SOME REMARKS ON THE UNIVERSAL COEFFICIENT THEOREM IN KK-THEORY

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Abstract. If a nuclear separable C*-algebra $A$ can be approximated by C*-subalgebras satisfying the UCT, then $A$ satisfies the UCT. It is also shown that the validity of the UCT for all separable nuclear C*-algebras is equivalent to a certain finite dimensional approximation property.

1. Introduction

Consider the category with objects separable C*-algebras and set of morphisms from $A$ to $B$ given by the Kasparov group $KK(A, B)$. Two C*-algebras that are isomorphic in this category are called KK-equivalent. It was shown by Rosenberg and Schochet [13] that the separable C*-algebras $A$ KK-equivalent to abelian C*-algebras are exactly those satisfying the following universal coefficient exact sequence

$$0 \to \text{Ext}(K_*(A), K_{*-1}(B)) \to KK_*(A, B) \to \text{Hom}(K_*(A), K_*(B)) \to 0$$

for any separable C*-algebra $B$. If $A$ has this property we say that $A$ satisfies the UCT.

While not all separable exact C*-algebras satisfy the UCT [15], it is an outstanding open question whether all separable nuclear C*-algebras satisfy the UCT. The class of separable nuclear C*-algebras satisfying the UCT is closed under inductive limits [13]. We use recent results in classification theory [9], [11], [5], [2] to prove the following.

Theorem 1.1. Let $A$ be a nuclear separable C*-algebra. Assume that for any finite set $F \subset A$ and any $\epsilon > 0$ there is a C*-subalgebra $B$ of $A$ satisfying the UCT and such that $\text{dist}(a, B) < \epsilon$ for all $a \in F$. Then $A$ satisfies the UCT.

Kirchberg proved that the UCT holds true for all nuclear separable C*-algebras if and only if any purely infinite simple unital separable nuclear C*-algebra with trivial K-theory is isomorphic to the Cuntz algebra $O_2$ [12, Cor. 8.4.6]. We prove an analogous result for tracially AF algebras, with $O_2$ replaced by the universal UHF algebra. The class of tracially AF algebras was introduced in [9] and it includes both the class of real rank zero AH algebras studied in [8] and the class of algebras constructed in [4]. A separable C*-algebra $A$ is called residually finite dimensional (abbreviated RFD) if it has a separating sequence of finite dimensional representations. Equivalently, $A$ embeds in a C*-algebra of

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the form $\prod_{n=1}^{\infty} M_{k(n)}$. The suspension of a C*-algebra $A$ is denoted by $SA = C_0(0,1) \otimes A$. The C*-algebra obtained by adding a unit to $A$ is denoted by $\tilde{A}$.

**Theorem 1.2.** The following assertions are equivalent.

(i) Every separable nuclear C*-algebra satisfies the UCT.

(ii) Every separable nuclear simple unital tracially AF algebra $A$ with $K_0(A) \cong \mathbb{Q}$ (as scaled ordered groups) and $K_1(A) = 0$ is isomorphic to the universal UHF algebra.

(iii) For any separable nuclear RFD C*-algebra $D$ with $K_* (D) = 0$, for any finite set $F \subset D$ and any $\epsilon > 0$, there exist a representation $\pi : D \to M_k(\mathbb{C})$ and a finite dimensional C*-subalgebra $B$ of $M_{k+1}(\tilde{D})$ such that $\text{dist}(\begin{pmatrix} a \\ 0 \\ \pi(a) \end{pmatrix}, B) < \epsilon$ for all $a \in F$.

The distance in the statement is computed in $M_{k+1}(\tilde{D})$.

2. REDUCTION TO RESIDUALLY FINITE DIMENSIONAL C*-ALGEBRAS

The following proposition gathers a number of useful facts proven in [13].

**Proposition 2.1.** (1) A separable C*-algebra $A$ satisfies the UCT if and only if $A$ is KK-equivalent to a commutative C*-algebra.

(2) If $0 \to J \to A \to B \to 0$ is a semisplit exact sequence of separable C*-algebras and if two of the C*-algebras $J, A$ and $B$ satisfy the UCT, then so does the third.

