with $F = C_\theta$), if $\widetilde{D}(\underline{\Gamma})$ satisfies the UCT, then $D = SC_\theta$ does so. Using (11) and (12) we obtain that F and hence A satisfy the UCT by Proposition 2.1. To verify that $\widetilde{D}(\underline{\Gamma})$ is AF it suffices to show that for any $r \geq 1$, any finite subset $\mathcal{F} \subset M_{H(r)}(\widetilde{D})$ and any $\epsilon > 0$ there is $n \geq r$ such that

$$\Gamma_{n+1,r}(\mathcal{F}) \subset_\epsilon B$$

for some finite dimensional C*-subalgebra B of $M_{H(n+1)}(\widetilde{D})$; (we write $\mathcal{G} \subset_\epsilon \mathcal{G}'$ if $dist(a, \mathcal{G}') < \epsilon$ for all $a \in \mathcal{G}$.) By (iv$_\sigma$) we find $n \geq r$ such that

$$(id_{H(r)} \otimes \gamma_n)(\mathcal{F}) \subset_\epsilon M_{H(r)}(B_n) \subset M_{H(r)h(n+1)}(\widetilde{D}).$$

Note that

$$\Gamma_{n,r}(\mathcal{F}) \subset u \begin{pmatrix} \mathcal{F} & 0 \\ 0 & M_{H(n)-H(r)}(\mathbb{C}1_{\widetilde{D}}) \end{pmatrix} u^*$$

for some unitary $u \in M_{H(n)}(\widetilde{D})$. Therefore

$$\Gamma_{n+1,r}(\mathcal{F}) = (id_{H(n)} \otimes \gamma_n)(\Gamma_{n,r}(\mathcal{F})) \subset_\epsilon B \subset M_{H(n+1)}(\widetilde{D})$$

where B is isomorphic to $M_{H(r)}(B_n) \oplus M_{H(n+1)-H(r)h(n+1)}(\mathbb{C}1)$. □

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