

LOCALIZATION C^* -ALGEBRAS AND K -THEORETIC DUALITY

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ABSTRACT. Based on the localization algebras of Yu, and their subsequent analysis by Qiao and Roe, we give a new picture of KK -theory in terms of time-parametrized families of (locally) compact operators that asymptotically commute with appropriate representations.

1. INTRODUCTION

Let A be a unital C^* -algebra, unitaly represented on a Hilbert space H . Let $(q_t)_{t \in [0, \infty)}$ be a continuous family of compact projections on H that asymptotically commutes with A , meaning that $[q_t, a] \rightarrow 0$ as $t \rightarrow \infty$ for all $a \in A$. Note that if p is a projection in A , then the family $t \mapsto pq_t$ of compact operators gets close to being a projection, and is thus close to a projection that is uniquely defined up to homotopy; in particular, there is a well-defined K -theory class $[pq_t] \in K_0(K(H)) = \mathbb{Z}$. It is moreover not difficult to see that this idea can be bootstrapped up to define a homomorphism

$$(1) \quad [q_t] : K_0(A) \rightarrow \mathbb{Z}, \quad [p] \mapsto [pq_t].$$

This suggests using such parametrized families $(q_t)_{t \in [0, \infty)}$ to define elements of K -homology.

Indeed, this has been done when $A = C(X)$ is commutative. In this case, the condition that $[q_t, a] \rightarrow 0$ is (up to an arbitrary approximation) equivalent to the condition that the ‘propagation’ of q_t (in the sense of Roe, [5, Ch. 6]) tends to zero. Motivated by the heat kernel approach to the Atiyah-Singer index theorem, Yu [13] described K -homology in terms of families with asymptotically vanishing propagation, at least when X is a simplicial complex, using his localization algebras. Subsequently, Qiao and Roe [9] gave a new approach to this result of Yu that works for all compact (in fact, all proper) metric spaces.

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In this paper, we present a new picture of Kasparov's KK groups based on asymptotically commuting families. Thanks to the relationship between asymptotically vanishing propagation and asymptotic commutation, our picture can be thought of as an extension of the results of Yu and Qiao-Roe from commutative to general (separable) C^* -algebras, and from K -homology to KK -theory. We think this gives an attractive picture of KK -theory. We also suspect that the ease with which the pairing in line (1) is defined — note that unlike in the case of Paschke duality, there is no dimension shift, and unlike in the case of E -theory, there is no suspension — should be useful for future applications.

We now give precise statements of our main results. For a C^* -algebra B , we denote by $C_u(T, B)$ the C^* -algebra of bounded and uniformly continuous functions from $T = [0, \infty)$ to B . Inspired by work of Yu [13] and Qiao and Roe [9], we define the localization algebra $\mathcal{C}_L(\pi)$ associated to a representation π of a separable C^* -algebra A on a separable Hilbert space to be the C^* -subalgebra of $C_u(T, L(H))$ consisting of all the functions f such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H)) \text{ and } \pi(a)f \in C_u(T, K(H)).$$

Let us recall that a representation π is ample if it is nondegenerate, faithful and $\pi(A) \cap K(H) = \{0\}$. One verifies that the isomorphism class of $\mathcal{C}_L(\pi)$ does not depend on the choice of an ample representation π . In this case, we write $\mathcal{C}_L(A)$ in place of $\mathcal{C}_L(\pi)$ and view A as a C^* -subalgebra of $L(H)$. Note that if A is unital, then

$$\mathcal{C}_L(A) = \{f \in C_u(T, K(H)) : [f, a] \in C_0(T, K(H)), \forall a \in A\}.$$

In this paper we establish canonical isomorphisms $K^i(A) \cong K_i(\mathcal{C}_L(A))$, $i = 0, 1$, between the K -homology of A and the K -theory of the localization algebra $\mathcal{C}_L(A)$. More generally, we use results of Thomsen [11] to show that for separable C^* -algebras A, B and any absorbing representation $\pi : A \rightarrow L(H_B)$ on the standard infinite dimensional countably generated right Hilbert B -module H_B , there are canonical isomorphisms of groups

$$(2) \quad KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1,$$

where the localization C^* -algebra $\mathcal{C}_L(\pi)$ consists of those functions $f \in C_u(T, L(H_B))$ such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H_B)) \text{ and } \pi(a)f \in C_u(T, K(H_B)).$$

The isomorphism in line (2) is defined and proved by combining Paschke duality with a generalization of the techniques used by Roe and Qiao in the

commutative case.

The paper is structured as follows. In Section 2, we discuss absorbing representations and give a version of Voiculescu’s theorem appropriate to localization algebras. In Section 3, we define the various dual algebras and localization algebras that we use, and show that they do not depend on the choice of absorbing representation. In Section 4, we prove the isomorphism in line (2). Finally, in Section 5, we construct maps $K_i(\mathcal{C}_L(\pi)) \rightarrow E_i(A, B)$ and show that they ‘invert’ the isomorphism in line (2) in the sense that the composition $KK_i(A, B) \rightarrow K_i(\mathcal{C}_L(\pi)) \rightarrow E_i(A, B)$ is the canonical natural transformation from KK -theory to E -theory.