(3) Let $(B_i, \eta_i)$ be an inductive system of separable C*-algebras and let $B = \lim_{\to} (B_i, \eta_i)$ be its inductive limit. If each $B_i$ satisfies the UCT and $B$ is nuclear, then $B$ satisfies the UCT.

**Definition 2.2.** A sequence $(A_i)_{i=1}^{\infty}$ of C*-subalgebras of $A$ is called exhausting if for any finite set $F \subset A$ and any $\epsilon > 0$ there is $i$ such that $\text{dist}(a, A_i) < \epsilon$ for all $a \in F$.

It is clear that any increasing sequence of C*-subalgebras of $A$ whose union is dense in $A$ is exhausting. Theorem 1.1 can be rephrased as follows.

**Theorem 2.3.** Let $A$ be a nuclear separable C*-algebra. Assume that there is an exhausting sequence $(A_i)_{i=1}^{\infty}$ of C*-subalgebras of $A$ such that each $A_i$ satisfies the UCT. Then $A$ satisfies the UCT.

Each $A_i$ is exact but not necessarily nuclear.

**Lemma 2.4.** It suffices to prove Theorem 2.3 under the additional assumptions that $A = \tilde{SF}$ and $A_i = \tilde{SF_i}$ where $F$ is a unital RFD algebra and $(F_i)_{i=1}^{\infty}$ is an exhausting sequence of unital C*-subalgebras of $F$ and each $F_i$ satisfies the UCT.

**Proof.** Let $A$ and $(A_i)$ be as in the statement of the Theorem 2.3. The C*-algebra $\tilde{SA}$ is quasidiagonal and in fact by a result of Voiculescu [16] there is a unital completely positive map $\theta : \tilde{SA} \to \prod M_{k(n)}$ such that the induced quotient map $\Theta : \tilde{SA} \to \prod M_{k(n)}/\sum M_{k(n)}$ is
a unital \(*\)-monomorphism. For each \(i\) consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & \prod M_{k(n)} & \longrightarrow & \prod M_{k(n)}/\sum M_{k(n)} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \Theta & & \uparrow & & \Theta \\
0 & \longrightarrow & \sum \tilde{M}_{k(n)} & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{SA} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \sum M_{k(n)} & \longrightarrow & \tilde{F}_i & \longrightarrow & \tilde{SA}_i & \longrightarrow & 0
\end{array}
\]

where the second and third row are successive pullbacks of the first row. The C*-algebra \(F\) is nuclear as an extension of nuclear C*-algebras and \(F\) is RFD since it embeds in \(\prod M_{k(n)}\). Since both \(\sum M_{k(n)}\) and \(\tilde{SA}_i\) satisfy the UCT, so does \(F_i\) by Proposition 2.1(2). Here we use the fact that the third row is a semisplit extension being a pullback of a semisplit extension.

The sequence \((\tilde{SF}_i)\) is exhausting in \(\tilde{SF}\) hence \(\tilde{SF}\) and also (by Proposition 2.1 (2)) \(F\) will satisfy the UCT by assumption. Finally we obtain that \(\tilde{SA}\) hence \(A\) satisfy the UCT by applying once more Proposition 2.1(2). □

3. The construction \(\tilde{SF}(\Gamma)\)

In this section we revisit a construction of [4] which embeds a given RFD algebra into a simple tracially AF algebra.

Let \(F\) be a separable RFD algebra. Let

\[
E = \tilde{SF} = \{a \in C([0,1], F) : a(0) = a(1) = 0\} + \mathbb{C}1_F \subset C([0,1], \tilde{F})
\]

and construct a sequence of unital finite dimensional representations \(\sigma_n : E \rightarrow M_{r(n)}(\mathbb{C})\) such that

(i) \(\sigma_n\) is homotopic to the evaluation map at 0, \(\sigma_n^0 : E \rightarrow M_{r(n)}(\mathbb{C}), \sigma_n^0(a) = a(0)1_{r(n)}\).

(ii) For any \(m \) the set \(\{\sigma_n : n \geq m\}\) separates the elements of \(E\).

(iii) If \(h(n)\) is defined by \(h(1) = 1, h(n+1) = r(n) + 1\), then \(h(n)\) is divisible by \(n\).

