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ON APPROXIMATELY INNER AUTOMORPHISMS OF CERTAIN CROSSED PRODUCT C^*-ALGEBRAS

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ABSTRACT. Let G be a compact connected topological group having a dense subgroup isomorphic to $\mathbf{Z}$. Let $C(G) \rtimes_{\alpha} \mathbf{Z}$ be the crossed product C^*-algebra of $C(G)$ with $\mathbf{Z}$, where $\mathbf{Z}$ acts on G by rotations. Automorphisms of $C(G) \rtimes_{\alpha} \mathbf{Z}$ leaving invariant the canonical copy of $C(G)$ are shown to be approximately inner iff they act trivially on $K_1(C(G) \rtimes_{\alpha} \mathbf{Z})$.

Let G be a compact abelian topological group. An element $s \in G$ is called a generator if the group algebraically generated by s is dense in G. G is called monothetic if it has at least one generator. If in addition G is connected, this is equivalent to saying that the topology of G has a base of cardinality $\leq c$. Moreover if G is second countable then the set of generators is measurable and its Haar measure equals 1. (See [4], Theorems 24.15, 24.27.)

From now on, G is a monothetic compact connected infinite topological group and $s \in G$ is a fixed generator. Let $A = C(G)$ be the C^*-algebra of all complex-valued continuous functions on G. We consider the action $\alpha\colon \mathbf{Z} \to \operatorname{Aut}(A)$ given by

$$(\alpha_k(a))(x) = a(s^{-k}x), \qquad \text{for } a \in A,\ x \in G$$

and the corresponding crossed product C^*-algebra $A \rtimes_{\alpha} \mathbf{Z}$ (see [5, 8]). Denote by $\operatorname{Aut}_A(A \rtimes_{\alpha} \mathbf{Z})$ the closed subgroup

$$\{\beta \in \operatorname{Aut}(A \rtimes_{\alpha} \mathbf{Z})\colon \beta(A) = A\}$$

where $\operatorname{Aut}(A \rtimes_{\alpha} \mathbf{Z})$ has the topology of pointwise norm convergence. Note that $\operatorname{Aut}_A(A \rtimes_{\alpha} \mathbf{Z}) = \{\beta \in \operatorname{Aut}(A \rtimes_{\alpha} \mathbf{Z})\colon \beta(A) \subset A\}$, since A is a maximal abelian self-adjoint subalgebra in $A \rtimes_{\alpha} \mathbf{Z}$ (see [8], Proposition 4.14). We prove the following.

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1. **Theorem.** *An automorphism* $\beta \in \mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$ *is approximately inner iff* β *induces the identity automorphism of* $K_1(A \rtimes_\alpha \mathbf{Z})$.

For G isomorphic to the one-dimensional torus $\mathbf{T}$, the corresponding result is due to Brenken [2].

The proof uses the description of $\mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$ which follows from more general results [3, Theorem 2.8].

Let u be the generator of $\mathbf{Z}$ in $A \rtimes_\alpha \mathbf{Z}$, i.e. $A \rtimes_\alpha \mathbf{Z} = C^*(A, u)$ with $uau^* = \alpha_1(a)$ for $a \in A$. Then each $\beta \in \mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$ is given by a unique triplet $(b, x, q) \in U(A) \times G \times \{-1, 1\}$ such that $\beta(u) = bu^q$ and $\beta(a)(y) = a(xy^q)$ for $a \in A$, $y \in G$. Here $U(A)$ denotes the unitary group of A (with the norm topology) and the correspondence $\beta \leftrightarrow (b, x, q)$ is a homeomorphism. It follows by ([3], Lemma 2.4) that such an automorphism is inner iff $q = 1$, $x = s^k$ for some $k \in \mathbf{Z}$ and b has the form $w(\cdot)w^*(s^{-1}\cdot)$ for some $w \in U(A)$. In this case $\beta(t) = wu^{-k}tu^kw^*$, $t \in A \rtimes_\alpha \mathbf{Z}$. Therefore if $\beta \in \mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$ is given by (b, x, q) then β is approximately inner provided that $q = 1$ and that b is in the closure of the set

$$\{w(\cdot)w^*(s^{-1}\cdot) \colon w \in U(A)\}.$$

Indeed, if $w_n(\cdot)w_n^*(s^{-1}\cdot)$ converges to b in $U(A)$ and s^{k_n} converges to x in G then, $\mathrm{ad}(w_n u^{-k_n})$ converges to β in $\mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$.