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2. ABSORBING REPRESENTATIONS

Let A, B be separable C^* -algebras. If E, F are countably generated right Hilbert B -modules, we denote by $L(E, F)$ the C^* -algebra of bounded B -linear adjointable operators from E to F . The corresponding C^* -algebra of ‘compact’ operators is denoted by $K(E, F)$, [6]. Set $L(E) = L(E, E)$ and $K(E) = K(E, E)$. Recall that H_B is the standard infinite dimensional countably generated right Hilbert B -module.

We shall use the notion of (unitaly) absorbing $*$ -representations $\pi : A \rightarrow L(H_B)$, see [11].

Definition 2.1. (i) Suppose that A is a unital separable C^* -algebra. A unital representation $\pi : A \rightarrow L(H_B)$ is called unitaly absorbing for the pair (A, B) if for any other unital representation $\sigma : A \rightarrow L(E)$, there is an isometry $v \in C_b(\mathbb{N}, L(E, H_B))$ such that $v\sigma(a) - \pi(a)v \in C_0(\mathbb{N}, K(E, H_B))$ for all $a \in A$.

(ii) Suppose that A is a separable C^* -algebra. We denote by \tilde{A} the unitalization of A , with the convention that $\tilde{A} = A$, if A is already unital. A representation $\pi : A \rightarrow L(H_B)$ is called absorbing for the pair (A, B) if its unitalization $\tilde{\pi} : \tilde{A} \rightarrow L(H_B)$ is unitaly absorbing for the pair (\tilde{A}, B) .

Note that in Definition 2.1, if we denote the components of v by v_n , we have $v_n\sigma(a) - \pi(a)v_n \in K(E, H_B)$ and $\lim_{n \rightarrow \infty} \|v_n\sigma(a) - \pi(a)v_n\| = 0$ for all $a \in A$.

Theorem 2.2 (Voiculescu, [12]). *Any ample representation of a separable C^* -algebra on a separable infinite dimensional Hilbert space is absorbing.*

Theorem 2.3 (Kasparov, [6]). *Let A be a unital separable C^* -algebra and let B be a σ -unital C^* -algebra. If either A or B are nuclear, then any unital ample representation $\pi : A \rightarrow L(H) \subset L(H_B)$ is absorbing for the pair (A, B) .*

Theorem 2.4 (Thomsen, [11]). *For any separable C^* -algebras A and B there exist absorbing representations $\pi : A \rightarrow L(H_B)$.*

Given two $*$ -representations $\pi_i : A \rightarrow L(E_i)$ we write that $\pi_1 \underset{v}{\preceq} \pi_2$ if there is an isometry $v \in C_u(T, L(E_1, E_2))$ such that

$$v\pi_1(a) - \pi_2(a)v \in C_0(T, K(E_1, E_2)).$$

If in addition $v \in C_u(T, L(E_1, E_2))$ is a unitary with the same property, then we write $\pi_1 \underset{v}{\approx} \pi_2$.

Let $w^\infty : E_1^\infty \rightarrow E_1 \oplus E_1^\infty$ be the unitary defined by $w^\infty(h_0, h_1, h_2, \dots) = h_0 \oplus (h_1, h_2, \dots)$.

Lemma 2.5 (Lemma 2.16, [3]). *Let $\pi_i : A \rightarrow L(E_i)$ be two representations and let $v \in L(E_1^\infty, E_2)$ be an isometry such that $v\pi_1^\infty(a) - \pi_2(a)v \in K(E_1^\infty, E_2)$ for all $a \in A$. Then $u = (1_{E_1} \oplus v)w^\infty v^* + (1_{E_2} - vv^*) \in L(E_2, E_1 \oplus E_2)$ is a unitary operator such that $\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^* \in K(E_1 \oplus E_2)$ for all $a \in A$ and moreover*

$$\|\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^*\| \leq 6\|v\pi_1^\infty(a) - \pi_2(a)v\| + 4\|v\pi_1^\infty(a^*) - \pi_2(a^*)v\|.$$

Proposition 2.6. *Let A, B be separable C^* -algebras and let $\pi_i : A \rightarrow L(E_i)$, $i = 1, 2$ be two representations where $E_i \cong H_B$. If π_2 is absorbing, then $\pi_1 \underset{v}{\preceq} \pi_2$ for some isometry $v \in C_u(T, L(E_1, E_2))$. If both π_1 and π_2 are absorbing, then $\pi_1 \underset{u}{\approx} \pi_2$ for some unitary $u \in C_u(T, L(E_1, E_2))$.*