The sequence \((\sigma_n)\) can be constructed by taking \(\sigma_n\) to be of the form

\[
\sigma_n(a) = \nu_n(a(t_n))
\]

with \(\nu_n\) a unital finite dimensional representation of \(\tilde{F}; t_n \in [0,1]\). By adding a suitable number of point evaluation maps to \(\sigma_n\) we may arrange that \(h(n)\) is divisible by \(n\). Using the inclusion \(M_{r(n)}(\mathbb{C}) \subset M_{r(n)}(\mathbb{C}1_E)\) we construct a sequence of unital \(*\)-homomorphisms \(\gamma_n : E \rightarrow M_{h(n+1)}(E)\) by

\[
\gamma_n(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix}
\]

Note that \(\gamma_n\) is homotopic to a unital \(*\)-homomorphism \(\gamma_n^0\) where \(\gamma_n^0(a) = \begin{pmatrix} a & 0 \\ 0 & a(0)1_{r(n)} \end{pmatrix} \).
Let $H(n) = h(1)h(2) \ldots h(n)$, $E_n = M_{H(n)}(E)$ and define $\Gamma_n, \Gamma^0_n : E_n \to E_{n+1}$ by

$$
\Gamma_n = id_{H(n)} \otimes \gamma_n, \quad \Gamma^0_n = id_{H(n)} \otimes \gamma^0_n.
$$

The systems of maps $(\gamma_n)$ and $(\gamma^0_n)$ are denoted by $\Gamma$ and $\Gamma^0$, respectively. Let $A = E(\Gamma) = \lim \rightarrow (E_n, \Gamma_n)$ and $A^0 = E(\Gamma^0) = \lim \rightarrow (E_n, \Gamma^0_n).

Assume now that $(F_i)_{i=1}^\infty$ is an exhausting sequence of unital C*-subalgebras of $F$. The construction which takes the pair $E, \Gamma = (\gamma_n)$ to $E(\Gamma)$ is functorial as described in [4, Remark 4]. Thus since $\gamma_n$ maps $\widetilde{SF}_i$ into $M_{H(n)}(\widetilde{SF}_i)$ we obtain unital embeddings $A_i = \widetilde{SF}_i(\Gamma) \subset A = \widetilde{SF}(\Gamma)$. Moreover it is clear that $(A_i)$ is an exhausting sequence of unital C*-subalgebras of $A$.

4. $F$ satisfies the UCT if and only if $A = \widetilde{SF}(\Gamma)$ does so

In this section we employ the construction $\widetilde{SF}(\Gamma)$ and an argument based on shape theory to reduce the proof of Theorem 2.3 to a certain class of simple tracially AF algebras. Let $F$ be a separable nuclear RFD C*-algebra and let $\Gamma = (\gamma_n)$ be constructed as above with $(\sigma_n)$ verifying the conditions $(i_\sigma)$–$(iii_\sigma)$ of Section 3.

**Proposition 4.1.** $F$ satisfies the UCT if and only if $A = \widetilde{SF}(\Gamma)$ does so.

**Proof.** The diagram

$$
\begin{array}{ccc}
E_n & \xrightarrow{\Gamma_n} & E_{n+1} \\
\downarrow & & \downarrow \\
E_n & \xrightarrow{\Gamma^0_n} & E_{n+1}
\end{array}
$$

commutes up to homotopy. Therefore $A$ is shape equivalent to $A^0 = \widetilde{SF}(\Gamma^0)$. It follows from [3, Theorem 3.9] that $A$ is isomorphic to $A^0$ in the asymptotic homotopy category of Connes and Higson. Since these algebras are nuclear, it follows that they are KK-equivalent [1]. Thus $A$ satisfies the UCT if and only if $A^0$ does so. We are now going to argue that $A^0$ satisfies the UCT if and only if $F$ does so. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M_{H(n)}(SF) \\
\downarrow \Gamma^0_n & & \downarrow \Gamma^0_n \\
0 & \longrightarrow & M_{H(n+1)}(SF)
\end{array}
\begin{array}{ccc}
E_n & \xrightarrow{\mu_n} & M_{H(n)}(\mathbb{C}) \\
\downarrow \theta_n & & \downarrow \theta_n \\
E_{n+1} & \xrightarrow{\mu_{n+1}} & M_{H(n+1)}(\mathbb{C}) \\
\end{array}
\begin{array}{ccc}
& & 0 \\
& & 0
\end{array}
$$

