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ON APPROXIMATELY INNER AUTOMORPHISMS OF CERTAIN CROSSED PRODUCT $C^*$-ALGEBRAS

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ABSTRACT. Let $G$ be a compact connected topological group having a dense subgroup isomorphic to $\mathbb{Z}$. Let $C(G) \times_{\alpha} \mathbb{Z}$ be the crossed product $C^*$-algebra of $C(G)$ with $\mathbb{Z}$, where $\mathbb{Z}$ acts on $G$ by rotations. Automorphisms of $C(G) \times_{\alpha} \mathbb{Z}$ leaving invariant the canonical copy of $C(G)$ are shown to be approximately inner iff they act trivially on $K_1(C(G) \times_{\alpha} \mathbb{Z})$.

Let $G$ be a compact abelian topological group. An element $s \in G$ is called a generator if the group algebraically generated by $s$ is dense in $G$. $G$ is called monothetic if it has at least one generator. If in addition $G$ is connected, this is equivalent to saying that the topology of $G$ has a base of cardinality $\leq c$. Moreover if $G$ is second countable then the set of generators is measurable and its Haar measure equals 1. (See [4], Theorems 24.15, 24.27.)

From now on, $G$ is a monothetic compact connected infinite topological group and $s \in G$ is a fixed generator. Let $A = C(G)$ be the $C^*$-algebra of all complex-valued continuous functions on $G$. We consider the action $\alpha: \mathbb{Z} \rightarrow \text{Aut}(A)$ given by

$$(\alpha_k(a))(x) = a(s^{-k}x), \quad \text{for } a \in A, \ x \in G$$

and the corresponding crossed product $C^*$-algebra $A \times_{\alpha} \mathbb{Z}$ (see [5, 8]). Denote by $\text{Aut}_{\alpha}(A \times_{\alpha} \mathbb{Z})$ the closed subgroup

$${\beta} \in \text{Aut}(A \times_{\alpha} \mathbb{Z}) : \beta(A) = A$$

where $\text{Aut}(A \times_{\alpha} \mathbb{Z})$ has the topology of pointwise norm convergence. Note that $\text{Aut}_{\alpha}(A \times_{\alpha} \mathbb{Z}) = \{ \beta \in \text{Aut}(A \times_{\alpha} \mathbb{Z}) : \beta(A) \subset A \}$, since $A$ is a maximal abelian self-adjoint subalgebra in $A \times_{\alpha} \mathbb{Z}$ (see [8], Proposition 4.14). We prove the following.
1. **Theorem.** An automorphism \( \beta \in \text{Aut}_A(A \times \mathbb{Z}) \) is approximately inner iff \( \beta \) induces the identity automorphism of \( K_1(A \times \mathbb{Z}) \).

For \( G \) isomorphic to the one-dimensional torus \( T \), the corresponding result is due to Brenken [2].

The proof uses the description of \( \text{Aut}_A(A \times \mathbb{Z}) \) which follows from more general results [3, Theorem 2.8].

Let \( u \) be the generator of \( \mathbb{Z} \) in \( A \times \mathbb{Z} \), i.e. \( A \times \mathbb{Z} = C^*(A, u) \) with \( uau^* = \alpha_1(a) \) for \( a \in A \). Then each \( \beta \in \text{Aut}_A(A \times \mathbb{Z}) \) is given by a unique triplet \( (b, x, q) \in U(A) \times G \times \{-1, 1\} \) such that \( \beta(u) = bu^q \) and \( \beta(a)(y) = a(xy^q) \) for \( a \in A, \ y \in G \). Here \( U(A) \) denotes the unitary group of \( A \) (with the norm topology) and the correspondence \( \beta \mapsto (b, x, q) \) is a homeomorphism.

It follows by ([3], Lemma 2.4) that such an automorphism is inner iff \( q = 1 \), \( x = s^k \) for some \( k \in \mathbb{Z} \) and \( b \) has the form \( w(\cdot)w^*(s^{-1}\cdot) \) for some \( w \in U(A) \).

In this case \( \beta(t) = wu^{-k}tu^k w^*, \ t \in A \times \mathbb{Z} \). Therefore if \( \beta \in \text{Aut}_A(A \times \mathbb{Z}) \) is given by \( (b, x, q) \) then \( \beta \) is approximately inner provided that \( q = 1 \) and that \( b \) is in the closure of the set

\[
\{ w(\cdot)w^*(s^{-1}\cdot) : w \in U(A) \}.
\]

Indeed, if \( w_n(\cdot)w_n^*(s^{-1}\cdot) \) converges to \( b \) in \( U(A) \) and \( s^{k_n} \) converges to \( x \) in \( G \) then, \( \text{ad}(w_nu^{-k_n}) \) converges to \( \beta \) in \( \text{Aut}_A(A \times \mathbb{Z}) \).