Proof. Since π_2 absorbs π_2^∞ there is an isometry $u = (u_n)_n \in C_b(\mathbb{N}, L(E_2^\infty, E_2))$ such that $u\pi_2^\infty(a) - \pi_2(a)u \in C_0(\mathbb{N}, K(E_2^\infty, E_2))$ for all $a \in A$. Since π_2 absorbs π_1 , there is a sequence of isometries $w_n \in L(E_1, E_2^\infty)$ with mutually orthogonal ranges such that $w_n\pi_1(a) - \pi_2^\infty(a)w_n \in K(E_1, E_2^\infty)$ and $\lim_{n \rightarrow \infty} \|w_n\pi_1(a) - \pi_2^\infty(a)w_n\| = 0$ for all $a \in A$. Then $v_n = u_n w_n \in L(E_1, E_2)$ is a sequence of isometries with orthogonal ranges such that the corresponding isometry $v \in C_b(\mathbb{N}, L(E_1, E_2))$ satisfies $v\pi_1(a) - \pi_2(a)v \in C_0(\mathbb{N}, K(E_1, E_2))$ for all $a \in A$. This follows from the identity

$$u_n w_n \pi_1(a) - \pi_2(a) u_n w_n = u_n (w_n \pi_1(a) - \pi_2^\infty(a) w_n) + (u_n \pi_2^\infty(a) - \pi_2(a) u_n) w_n.$$

Since $v_n^*v_m = 0$ for $n \neq m$, one observes that $\mathbf{v}(n+s) = (1-s)^{1/2}v_n + s^{1/2}v_{n+1}$, $0 \leq s \leq 1$, extends v to a uniformly continuous isometry $\mathbf{v} \in C_u(T, L(E_1, E_2))$ that satisfies $\pi_1 \underset{\mathbf{v}}{\preceq} \pi_2$.

For the second part of the statement, we note that by the first part $\pi_1^\infty \underset{v}{\preceq} \pi_2$. Thus, $v\pi_1^\infty(a) - \pi_2(a)v \in C_0(T, K(E_1^\infty, E_2))$, for all $a \in A$ where $v = (v_t)_{t \in T}$ is a uniformly continuous isometry with $v_t \in L(E_1^\infty, E_2)$. It follows by Lemma 2.5 that

$$u_t = (1_{E_1} \oplus v_t)w^\infty v_t^* + (1_{E_2} - v_t v_t^*)$$

is a uniformly continuous unitary such that $\pi_1 \oplus \pi_2 \underset{u}{\approx} \pi_2$. By symmetry we have that $\pi_1 \oplus \pi_2 \underset{u}{\approx} \pi_1$ and hence $\pi_1 \underset{u}{\approx} \pi_2$. \square

As a corollary we have the following strengthened variation of Voiculescu's theorem [12]. This result appears in [2] as Theorem 3.11, except that the uniform continuity of the isometry v and the unitary u were not addressed explicitly in the statement.

Theorem 2.7. *Let A, B be separable C^* -algebras and let $\pi_i : A \rightarrow L(E_i)$, $i = 1, 2$ be two representations where $E_i \cong H_B$. If π_2 is absorbing, then $\pi_1 \underset{v}{\preceq} \pi_2$ for some isometry $v \in C_u(T, L(E_1, E_2))$. If both π_1 and π_2 are absorbing, then $\pi_1 \underset{u}{\approx} \pi_2$ for some unitary $u \in C_u(T, L(E_1, E_2))$.*

3. DUAL ALGEBRAS

Let A, B be separable C^* -algebras and let $\pi : A \rightarrow L(H_B)$ be a $*$ -representation. Consider the following pairs of C^* -algebras and ideals:

$$\mathcal{D}(\pi) = \{b \in L(H_B) : [b, \pi(a)] \in K(H_B), \forall a \in A\},$$

$$\mathcal{C}(\pi) = \{b \in L(H_B) : \pi(a)b \in K(H_B), \forall a \in A\},$$

$$\mathcal{D}_T(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_u(T, K(H_B)), \forall a \in A\} \cong C_u(T, \mathcal{D}(\pi)),$$

$$\mathcal{C}_T(\pi) = \{f \in C_u(T, L(H_B)) : \pi(a)f \in C_u(T, K(H_B)), \forall a \in A\} \cong C_u(T, \mathcal{C}(\pi)),$$

$$\mathcal{D}_T^0(\pi) = \{f \in \mathcal{D}_T(\pi) : f(0) = 0\},$$

$$\mathcal{C}_T^0(\pi) = \{f \in \mathcal{C}_T(\pi) : f(0) = 0\},$$

$$\mathcal{D}_L(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_0(T, K(H_B)), \forall a \in A\},$$

$$\mathcal{C}_L(\pi) = \{f \in \mathcal{D}_L(\pi) : \pi(a)f \in C_u(T, K(H_B)), \forall a \in A\}.$$

To simplify some of the statements it is useful to introduce the following notation: $A_1(\pi) = \mathcal{D}_T(\pi)$, $A_2(\pi) = \mathcal{D}_L(\pi)$, $A_3(\pi) = \mathcal{D}_T^0(\pi)$, $A_4(\pi) = \mathcal{C}_T^0(\pi)$, $A_5(\pi) = \mathcal{D}_L(\pi)$ and $A_6(\pi) = \mathcal{C}_L(\pi)$. We are going to see that the isomorphism classes of these C^* -algebras are independent of π , provided that π

is an absorbing representation. We follow the presentation from [5, Section 5.2] where analogous properties of $\mathcal{D}(\pi)$ and $\mathcal{C}(\pi)$ are established, except that we need to employ a strengthened version of Voiculescu's theorem, see Theorem 2.7 and Proposition 2.6 below.

Let $\pi_1, \pi_2 : A \rightarrow L(H_B)$ be two representations.