where $\theta_n$ is a unital $*$-homomorphism, $\mu_n : E_n = M_{H(n)}(\widetilde{SF}) \to M_{H(n)}(\mathbb{C})$ is the evaluation map at 0 and there is a permutation unitary $u_n \in M_{H(n+1)}(\mathbb{C})$ such that

$$
\Gamma^0_n(a) = u_n \begin{pmatrix} a \\ 0 \end{pmatrix} u^*_n.
$$
Passing to the inductive limit we obtain an exact sequence
\[(3) \quad 0 \to SF \otimes K \to A^0 \to \mathcal{U} \to 0\]
where $\mathcal{U}$ is the universal UHF algebra.

From (3) and Proposition 2.1 we see that $A$ satisfies the UCT if and only if $F$ does so. \(\square\)

Using the extension (3) and its construction we see that $K_*(A) \cong K_*(A^0) \cong K_*(SA) \oplus K_*(\mathcal{U})$. The order on $K_0(A) \cong K_0(SF) \oplus \mathbb{Q}$ is given by
\[(4) \quad K_0(A)^+ = K_0(A^0)^+ = \{(x, r) \in K_0(SF) \oplus \mathbb{Q} : r > 0\} \cup \{0\}.
\]
This follows from the observation that if $p, q$ are projections in (matrices over) $E_n$ and the rank of $\mu_n(p)$ is strictly greater than the rank of $\mu_n(q)$, then $\Gamma_n^0 \succ \Gamma_{n+r,n}^0$ for some large $r$. The description of positive elements is useful when we construct an AH algebra whose ordered K-theory is isomorphic to that of $A$.

5. $A = \widetilde{SF}(\Gamma)$ is isomorphic to an AH algebra

Let $F$ and $(F_n)$ be as in Lemma 2.4 and let $A = \widetilde{SF}(\Gamma)$ and $A_i = \widetilde{SF}_i(\Gamma)$ be constructed as in the previous section.

Let $(a_i)$ be a sequence dense in $A$. Since the sequence $(A_i)$ is exhausting, after passing to a subsequence, we may arrange that for each $i$ there is a set $\{a_1^{(i)}, a_2^{(i)}, \ldots, a_j^{(i)}\} \subset A_i$ such that
\[(5) \quad \|a_j - a_j^{(i)}\| < 1/i, \quad \text{for all } 1 \leq j \leq i.
\]
Assume that we are given a sequence $\varphi_i : A_i \to B$ of unital nuclear $*$-homomorphisms. By nuclearity, for each $i$ there are unital completely positive maps $\mu_i : A_i \to M_k(i)(\mathbb{C})$ and $\nu_i : M_k(i)(\mathbb{C}) \to B$ such that $\|\varphi_i(a_j^{(i)}) - \nu_i\mu_i(a_j^{(i)})\| < 1/i$ for all $1 \leq j \leq i$. By Arveson’s extension theorem, $\mu_i$ extends to a unital completely positive map $\tilde{\mu}_i : A \to M_k(i)(\mathbb{C})$. Define $\phi_i : A \to B$ by $\phi_i = \nu_i\tilde{\mu}_i$. Then
\[(6) \quad \|\phi_i(a_j^{(i)}) - \varphi_i(a_j^{(i)})\| = \|\nu_i\mu_i(a_j^{(i)}) - \varphi_i(a_j^{(i)})\| < 1/i
\]
for all $1 \leq j \leq i$.

**Lemma 5.1.** (a) If $x_i \in A_i$, $x \in A$ and $\|x_i - x\| \to 0$, then $\|\phi_i(x) - \varphi_i(x_i)\| \to 0$, as $i \to \infty$.

(b) If $x, y \in A$, then $\|\phi_i(xy) - \phi_i(x)\phi_i(y)\| \to 0$.