2. **Lemma.** *Let* $\beta \in \mathrm{Aut}_A(A \rtimes_\alpha \mathbf{Z})$ *be given by* (b, x, q). *If* β *induces the identity automorphism of* $K_1(A \rtimes_\alpha \mathbf{Z}))$ *then* $q = 1$ *and* $b \in U_0(A)$ (*the connected component of the identity in* $U(A)$).

Proof. Since G is connected it follows that α_1 induces the identity automorphism of $K_1(A)$. Using the Pimsner–Voiculescu exact sequence [6] one sees that the canonical map $K_1(A) \to K_1(A \rtimes_\alpha \mathbf{Z})$ is injective. The obvious map $\pi^1(G) := [G, \mathbf{T}] \to K_1(A)$ is also injective (use for instance the determinant map). Consequently, if $a \in U(A)$ then $a \in U_0(A)$ iff $[a] = 0$ in $K_1(A \rtimes_\alpha \mathbf{Z})$.

For $\gamma \in \widehat{G}$ (the Pontrjagin dual of G) we have $\beta(\gamma) = \gamma(x)\gamma^q$. Therefore $[\gamma] = [\gamma^q]$ in $K_1(A \rtimes_\alpha \mathbf{Z})$ and by the above remarks γ is homotopic to γ^q as maps $G \to \mathbf{T}$. By a result of Scheffer [7] this is possible only if $q = 1$. The equation $\beta(u) = bu$ implies that $[\beta(u)] = [b] + [u]$ in $K_1(A \rtimes_\alpha \mathbf{Z})$ hence using the hypothesis on β and the above remarks we find that $b \in U_0(A)$.

3. **Lemma.** *The map* $w \to w(\cdot)w^*(s^{-1}\cdot)$ *from* $U(A)$ *to* $U_0(A)$ *has dense range* (*compare with Theorem* 4 *in* [2]).

Proof. Let $A_s = \{a(\cdot) - a(s^{-1}\cdot), a \in A\}$. Our first aim is to prove that $A_s + \mathbf{C}.1$ is a dense (linear, self-adjoint) subspace of A. This is accomplished by showing that it contains the $*$-subalgebra of $C(G)$ generated by the characters of G (which is dense in $C(G)$ by the Stone–Weierstrass Theorem). We use the fact that

$$S = \{\chi(s), \chi \in \widehat{G}\backslash\{1\}\}$$

is a dense subset of $\mathbf{T}$ and $1 \notin S$ (see [4], Theorem 25.11). Thus if $\gamma \in \widehat{G}\backslash\{1\}$ then $a = (1-\gamma(s^{-1}))^{-1}\gamma$ is such that $\gamma = a(\cdot) - a(s^{-1}\cdot) \in A_s$.

Any $v \in U_0(A)$ has the form $v = \exp(ih)$ for some $h \in C(G, \mathbf{R})$. By the above discussion we can find $a \in C(G, \mathbf{R})$ and $\lambda \in \mathbf{R}$ such that $a(\cdot)-a(s^{-1}\cdot)+\lambda$ is arbitrarily close to h in norm. Also there is $\gamma \in \widehat{G}\backslash\{1\}$ such that $|e^{i\lambda} - \gamma(s)|$ is arbitrarily small. Then for $w = \gamma\exp(ia)$,

$$w(\cdot)w^*(s^{-1}\cdot) = \gamma(s)\cdot\exp i(a(\cdot) - a(s^{-1}\cdot))$$

will approximate v as well as we want.

Proof of the theorem. If $\beta \in \mathrm{Aut}_A(A \underset{\alpha}{\rtimes} \mathbf{Z})$ given by (b, x, q) induces the identity automorphism of $K_1(A \underset{\alpha}{\rtimes} \mathbf{Z})$ then by Lemma 2, $b \in U_0(A)$ and $q = 1$. Using Lemma 3 we can find a sequence $w_n \in U(A)$ such that $w_n(\cdot)w_n^*(s^{-1}\cdot)$ converges to b in $U_0(A)$. The discussion before Lemma 2 shows that β is approximately inner. The reverse implication is a general fact.

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