Lemma 3.1. *If $\pi_1 \underset{v}{\preceq} \pi_2$, then the equation $\Phi_v(f) = vfv^*$ defines a $*$ -homomorphism $\Phi_v : \mathcal{D}_T(\pi_1) \rightarrow \mathcal{D}_T(\pi_2)$ with the property that $\Phi_v(A_j(\pi_1)) \subset A_j(\pi_2)$ for all $1 \leq j \leq 6$.*

Proof. This follows from the identities:

$$\begin{aligned} [vfv^*, \pi_2(a)] &= v[f, \pi_1(a)]v^* + (v\pi_1(a) - \pi_2(a)v)fv^* - vf(v\pi_1(a^*) - \pi_2(a^*)v)^*, \\ \pi_2(a)vfv^* &= v\pi_1(a)fv^* - (v\pi_1(a) - \pi_2(a)v)fv^*. \quad \square \end{aligned}$$

Corollary 3.2. *Let $\pi_1, \pi_2 : A \rightarrow L(H_B)$ be two absorbing representations. Then $A_j(\pi_1) \cong A_j(\pi_2)$ for all $1 \leq j \leq 6$.*

Proof. Proposition 2.6 yields a unitary $v \in C_u(T, L(H_B))$ such that $\pi_1 \underset{v}{\approx} \pi_2$. The corresponding maps $\Phi_v : A_j(\pi_1) \rightarrow A_j(\pi_2)$ are isomorphisms. \square

Lemma 3.3. *Let $\pi_1, \pi_2 : A \rightarrow L(H_B)$ be two representations of A and suppose that v_1, v_2 are two isometries such that $\pi_1 \underset{v_i}{\preceq} \pi_2$, $i = 1, 2$. Then $(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \rightarrow K_*(A_j(\pi_2))$ for all $1 \leq j \leq 6$.*

Proof. The unitary $u = \begin{pmatrix} 1 - v_1v_1^* & v_1v_2^* \\ v_2v_1^* & 1 - v_2v_2^* \end{pmatrix} \in M_2(\mathcal{D}_L(\pi_2))$ conjugates $\begin{pmatrix} \Phi_{v_1} & 0 \\ 0 & 0 \end{pmatrix}$ over $\begin{pmatrix} 0 & 0 \\ 0 & \Phi_{v_1} \end{pmatrix}$. It follows that $(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(\mathcal{D}_T(\pi_1)) \rightarrow K_*(\mathcal{D}_T(\pi_2))$. Similarly, one verifies that the equality $(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \rightarrow K_*(A_j(\pi_2))$ holds for all $1 \leq j \leq 6$. \square

Denote by π^∞ the direct sum $\pi^\infty = \bigoplus_{n=1}^\infty \pi : A \rightarrow L(H_B^\infty) = L(\bigoplus_{n=1}^\infty H_B)$.

Corollary 3.4. *If $\pi : A \rightarrow L(H_B)$ is an absorbing representation, then the inclusion $\mathcal{D}_T(\pi) \rightarrow \mathcal{D}_T(\pi^\infty)$, $f \mapsto (f, 0, 0, \dots)$ induces isomorphisms on K -theory: $K_*(A_j(\pi)) \rightarrow K_*(A_j(\pi^\infty))$, for all $1 \leq j \leq 6$.*

Proof. We have $\pi \underset{v}{\preceq} \pi^\infty$, where $v \in C_u(T, L(H_B, H_B^\infty))$ is the constant isometry defined by $v(t)(h) = (h, 0, 0, \dots)$ for any $t \in T$ and $h \in H_B$. The inclusion map from the statement coincides with Φ_v . On the other hand $\pi \underset{u}{\approx} \pi^\infty$ since π is absorbing and hence Φ_u is an isomorphism. We conclude the proof by noting that $(\Phi_v)_* = (\Phi_u)_*$ by Lemma 3.3. \square

4. A DUALITY ISOMORPHISM

Let A and B be separable C^* -algebras. We are going to show that when we fix an absorbing representation $\pi: A \rightarrow L(H_B)$ (the existence of such an absorbing representation is guaranteed by Theorem 2.4), the K -theory of $\mathcal{C}_L(\pi)$ is canonically isomorphic to the KK -theory of the pair (A, B) .

We start with a technical lemma that will be used several times later.

Lemma 4.1. *For any separable C^* -algebra $D \subset C_u(T, L(H_B))$ there is a positive contraction $x \in C_u(T, K(H_B))$ such that:*

- (a) $[x, d] \in C_0(T, K(H_B))$ for all $d \in D$, and
- (b) $(1 - x)d \in C_0(T, K(H_B))$ for all $d \in D \cap C_u(T, K(H_B))$.