**Proof.** Since $\varphi_i$ are $*$-homomorphisms, (b) follows from (a). To prove (a), for any $\epsilon > 0$ we are going to find $n$ such that
\[(7) \quad \|\phi_i(x) - \varphi_i(x_i)\| < 8\epsilon \quad \text{for all } i \geq n.
\]
Since $(a_i)$ is dense in $A$, there is $m$ such that $\|x - a_m\| < \epsilon$. Let $n \geq \max(m, 1/\epsilon)$ be such that $\|x - x_i\| < \epsilon$ for all $i \geq n$. If $i \geq n$, then from (5), we have
\[
\|x_i - a_m^{(i)}\| \leq \|x_i - x\| + \|x - a_m\| + \|a_m - a_m^{(i)}\| < \epsilon + \epsilon + 1/i < 3\epsilon.
\]
Using (6), and assuming that \( i \geq n \) we have
\[
\| \phi_i(x) - \varphi_i(x_i) \| \leq \| \phi_i(x) - \phi_i(x_i) \| + \| \phi_i(x_i) - \phi_i(a_m^{(i)}) \| +
\| \phi_i(a_m^{(i)}) - \varphi_i(a_m^{(i)}) \| + \| \varphi_i(a_m^{(i)}) - \varphi_i(x_i) \| \leq
\| x - x_i \| + 2 \| x_i - a_m^{(i)} \| + 1/i < \epsilon + 6\epsilon + 1/i < 8\epsilon.
\]

The reader is referred to [5] for a background discussion on the total K-theory group \( K(A) \) of a C*-algebra \( A \) and the partial maps induced on \( K(A) \) by approximate morphisms. The graded group \( K(A) \) is acted un by a natural set of coefficient and Bockstein operations [14], denoted here by \( \Lambda \). We need the following.

**Theorem 5.2.** [5] Let \( A \) be a unital separable C*-algebra. Suppose that there is an exhausting sequence \( (A_i)_{i=1}^\infty \) of unital C*-subalgebras of \( A \) such that each \( A_i \) is simple, exact, tracially AF, and satisfies the UCT. Then for any finite subset \( F \subset A \) and any \( \epsilon > 0 \), there exists a \( K \)-triple \( (\mathcal{P}, \mathcal{G}, \delta) \) with the following property. For any unital simple infinite-dimensional tracially AF algebra \( B \), and any two unital nuclear completely positive contractions \( \varphi, \psi : A \rightarrow B \) which are \( \delta \)-multiplicative on \( \mathcal{G} \), with \( \varphi(p) = \psi(p) \) for all \( p \in \mathcal{P} \), there exists a unitary \( u \in U(B) \) such that \( \| u\varphi(a)u^* - \psi(a) \| < \epsilon \) for all \( a \in F \).

**Proof.** If \( A = A_i \) for some \( i \) the result was proved in [5]. To derive Theorem 5.2, it suffices to apply [5, Thm. 6.7] for a finite set \( F' \) that approximates \( F \) and is contained in some \( A_i \). □

The classification of separable simple unital nuclear tracially AF algebras satisfying the UCT was completed by Lin [11], who succeeded in proving a key lifting result. Lin’s lifting result extends to certain exact C*-algebras as follows (the case when \( A \) is nuclear is due to Lin).

**Theorem 5.3.** [2] Let \( A, B \) be infinite dimensional separable simple unital tracially AF C*-algebras. Suppose that \( A \) is exact and satisfies the UCT. Then for any \( \alpha \in KK(A, B) \) such that the induced map \( \alpha_* : K_0(A) \rightarrow K_0(B) \) is order preserving and \( \alpha_*[\mathbb{1}_A] = [\mathbb{1}_B] \) there is a nuclear unital \( * \)-homomorphism \( \varphi : A \rightarrow B \) such that \( \varphi(x) = \alpha_*(x) \) for all \( x \in K(A) \). If \( \psi : A \rightarrow B \) is another nuclear \( * \)-homomorphism with \( \psi_* = \varphi_* : K(A) \rightarrow K(B) \), then there is a sequence of unitaries \( u_n \in B \) such that \( \| \varphi(a) - u_n\psi(a)u_n^* \| \rightarrow 0 \) for all \( a \in A \).

By Theorem 5.2, if \( \psi : A \rightarrow B \) is another \( * \)-homomorphism with \( \psi_* = \varphi_* : K(A) \rightarrow K(B) \), then there is a sequence of unitaries \( u_n \in B \) such that \( \| \varphi(a) - u_n\psi(a)u_n^* \| \rightarrow 0 \) for all \( a \in A \).