2. **Lemma.** Let \( \beta \in \text{Aut}_A(A \times \mathbb{Z}) \) be given by \( (b, x, q) \). If \( \beta \) induces the identity automorphism of \( K_1(A \times \mathbb{Z}) \) then \( q = 1 \) and \( b \in U_0(A) \) (the connected component of the identity in \( U(A) \)).

**Proof.** Since \( G \) is connected it follows that \( \alpha_1 \) induces the identity automorphism of \( K_1(A) \). Using the Pimsner–Voiculescu exact sequence [6] one sees that the canonical map \( K_1(A) \to K_1(A \times \mathbb{Z}) \) is injective. The obvious map \( \pi^1(G) := [G, T] \to K_1(A) \) is also injective (use for instance the determinant map). Consequently, if \( a \in U(A) \) then \( a \in U_0(A) \) iff \( [a] = 0 \) in \( K_1(A \times \mathbb{Z}) \).

For \( \gamma \in \hat{G} \) (the Pontrjagin dual of \( G \)) we have \( \beta(\gamma) = \gamma(x)\gamma^q \). Therefore \( [\gamma] = [\gamma^q] \) in \( K_1(A \times \mathbb{Z}) \) and by the above remarks \( \gamma \) is homotopic to \( \gamma^q \) as maps \( G \to T \). By a result of Scheffer [7] this is possible only if \( q = 1 \). The equation \( \beta(u) = bu \) implies that \( [\beta(u)] = [b] + [u] \) in \( K_1(A \times \mathbb{Z}) \) hence using the hypothesis on \( \beta \) and the above remarks we find that \( b \in U_0(A) \).

3. **Lemma.** The map \( w \to w(\cdot)w^*(s^{-1}\cdot) \) from \( U(A) \) to \( U_0(A) \) has dense range (compare with Theorem 4 in [2]).

**Proof.** Let \( A_s = \{ a(\cdot) - a(s^{-1}\cdot), \ a \in A \} \). Our first aim is to prove that \( A_s + C.1 \) is a dense (linear, self-adjoint) subspace of \( A \). This is accomplished by showing
that it contains the *-subalgebra of $C(G)$ generated by the characters of $G$ (which is dense in $C(G)$ by the Stone–Weierstrass Theorem). We use the fact that

$$S = \{ \chi(s) : \chi \in \hat{G}\setminus\{1\} \}$$

is a dense subset of $T$ and $1 \notin S$ (see [4], Theorem 25.11). Thus if $\gamma \in \hat{G}\setminus\{1\}$ then $a = (1 - \gamma(s^{-1}))^{-1}\gamma$ is such that $\gamma = a(\cdot) - a(s^{-1}\cdot) \in A_1$.

Any $v \in U_0(A)$ has the form $v = \exp(ih)$ for some $h \in C(G, \mathbb{R})$. By the above discussion we can find $a \in C(G, \mathbb{R})$ and $\lambda \in \mathbb{R}$ such that $a(\cdot) - a(s^{-1}\cdot) + \lambda$ is arbitrarily close to $h$ in norm. Also there is $\gamma \in \hat{G}\setminus\{1\}$ such that $|e^{i\lambda} - \gamma(s)|$ is arbitrarily small. Then for $w = \gamma \exp(i\lambda)$,

$$w(\cdot)w^*(s^{-1}\cdot) = \gamma(s) \cdot \exp i(a(\cdot) - a(s^{-1}\cdot))$$

will approximate $v$ as well as we want.

**Proof of the theorem.** If $\beta \in \text{Aut}_A(A \rtimes \mathbb{Z})$ given by $(b, x, q)$ induces the identity automorphism of $K_1(A \rtimes \mathbb{Z})$ then by Lemma 2, $b \in U_0(A)$ and $q = 1$.

Using Lemma 3 we can find a sequence $w_n \in U(A)$ such that $w_n(\cdot)w_n^*(s^{-1}\cdot)$ converges to $b$ in $U_0(A)$. The discussion before Lemma 2 shows that $\beta$ is approximately inner. The reverse implication is a general fact.

**References**


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**References**

7. **Maps Between Topological Groups That are Homotopic to Homomorphisms**

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