Proof. Our arguments will in fact show that the statement holds true in the more general situation where $L(H_B)$ is replaced by a C^* -algebra L and $K(H_B)$ is replaced by a two-sided closed ideal I of L . Let \dot{D} denote the C^* -subalgebra of L generated by all images $d(t)$ as d ranges over D and t over T . This is separable, and contains $\dot{C} = \dot{D} \cap I$ as an ideal. Let $(x_n)_n$ be a positive contractive approximate unit for \dot{C} which is quasi-central in \dot{D} . Choose countable dense subsets $(d_k)_{k=1}^\infty$ and $(c_k)_{k=1}^\infty$ of D and $D \cap C_u(T, I)$ respectively. As for each n , the subsets $\bigcup_{k=1}^n \{d_k(t) : t \in [0, n+1]\}$ and $\bigcup_{k=1}^n \{c_k(t) : t \in [0, n+1]\}$ of \dot{D} and \dot{C} respectively are compact, we may assume on passing to a subsequence of (x_n) that

- (i) $\|[d_k(t), x_n]\| < \frac{1}{n+1}$ for all $1 \leq k \leq n$ and all $t \in [0, n+1]$, and
- (ii) $\|(1 - x_n)c_k(t)\| < \frac{1}{n+1}$ for all $1 \leq k \leq n$ and all $t \in [0, n+1]$.

For $t \in [n, n+1)$, write $s = t - n$ and set $x(t) = (1 - s)x_n + sx_{n+1}$; note that the function $x : t \mapsto x(t)$ is uniformly continuous. Then from (i) and (ii) above we have

- (i) $\|[d_k(t), x(t)]\| < \frac{1}{n+1}$ for all $1 \leq k \leq n$ and all $t \in [n, n+1)$, and
- (ii) $\|(1 - x(t))c_k(t)\| < \frac{1}{n+1}$ for all $1 \leq k \leq n$ and all $t \in [n, n+1)$.

This implies that x has the right properties. \square

We have obvious inclusions $\mathcal{D}_L(\pi) \subset \mathcal{D}_T(\pi)$ and $\mathcal{C}_L(\pi) \subset \mathcal{C}_T(\pi)$ which induce a $*$ -homomorphism

$$\eta : \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \rightarrow \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi).$$

Proposition 4.2. *For any separable C^* -algebras A, B and any representation $\pi : A \rightarrow L(H_B)$, the map η is a $*$ -isomorphism.*

Proof. It is clear from the definitions that $\mathcal{C}_L(\pi) = \mathcal{D}_L(\pi) \cap \mathcal{C}_T(\pi)$ and hence η is injective. It remains to prove that η is surjective. It suffices to show

that for any $f \in \mathcal{D}_T(\pi)$ there is $\tilde{f} \in \mathcal{D}_L(\pi)$ such that $\tilde{f} - f \in \mathcal{C}_T(\pi)$. Let $f \in \mathcal{D}_T(\pi)$ be given.

Let D be the C^* -subalgebra of $C_u(T, L(H_B))$ generated by $\pi(A)$ (embedded as constant functions) and f , and let x be as in Lemma 4.1. With this choice of x (that depends on f) we define $\tilde{f} = (1 - x)f$. Note that $\tilde{f} = f - xf \in \mathcal{D}_T(\pi)$ since $f, x \in \mathcal{D}_T(\pi)$, and $\tilde{f} - f = -xf \in C_u(T, K(H_B))$ since $x \in C_u(T, K(H_B))$. In particular it follows that $\tilde{f} - f \in \mathcal{C}_T(\pi)$.

It remains to verify that $\tilde{f} \in \mathcal{D}_L(\pi)$. This follows as for any $a \in A$,

$$[\tilde{f}, \pi(a)] = [(1 - x)f, \pi(a)] = [\pi(a), x]f + (1 - x)[f, \pi(a)]. \quad \square$$

Note now that there is a split exact sequence:

$$0 \longrightarrow \mathcal{D}_T^0(\pi)/\mathcal{C}_T^0(\pi) \longrightarrow \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi) \longrightarrow \mathcal{D}(\pi)/\mathcal{C}(\pi) \longrightarrow 0.$$

An adaptation of the arguments from the paper [9] of Qiao and Roe gives:

Proposition 4.3. *Let A, B be separable C^* -algebras and let $\pi : A \rightarrow L(H_B)$ be an absorbing representation. Then*

- (a) $K_*(\mathcal{D}_L(\pi)) = 0$ and hence the boundary map $\partial : K_*(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \rightarrow K_{*+1}(\mathcal{C}_L(\pi))$ is an isomorphism.
- (b) The evaluation map at $t = 0$ induces an isomorphism. $e_* : K_*(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \rightarrow K_*(\mathcal{D}(\pi)/\mathcal{C}(\pi))$.

Proof. Fix an ample representation π of A . One verifies that if $f \in \mathcal{D}_L(\pi)$, then the formula

$$F(t) := (f(t), f(t+1), \dots, f(t+n), \dots)$$

defines an element $F \in \mathcal{D}_L(\pi^\infty)$. Indeed,

$$[F(t), \pi(a)] = ([f(t), \pi(a)], [f(t+1), \pi(a)], \dots, [f(t+n), \pi(a)], \dots)$$

and each entry belongs to $C_0(T, K(H_B))$ and is bounded by $\|[f, \pi(a)]\|$. This shows that $F \in C_u(T, K(H_B^\infty))$. Since $[f, \pi(a)] \in C_0(T, K(H_B))$, it follows immediately that in fact $[F, \pi(a)] \in C_0(T, K(H_B^\infty))$.

