NONNUCLEAR SUBALGEBRAS OF AF ALGEBRAS

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Abstract. We show that any non-type I separable unital AF algebra B can be modeled from inside by a nonnuclear C*-algebra and from outside by a nonexact C*-algebra. More precisely there exist unital separable quasidiagonal C*-algebras $A \subset B \subset C$ of real rank zero, stable rank one, such that A is nonnuclear, C is nonexact, and both A and C are asymptotically homotopy equivalent to B. In particular A, B and C have the same ordered K-theory groups, hence isomorphic ideal lattices, and, A and B have (affinely) homeomorphic trace spaces.

1. Introduction. The type I C*-algebras are nuclear [Ta$_1$]. Every C*-subalgebra of a type I C*-algebra is type I hence nuclear. On the other hand Blackadar showed that any non-type I C*-algebra contains a nonnuclear C*-subalgebra [B$_1$]. Thus a C*-algebra is type I if and only if all its C*-subalgebras are nuclear.

The main result of this paper shows that every non-type I unital AF algebra B contains a nonnuclear unital C*-subalgebra A and is contained in a unital nonexact C*-algebra C such that both A and C "model" B (see Proposition 9 and Theorem 11).

In [Ha], Hadwin raises the question of whether every strongly quasidiagonal C*-algebra is nuclear. In [R], Rosenberg shows that if the reduced C*-algebra of a discrete countable group is quasidiagonal then G is amenable, and asks whether the strongly quasidiagonal C*-algebras are exact.

The examples of nonnuclear quasidiagonal C*-subalgebras of non-type I AF algebras exhibited in [B$_1$] are not strongly quasidiagonal since they have a non-quasidiagonal quotient, the Choi algebra. They are also nonsimple, as noticed in [Po]. Let $\mathcal{C}$ be the class of all separable simple unital quasidiagonal real rank zero C*-algebras with unique unital trace. Popa pointed out that Connes' Følner type condition for a II$_1$ factor M (which implies that M is hyperfinite [C, p. 501]), coincides with the Hilbert-Schmidt-norm version of the local finite dimensional approximation property that characterizes the quasidiagonality of a simple C*-algebra. Actually, using an adaptation to the C*-algebra context of his short proof of Connes' theorem of equivalence between injectiveness and hyperfiniteness, Popa has shown that any C*-algebra A in the class $\mathcal{C}$ satisfies the following local finite dimensional approximation property. For any finite subset F of A and

any $\epsilon > 0$, there is a nonzero finite dimensional matrix subalgebra C of A with unit e such that $\|ex - xe\| < \epsilon$ and $dist(exe, C) < \epsilon$ for all $x \in F$. The question of whether this local approximation could be extended to a suitable global approximation that would imply the nuclearity of A was left open. In this context, Popa has asked whether every C*-algebra in $\mathcal{C}$ is nuclear.

We answer Popa's question by showing that $\mathcal{C}$ contains C*-algebras which are not exact (or subnuclear), and exact C*-algebras which are not nuclear. Since any simple quasidiagonal C*-algebra is strongly quasidiagonal, this also yields a negative answer to the questions of Hadwin and Rosenberg.

In the very last part of the paper, Example 13 (which is joint work with Chris Phillips), we exhibit two nonisomorphic separable simple exact C*-algebras of real rank zero and stable rank one, with the same Elliott and Haagerup invariants. Real rank one examples with similar properties have recently been exhibited in [GePh].

Thanks are due to the referee for a number of suggestions that improved the exposition.

2. The C*-algebra of a Bratteli system.   All the C*-algebras in this paper are assumed to be separable. The identity map of $M_r(\mathbb{C})$ will be denoted by id_r.

Definition 1. A Bratteli system $(A, \underline{\pi})$ consists of a unital C*-algebra A and a sequence $\underline{\pi} = (\pi_n)_{n=1}^{\infty}$ where each $\pi_n = (\pi_n^{i,j})$, $1 \leq i \leq t_{n+1}$, $1 \leq j \leq t_n$ is a $t_{n+1} \times t_n$ matrix of unital *-homomorphisms

$$\pi_n^{i,j}\colon A \to M_{k_n^{i,j}}(A), \quad k_n^{i,j} \in \mathbb{N}.$$

We always assume that $t_1 = 1$.