Let \( F' \) be an abelian C*-algebra that has the same K-theory groups as \( F \) and construct as in the previous section a C*-algebra \( B = SF'(\Gamma') \) with exactly the same integers \( r(n) \). This is certainly possible since all the irreducible representations of an abelian C*-algebra are one-dimensional. Reasoning as above \( K_0(B) \cong K_0(SF') \oplus \mathbb{Q} \) with \( K_0(SF) \cong K_0(SF') \) and

\[
K_0(B)^+ = \{ (x, r) \in K_0(SF') \oplus \mathbb{Q} : r > 0 \} \cup \{ 0 \}.
\]
Therefore there is an isomorphism of groups \( \beta_* : K_*(B) \to K_*(A) \) whose \( K_0 \)-component is an isomorphism of scaled ordered groups.

**Proposition 5.4.** The \( C^* \)-algebra \( A = SF(\Gamma) \) is isomorphic to the AH algebra \( B = SF'(\Gamma') \); hence it satisfies the UCT.

**Proof.** By construction of \( B \), there is an isomorphism of groups \( \beta_* : K_*(B) \to K_*(A) \) whose \( K_0 \)-component is an isomorphism of scaled ordered groups. Since \( B \) satisfies the UCT, \( \beta_* \) lifts to an element \( \beta \in KK(B, A) \). Since \( B \) is also nuclear, by Theorem 5.3 there is a unital \(*\)-homomorphism \( \psi : B \to A \) with \( \psi_* = \beta_* : K(B) \to K(A) \). Let \( \alpha_* \in Hom_A(K(A), K(B)) \) be defined by \( \alpha_* = \psi_*^{-1} \). Let \( \alpha_*^{(i)} \in Hom_A(K(A_i), K(B)) \) be obtained by composing \( \alpha_* \) with the map \( K(A_i) \to K(A) \) induced by the inclusion \( A_i \hookrightarrow A \). We have that \( \alpha_*^{(i)}[1] = [1] \) and \( \alpha_*^{(i)} : K_0(A_i) \to K_0(B) \) is positive since it is given by the composition of \( \psi_*^{-1} : K_0(A) \to K_0(B) \) with \( K_0(A_i) \to K_0(A) \). Since \( A_i \) is exact and satisfies the UCT, by Theorem 5.3 there is a unital \(*\)-homomorphism \( \varphi_i : A_i \to B \) with \( \varphi_i \circ \alpha_*^{(i)} : K(A_i) \to K(B) \). Let \( (\phi_i) \) be constructed as discussed before Lemma 5.1. We claim that

\[
(9) \quad \phi_i(z(p)) \to \alpha_*(p) \quad \text{for all } p \in Proj_\infty(A \otimes C),
\]

where \( C \) is a fixed unital abelian \( C^* \)-algebra with \( K(A) = K_0(A \otimes C) \). For simplicity we are going to prove (9) only for \( p \in A \) a projection. The general case is similar. Let \( p_i \in A_i \) be a sequence of projections with \( \|p_i - p\| \to 0 \). Then \( \|\phi_i(p) - \varphi_i(p_i)\| \to 0 \) by Lemma 5.1. Therefore there is \( n \) such that \( \|p_i - p\| < 1 \) and \( \|\chi(\phi_i(p)) - \varphi_i(p_i)\| < 1 \), where \( \chi \) is the characteristic map of the interval \([2/3, 1]\), and \( i \geq n \). Consequently, if \( i \geq n \), then

\[
(10) \quad \phi_i(z(p)) = [\chi(\phi_i(p))] = [\varphi_i(p_i)] = \varphi_i([p_i]) = \alpha_*^{(i)}[p_i] = \alpha_*[p_i] = \alpha_*(p).
\]