With these remarks, the proof of (a) goes just like the proof of Proposition 3.5 from [9]. Indeed, define $*$ -homomorphisms $\alpha_i : \mathcal{D}_L(\pi) \rightarrow \mathcal{D}_L(\pi^\infty)$, $i = 1, 2, 3, 4$ by

$$\begin{aligned} \alpha_1(f) &= (f(t), 0, 0, \dots), \\ \alpha_2(f) &= (0, f(t+1), f(t+2), \dots), \\ \alpha_3(f) &= (0, f(t), f(t+1), \dots) \\ \alpha_4(f) &= (f(t), f(t+1), f(t+2), \dots). \end{aligned}$$

It is clear that $\alpha_1 + \alpha_2 = \alpha_4$. The isometry $v \in L(H_B^\infty)$ defined by $v(h_0, h_1, h_2, \dots) = (0, h_0, h_1, h_2, \dots)$ commutes with $\pi^\infty(A)$ and hence $v \in \mathcal{D}_L(\pi^\infty)$. Moreover $\alpha_4(a) = v\alpha_3(a)v^*$ and hence $(\alpha_4)_* = (\alpha_3)_*$ by [5, Lemma 4.6.2]. Using uniform continuity, one shows that α_3 is homotopic to α_2 , via the homotopy $f(t) \mapsto (0, f(t+s), f(t+s+1), \dots)$, $0 \leq s \leq 1$. We deduce that

$$(\alpha_1)_* + (\alpha_2)_* = (\alpha_1 + \alpha_2)_* = (\alpha_4)_* = (\alpha_3)_* = (\alpha_2)_*$$

and hence $(\alpha_1)_* = 0$. This concludes the proof of (a), since $(\alpha_1)_*$ is an isomorphism by Corollary 3.4.

(b) One follows the proof of Proposition 3.6 from [9]. Any $f \in \mathcal{D}_T^0(\pi)$ can be extended by 0 to an element of $C_u(\mathbb{R}, L(H_B))$. With this convention, define four maps $\beta_i : \mathcal{D}_T^0(\pi) \rightarrow \mathcal{D}_T^0(\pi^\infty)$, $i = 1, 2, 3, 4$ by

$$\beta_1(f) = (f(t), 0, 0, \dots),$$

$$\beta_2(f) = (0, f(t-1), f(t-2), \dots),$$

$$\beta_3(f) = (0, f(t), f(t-1), \dots)$$

$$\beta_4(f) = (f(t), f(t-1), f(t-2), \dots).$$

This definition requires that one verifies that if $f \in \mathcal{D}_T^0(\pi)$, then

$$F'(t) := (f(t), f(t-1), \dots, f(t-n), \dots)$$

defines an element of $\mathcal{D}_T^0(\pi^\infty)$. This is clearly the case, since if f is uniformly continuous, then so is F' and moreover, just as argued in [9], for each t in a fixed bounded interval only finitely many components of $F'(t)$ are non-zero, and hence $[F'(t), \pi^\infty(a)] \in K(H_B^\infty)$ if $[f(t), \pi(a)] \in K(H_B)$ for all $t \in T$. Similar arguments show that if $f \in \mathcal{C}_T^0(\pi)$ then $F' \in \mathcal{C}_T^0(\pi^\infty)$. Note that $(\beta_4)_* = (\beta_3)_*$ since $\beta_4(a) = v\beta_3(a)v^*$ where $v \in \mathcal{D}_T(\pi^\infty)$ is the same isometry as in part (a). Using uniform continuity, one observes that β_3 is homotopic to β_2 , via the homotopy $f(t) \mapsto (0, f(t-s), f(t-s-1), \dots)$, $0 \leq s \leq 1$. We deduce that

$$(\beta_1)_* + (\beta_2)_* = (\beta_1 + \beta_2)_* = (\beta_4)_* = (\beta_3)_* = (\beta_2)_*$$

and hence $(\beta_1)_* = 0$. This concludes the proof, since $(\beta_1)_*$ is an isomorphism by Corollary 3.4. \square

Theorem 4.4. *Let A, B be separable C^* -algebras and let $\pi : A \rightarrow L(H_B)$ be an absorbing representation. There are canonical isomorphisms of groups*

$$\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} KK_i(A, B) & \xrightarrow{P} & K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi)) & \xrightarrow{\iota_*} & K_{i+1}(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \\ & & & & \downarrow \eta_*^{-1} \\ & & K_i(\mathcal{C}_L(\pi)) & \xleftarrow{\partial} & K_{i+1}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \end{array}$$

where P is the Paschke duality isomorphism, see [8], [10, Remarque 2.8], [11, Theorem 3.2], and ι is the canonical inclusion. The maps ∂ and $\iota_* = e_*^{-1}$ are isomorphisms by Proposition 4.3 and η_* is an isomorphism by Proposition 4.2. \square

As a corollary we obtain the following duality theorem, mentioned in the introduction. Recall from the introduction that $\mathcal{C}_L(A)$ stands for $\mathcal{C}_L(\pi)$, where π is ample (and thus absorbing, by Theorem 2.2), and A is identified with $\pi(A)$.