There is an inductive system (A_n, ϕ_n) associated in a canonical way to a Bratteli system. For every $n \in \mathbb{N}$ define integers $[n, i]$, $1 \leq i \leq t_n$, recursively by

$$[n + 1, i] = \sum_{j=1}^{t_n} [n, j]\, k_n^{i,j}, \quad 1 \leq i \leq t_{n+1}, \quad [1, 1] = 1.$$

For every $n \geq 1$ let $A_n^i = M_{[n,i]}(A)$, and

$$A_n = \bigoplus_{i=1}^{t_n} A_n^i = \bigoplus_{i=1}^{t_n} M_{[n,i]}(A).$$

Let $\phi_n\colon A_n \to A_{n+1}$ be the unital *-homomorphism given by a matrix $(\phi_n^{i,j})$ of

partial $*$-homomorphisms, where

$$\phi_n^{i,j} = id_{[n,j]} \otimes \pi_n^{i,j} \colon M_{[n,j]}(A) \to M_{[n,j]k_n^{i,j}}(A) \subset M_{[n+1,i]}(A).$$

The corestriction of ϕ_n to $M_{[n+1,i]}(A)$, denoted by ϕ_n^i, maps $a = (a_1, \ldots, a_{t_n}) \in A_n$ to $\phi_n^i(a) = \oplus_{j=1}^{t_n} \phi_n^{i,j}(a_j)$. Note that $A_1 = A$ since $t_1 = 1$. If $n < m$ we set

$$\phi_{m,n} = \phi_{m-1} \circ \cdots \circ \phi_n \colon A_n \to A_m$$

and in particular $\phi_{n+1,n} = \phi_n$.

Definition 2. The C*-algebra of a Bratteli system $(A, \underline{\pi})$ is defined by $A(\underline{\pi}) = \varinjlim (A_n, \phi_n)$.

The induced map $A_n \to A(\underline{\pi})$ is denoted by $\phi_{\infty,n}$. The definition of $A(\underline{\pi})$ was inspired by similar constructions involving homogeneous C*-algebras, and in particular by [Go]. Note that any Bratteli system $(A, \underline{\pi})$ determines a Bratteli diagram (see [Br]) with incidence matrices $(k_n^{i,j})$. The corresponding AF algebra is denoted by $AF(\underline{\pi})$.

Definition 3. A Bratteli system $(A, \underline{\pi})$ is called *restricted* if the following two conditions are satisfied for every $n \geq 1$.
   (i) If $(i,j) \neq (1,1)$, then $\pi_n^{i,j}$ is a finite dimensional unital representation $\pi_n^{i,j} \colon A \to M_{k_n^{i,j}}(\mathbb{C})$, which is to be regarded as a map into $M_{k_n^{i,j}}(A)$ by identifying $M_{k_n^{i,j}}(\mathbb{C})$ with $M_{k_n^{i,j}}(\mathbb{C}1_A)$.
   (ii) $\pi_n^{1,1} \colon A \to M_{k_n^{1,1}}(A)$ is of the form

$$\pi_n^{1,1}(a) = \begin{pmatrix} a & 0 \\ 0 & \gamma_n(a) \end{pmatrix}$$

where $\gamma_n \colon A \to M_{k(n)}(\mathbb{C}1_A)$ is a finite dimensional unital representation, and $k(n) = k_n^{1,1} - 1$. The sequence $(\gamma_n)_{n=1}^\infty$ will be called the *distinguished* component of $\underline{\pi}$.

Let A be a separable C*-algebra and let $(\gamma_n)_{n=1}^\infty$ be a sequence of finite dimensional representations of A. We say that $(\gamma_n)_{n=1}^\infty$ is a *separating* sequence if for every $a \in A$ there is n such that $\gamma_n(a) \neq 0$. If this is the case we say that A is *residually finite dimensional* (abbreviated RFD). A sequence $(\gamma_n)_{n=1}^\infty$ is called *strongly separating* if for every $a \in A$, and any $m \geq 1$, there is $n \geq m$ such that $\gamma_n(a) \neq 0$. We say that $(\gamma_n)_{n=1}^\infty$ has *infinite multiplicity* if each representation in the sequence repeats itself infinitely many times. It is clear that if $(\gamma_n)_{n=1}^\infty$ is separating and has infinite multiplicity, then $(\gamma_n)_{n=1}^\infty$ is strongly separating. Note that

for any unital separable RFD C*-algebra there exist restricted Bratteli systems whose distinguished component $(\gamma_n)_{n=1}^{\infty}$ is strongly separating.