Using Theorems 5.3 and 5.2 we construct a unital \(*\)-homomorphism \( \varphi : A \to B \) such that \( \varphi_* = \alpha_* \). This goes as follows. Let \( F_n = \{a_1, \ldots, a_n\} \) and let \( \epsilon_n = 2^{-n} \). Let \( (\mathcal{P}_n, \mathcal{G}_n, \delta_n) \) be a \( K(A) \)-triple given by Theorem 5.2 for the input \( F_n, \epsilon_n \). One may assume that \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \), \( \mathcal{G}_n \subset \mathcal{G}_{n+1} \) and \( \delta_n > \delta_{n+1} \). Passing to a subsequence of \( (\phi_i) \) we may further assume that \( \phi_n(p) = \alpha_*[p] \) for all \( p \in \mathcal{P}_n \) and all \( n \). By Theorem 5.2 there is a sequence of unitaries \( (u_n) \) in \( B \) such that \( \|u_n^* \phi_n(a) u_n - \phi_{n+1}(a)\| < \epsilon_n \) for \( a \in F_n \). It follows that \( \Phi_n = u_1 u_2 \ldots u_{n-1} \phi_n u_n^* \) is a sequence of unitarily completely positive and asymptotically multiplicative maps such that \( (\Phi_n(a)) \) is a Cauchy sequence for all \( a \in \{a_1, a_2, \ldots\} \), a dense set in \( A \). Thus \( \Phi_n \) converges to a unital \(*\)-homomorphism \( \varphi : A \to B \) with \( \varphi_* = \alpha_* : K(A) \to K(B) \). Now \( \varphi \) and \( \psi \) are \(*\)-homomorphisms such that \( \psi \circ \varphi \) is approximately unitarily equivalent to \( id_A \) and \( \varphi \circ \psi \) is approximately unitarily equivalent to \( id_B \) by Theorem 5.2. One concludes that \( A \cong B \) by Elliott’s intertwining argument [7].

\[\square\]

6. Conclusion of proofs

**Proof of Theorems 1.1, 2.3.**

The result follows by putting together Lemma 2.4, Proposition 4.1 and Proposition 5.4.
Proof of Theorem 1.2.

(i) ⇒ (ii) Let $A$ be as in (ii). By assumption $A$ satisfies the UCT and it has the same ordered K-theory as the universal UHF algebra $\mathcal{U}$. By the isomorphism theorem of [9] or [5], $A$ is isomorphic to $\mathcal{U}$.

(ii) ⇒ (iii) Assume that $D$ is a nuclear separable RFD C*-algebra and $K_*(D) = 0$. Construct a simple tracially AF algebra $\tilde{D}(\Gamma)$ as in Section 3 where $\Gamma = (\gamma_n)$, $\gamma_n : \tilde{D} \to M_{r(n)+1}(\tilde{D})$, $\gamma_n(a) = \begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix}$ and $(\sigma_n)$ is a sequence of unital finite dimensional representations of $\tilde{D}$ satisfying the conditions (ii$_a$)–(iii$_a$) of Section 3. By construction $\tilde{D}(\Gamma)$ has the same ordered K-theory as the universal UHF algebra $\mathcal{U}$. By (ii) $\tilde{D}(\Gamma)$ is isomorphic to $\mathcal{U}$, hence $D$ has the desired approximation property.

(iii) ⇒ (i) By [13] it suffices to prove that all nuclear separable C*-algebras $A$ with $K_*(A) = 0$ satisfy the UCT. Fix such an $A$ and argue as in the proof of Lemma 2.4 to produce an extension

$$0 \to \sum M_{k(n)} \xrightarrow{\theta} F \to SA \to 0 \tag{11}$$

with $F$ RFD. Since $K_*(A) = 0$, $\theta$ induces an isomorphism of groups $\theta_* : K_*(\sum M_{k(n)}) \to K_*(F)$. The mapping cone C*-algebra $C_\theta = \{(f, x) \in C_0[0,1) \otimes F \oplus \sum M_{k(n)} : f(0) = \theta(x)\}$ is RFD since both $\sum M_{k(n)}$ and $F$ are so. The boundary map $\delta : K_*(\sum M_{k(n)}) \to K_{*-1}(SF)$ associated with the exact sequence

$$0 \to SF \to C_\theta \to \sum M_{k(n)} \to 0 \tag{12}$$

is an isomorphism since it identifies with $\theta_*$ modulo the isomorphism $K_{*-1}(SF) \cong K_*(F)$. This shows that $K_*(C_\theta) = 0$. Set $D = SC_\theta$ and construct a C*-algebra $\tilde{D}(\Gamma)$ as in Section 3 by using a sequence of unital finite dimensional representations $\sigma_n : \tilde{D} \to M_{r(n)}(\mathbb{C}1_{\tilde{D}})$ satisfying the conditions (i$_a$)–(iii$_a$) of Section 3 and the following