Theorem 4.5. *For any separable C^* -algebra A there are canonical isomorphisms of groups $K^i(A) \cong K_i(\mathcal{C}_L(A))$, $i = 0, 1$.* \square

5. AN INVERSE MAP

Let $\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi))$ be the isomorphism of Theorem 4.4. Recall that $K(H_B) \cong B \otimes K(H)$. Consider the $*$ -homomorphism

$$\Phi : \mathcal{D}_L(\pi) \otimes_{\max} A \rightarrow \frac{C_u(T, L(H_B))}{C_0(T, K(H_B))}$$

defined by $\Phi(f \otimes a) = f\pi(a)$ and its restriction to $\mathcal{C}_L(\pi) \otimes_{\max} A$

$$\varphi : \mathcal{C}_L(\pi) \otimes_{\max} A \rightarrow \frac{C_u(T, K(H_B))}{C_0(T, K(H_B))}.$$

We want φ to define a class in E -theory that we can take products with, but have to be a little careful due to the non-separability of the C^* -algebra $\mathcal{C}_L(\pi) \otimes_{\max} A$. Just as in the case of the KK -groups [10], if C is any C^* -algebra and B is a non-separable C^* -algebra one defines $E_{sep}(B, C) = \varprojlim_{B_1} E(B_1, C)$, with $B_1 \subset B$ and B_1 separable. Moreover if D is separable, then $E(D, B) = \varinjlim_{B_1} E(D, B_1)$, with $B_1 \subset B$ and B_1 separable. With these adjustments, one has a well-defined product

$$E(D, B) \times E_{sep}(B, C) \rightarrow E(D, C).$$

Moreover, it is clear that $[[\varphi]]$ defines an element of the group $E_{sep}(\mathcal{C}_L(\pi) \otimes_{\max} A, B)$.

Recall the isomorphism $K_i(\mathcal{C}_L(\pi)) \cong E_i(\mathbb{C}, \mathcal{C}_L(\pi))$. We use the product

$$E_i(\mathbb{C}, \mathcal{C}_L(\pi)) \times E_{sep}(\mathcal{C}_L(\pi) \otimes_{\max} A, B) \rightarrow E_i(A, B)$$

to define a map $\beta : K_i(\mathcal{C}_L(\pi)) \rightarrow E_i(A, B)$ by $\beta(z) = [[\varphi]] \circ (z \otimes \text{id}_A)$.

The map β is an inverse of α in the following sense.

Theorem 5.1. *The composition $\beta \circ \alpha$ coincides with the natural map $KK_i(A, B) \rightarrow E_i(A, B)$, $i = 0, 1$.*

Proof. We will give the proof for the odd case $i = 1$ and leave the even case for the reader. Recall that the E -theory group $E_1(A, B)$ of Connes and Higson [1] is isomorphic to $[[SA, K(H_B)]]$ by a desuspension result from [4].

For two continuous functions $f, g : T \rightarrow L(H_B)$ we will write $f(s) \sim g(s)$ (or $f(t) \sim g(t)$) if $f - g \in C_0(T, K(H_B))$. Let $\{\varphi_s : \mathcal{C}_L(\pi) \otimes_{\max} A \rightarrow K(H_B)\}_{s \in T}$ be an asymptotic homomorphism representing φ . More precisely take φ to be a set-theoretic lifting of φ . This means that $\varphi_s(f \otimes a) \sim f(s)\pi(a)$.

The composition $\beta \circ \alpha : KK_1(A, B) \rightarrow E_1(A, B)$ is computed as follows. Let $z \in KK_1(A, B)$ be represented by a self-adjoint element $e \in \mathcal{D}(\pi) \subset \mathcal{D}_T(\pi)$ whose image in $\mathcal{D}(\pi)/\mathcal{C}(\pi)$ is an idempotent \dot{e} . We identify $\mathcal{D}(\pi)$ with the C^* -subalgebra of constant functions in $\mathcal{D}_T(\pi)$. Choose an element $x \in C_u(T, K(H_B))$ as in Lemma 4.1 with respect to the (separable) C^* -subalgebra D of $C_u(T, L(H_B))$ generated by $\pi(A)$, e , and $K(H_B)$. Therefore both $[x, \pi(a)]$ and $(1-x)[e, \pi(a)]$ belong to $C_0(T, K(H_B))$ for all $a \in A$, and moreover $(1-x)e \in \mathcal{D}_L(\pi)$ as

$$[(1-x)e, \pi(a)] = [1-x, \pi(a)]e + (1-x)[e, \pi(a)] \in C_0(T, K(H_B))$$

for all $a \in A$. Let $e_L = (1-x)e$ and let \dot{e}_L be its image in $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$. Under the isomorphism $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \cong \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)$ we see that \dot{e}_L is just the image of $e \in \mathcal{D}_T(\pi)$ in the quotient, which is an idempotent since \dot{e} is.

Define a $*$ -homomorphism $\ell : \mathbb{C} \rightarrow \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$ by $\ell(1) = \dot{e}_L$ and set $S = C_0(0, 1)$. Then $(\beta \circ \alpha)(z)$ is represented by the composition of the asymptotic homomorphisms from the following diagram.

(3)

$$S \otimes \mathbb{C} \otimes A \xrightarrow{1 \otimes \ell \otimes 1} S \otimes \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \otimes A \xrightarrow{\delta_t \otimes 1} \mathcal{C}_L(\pi) \otimes A \xrightarrow{\varphi_s} K(H_B),$$

where the map labelled δ_t is defined by taking the product with a canonical element δ of $E_{1, \text{sep}}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi), \mathcal{C}_L(\pi))$ associated to the extension

$$0 \rightarrow \mathcal{C}_L(\pi) \rightarrow \mathcal{D}_L(\pi) \rightarrow \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \rightarrow 0.$$

Fixing a separable C^* -subalgebra \dot{M} of $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$, the image of δ in $E_1(\dot{M}, \mathcal{C}_L(\pi))$ is defined as follows. Choose a separable C^* -subalgebra M of $\mathcal{D}_L(\pi)$ that surjects onto \dot{M} , and for each $\dot{m} \in \dot{M}$ choose a lift $m \in M$. Let $(v_t)_{t \in T}$ be a positive, contractive, and continuous approximate unit for

$H \cap \mathcal{C}_L(\pi)$ which is quasentral in M . Then for $g \in S = C_0(0, 1)$, $\delta_t(g \otimes \dot{m})$ satisfies

$$\delta_t(g \otimes \dot{m}) \sim g(v_t)m$$

(the choices of (v_t) and the various lifts do not matter up to homotopy). In our case, to compute the composition we need, let M be a separable C^* -subalgebra of $\mathcal{D}_L(\pi)$ containing e and x , and let (v_t) be an approximate unit for $M \cap \mathcal{C}_L(\pi)$ that is quasentral in M .

On the level of elements, we can now concretely describe the composition in line (3) as follows. If $g \in S = C_0(0, 1)$ and $a \in A$, then under the asymptotic morphism $\{\mu_t : SA \rightarrow K(H_B)\}_t$ defined by diagram (3), elementary tensors $g \otimes a$ are mapped as follows

$$(4) \quad g \otimes a \mapsto g \otimes \dot{e}_L \otimes a \xrightarrow{\delta_t} g(v_t)(1-x)e \otimes a \xrightarrow{\varphi_{s(t)}} g(v_t(s(t)))(1-x(s(t)))e\pi(a)$$

for any positive map $t \mapsto s(t)$ which increases to ∞ sufficiently fast. Since the map $t \mapsto x(t)$ is an approximate unit of $K(H_B)$, $(1-x)y \in C_0(T, K(H_B))$ for all $y \in K(H_B)$. In particular it follows that $(1-x(s(t)))e[e, \pi(a)] \sim 0$ since $[e, \pi(a)] \in K(H_B)$. Since $e\pi(a) = e\pi(a)e + e[e, \pi(a)]$, it follows from (4) that

$$(5) \quad \mu_t(g \otimes a) \sim g(v_t(s(t)))(1-x(s(t)))e\pi(a)e.$$

On the other hand, the natural map $KK_1(A, B) \rightarrow E_1(A, B)$, maps z to $[[\gamma_t]]$, where $\{\gamma_t : S \otimes A \rightarrow K(H_B)\}_t$ is described in [1] as follows. Consider the extension:

$$0 \rightarrow K(H_B) \rightarrow e\pi(A)e + K(H_B) \rightarrow A \rightarrow 0.$$

Let $(u_t)_{t \in T}$ be a contractive, positive, and continuous approximate unit of $K(H_B)$ which is quasentral in $e\pi(A)e + K(H_B)$. Then

$$\gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e.$$

Applying Lemma 4.1 (this time with D the C^* -subalgebra of $C_u(T, L(H_B))$ generated by e , $\pi(A)$, $K(H_B)$, and $t \mapsto x(s(t))$), we can choose $(u_t)_t$ such that $\lim_{t \rightarrow \infty} (1-u_t)x(s(t)) = 0$. Since $C_0(0, 1)$ is generated by the function $f(\theta) = \theta(1-\theta)$, it follows that $\lim_{t \rightarrow \infty} g(u_t)x(s(t)) = 0$ for all $g \in C_0(0, 1)$.

Our goal now is to verify that $(\mu_t)_t$ is homotopic to $(\gamma_t)_t$. Due to the choice of $(u_t)_t$ and the comments above, we have that

$$(6) \quad \gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e \sim g(u_t)(1-x(s(t)))e\pi(a)e,$$

for all $a \in A$ and $g \in C_0(0, 1)$. Finally, define $w_t^{(r)} = (1 - r)v_t(s(t)) + ru_t$, $0 \leq r \leq 1$. As

$$\left[g(w_t^{(r)}), (1 - x(s(t)))e\pi(a)e \right] \rightarrow 0 \text{ as } t \rightarrow \infty$$

for all $r \in [0, 1]$ and $a \in A$, there is an asymptotic morphism $H_t : SA \rightarrow C[0, 1] \otimes K(H_B)$ defined by the condition

$$H_t^{(r)}(g \otimes a) \sim g(w_t^{(r)})(1 - x(s(t)))e\pi(a)e.$$

This gives the desired homotopy joining $(\mu_t)_t$ with $(\gamma_t)_t$. \square

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