(iv$_a$) There is a sequence of finite dimensional C*-algebras $B_n \subset M_{r(n)+1}(\tilde{D})$ such that

$$\lim_{n \to \infty} \text{dist}(\begin{pmatrix} a & 0 \\ 0 & \sigma_n(a) \end{pmatrix}, B_n) = 0, \text{ for all } a \in \tilde{D}.$$ 

The existence of the sequence $(\sigma_n)$ follows from our assumptions in condition (iii) of Theorem 1.2 and the following discussion. Let $\mathcal{F}$, $\epsilon$, $\pi$ and $B$ be as in condition (iii). Since $D$ is RFD, after adding to $\pi$ a finite dimensional representation we may assume that $||\pi(a)|| \geq ||a||(1-\epsilon)$ for all $a \in \mathcal{F}$. By replacing $\pi$ by $\pi \oplus \pi \circ \iota$ where $\iota : D = SC_\theta \to D$ is given be $\iota(a)(t) = a(1-t)$, $t \in [0,1]$, we may arrange that $\pi$ is null-homotopic. If $h > k$ and $\tilde{\pi} : \tilde{D} \to M_h(\mathbb{C} \tilde{D})$ is the unitalization of $\pi$, then $\tilde{\pi}$ is homotopic to the evaluation map $\sigma^0(a + \lambda 1) = \lambda 1_h$, $a \in D$ and

$$\text{dist}(\begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{\pi}(a) \end{pmatrix}, B + \mathbb{C}1) < \epsilon, \text{ for all } \tilde{a} \in \mathcal{F} + \mathbb{C}1.$$
By applying this construction for an increasing sequence of finite sets \( (\mathcal{F}_n) \) whose union is dense in \( D \) and \( \epsilon_n = 1/n \), we obtain a sequence \( \sigma_n = \tilde{\pi}_n \) with the desired properties.

Using the approximation property (iv\( _\sigma \)) it is not hard to see that \( \tilde{D} (\Gamma) \) is isomorphic to an AF algebra (see the discussion below), and so it satisfies the UCT. By Proposition 4.1 (with \( F = C_\theta \)), if \( \tilde{D} (\Gamma) \) satisfies the UCT, then \( D = SC_\theta \) does so. Using (11) and (12) we obtain that \( F \) and hence \( A \) satisfy the UCT by Proposition 2.1. To verify that \( \tilde{D} (\Gamma) \) is AF it suffices to show that for any \( r \geq 1 \), any finite subset \( F \subset M_{H(r)} (\tilde{D}) \) and any \( \epsilon > 0 \) there is \( n \geq r \) such that

\[
\Gamma_{n+1,r} (\mathcal{F}) \subset_\epsilon B
\]

for some finite dimensional C*-subalgebra \( B \) of \( M_{H(n+1)} (\tilde{D}) \); (we write \( \mathcal{G} \subset_\epsilon \mathcal{G}' \) if \( \text{dist}(a, \mathcal{G}') < \epsilon \) for all \( a \in \mathcal{G} \).) By (iv\( _\sigma \)) we find \( n \geq r \) such that

\[
(id_{H(r)} \otimes \gamma_n) (\mathcal{F}) \subset_\epsilon M_{H(r)} (B_n) \subset M_{H(r)h(n+1)} (\tilde{D}).
\]

Note that

\[
\Gamma_{n,r} (\mathcal{F}) \subset u \begin{pmatrix} \mathcal{F} & 0 \\ 0 & M_{H(n) - H(r)} (\mathbb{C}1_{\tilde{D}}) \end{pmatrix} u^*
\]

for some unitary \( u \in M_{H(n)} (\tilde{D}) \). Therefore

\[
\Gamma_{n+1,r} (\mathcal{F}) = (id_{H(n)} \otimes \gamma_n) (\Gamma_{n,r} (\mathcal{F})) \subset_\epsilon B \subset M_{H(n+1)} (\tilde{D})
\]

where \( B \) is isomorphic to \( M_{H(r)} (B_n) \oplus M_{H(n+1) - H(r)h(n+1)} (\mathbb{C}1) \). \( \square \)

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REFERENCES


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