Remark 4. The construction $A(\underline{\pi})$ has a certain functorial property. Suppose that B is a unital C*-subalgebra of A, with $1_B = 1_A$, and such that $\pi_n^{i,j}(B) \subset M_{k_n^{i,j}}(B)$ for all $\pi_n^{i,j}$. If $\iota_0 \colon B \to A$ denotes the inclusion map, then all the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;\pi_n^{i,j}\;\;} & M_{k_n^{i,j}}(A) \\[4pt]
\iota_0 \big\uparrow & & \big\uparrow id \otimes \iota_0 \\[4pt]
B & \xrightarrow[\pi_n^{i,j} \circ \iota_0]{} & M_{k_n^{i,j}}(B)
\end{array}
$$

are commutative. It is then clear that $(B, \underline{\pi} \circ \iota_0)$ is a Bratteli system and ι_0 induces a unital $*$-monomorphism $\iota \colon B(\underline{\pi} \circ \iota_0) \to A(\underline{\pi})$.

Note that if $(A, \underline{\pi})$ is a restricted Bratteli system, then the above diagram commutes for any unital C*-subalgebra B of A, with $1_B = 1_A$. In this case $(B, \underline{\pi} \circ \iota_0)$ is also a restricted Bratteli system. Note that if $(\gamma_n)_{n=1}^{\infty}$ is (strongly) separating, then so is $(\gamma_n \circ \iota_0)_{n=1}^{\infty}$.

In particular if $B = \mathbb{C}1_A$, $\iota_0 \colon \mathbb{C} \cong \mathbb{C}1_A \hookrightarrow A$, then $AF(\underline{\pi}) \cong \mathbb{C}(\underline{\pi} \circ \iota_0) \hookrightarrow A(\underline{\pi})$.

The notions of (topological) stable rank (sr) and real rank (RR) of a C*-algebra were defined in [BP] and respectively [Ri].

PROPOSITION 5. *Let $(A, \underline{\pi})$ be a restricted Bratteli system. If the sequence $(\gamma_n)_{n=1}^{\infty}$ is strongly separating, then $A(\underline{\pi})$ has real rank zero and stable rank one.*

Proof. We prove first that $RR(A) = 0$. Since the finite dimensional C*-algebras have real rank zero, and since ϕ_n^i has finite dimensional image for $i \neq 1$, it suffices to show that for every $n \geq 1$, every $\epsilon > 0$, and every noninvertible element $a = a^* \in A_n$, there is $m > n$ and $b = b^* \in GL(A_m^1)$ such that $\|\phi_{m,n}^1(a) - b\| < \epsilon$. After replacing a by $\phi_n(a)$ and after slightly perturbing the scalar components of $\phi_n(a)$, we may assume that if $a = (a_1, \ldots, a_{t_n})$, then a_i is invertible in A_n^i for $i \neq 1$ and $x = x^* = a_1$ is not invertible in A_n^1. Fix n, ϵ and $a \in A_n$ as above and let $\sigma_m = id_{[n,1]} \otimes \pi_m \colon M_{[n,1]}(A) \to M_{[n,1]k(m)}(\mathbb{C})$ for all $m > n$. Then $(\sigma_m)_{m>n}$ is a separating sequence of unital representations for A_n^1, hence

$$
\sigma = \sigma_{n+1} \oplus \sigma_{n+2} \oplus \ldots
$$

is a unital faithful representation of A_n^1. Since $\sigma(x)$ is necessarily noninvertible, it follows that if $x_m = \sigma_m(x)$ then either (i) there is $m > n$ such that x_m is not invertible, or (ii) x_m is invertible for all $m > n$ and $\overline{\lim}_{m \to \infty} \|x_m^{-1}\| = \infty$. Now x_m

is a selfadjoint element in $M_{[n,1]k(m)}(\mathbb{C})$. Thus there is a unitary u_m such that

$$x_m = u_m \begin{pmatrix} \lambda_m 1_A & 0 \\ 0 & y_m \end{pmatrix} u_m^*$$

where λ_m is an eigenvalue of x_m of minimum absolute value and $y_m = y_m^* \in M_{[n,1]k(m)-1}(\mathbb{C})$. Note that in case (i) there is $m > n$ such that $\lambda_m = 0$, and in case (ii) there is $m > n$ such that $|\lambda_m| < \epsilon$, since the sequence $\|x_m^{-1}\| = |\lambda_m|^{-1}$ is unbounded. Thus we can always find $m > n$ such that $|\lambda_m| < \epsilon$. Note that up to a conjugation by a unitary, $\phi_{m-1,n}^1(a)$ and $\phi_{m,n}^1(a)$ are of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \text{and respectively} \quad \begin{pmatrix} x & 0 & 0 \\ 0 & x_m & 0 \\ 0 & 0 & z \end{pmatrix}$$

where $y = y^*$ and $z = z^*$ are matrices over $\mathbb{C}1_A$. Therefore there is a unitary $v \in A_m^1$ such that

$$\phi_{m,n}^1(a) = v \begin{pmatrix} x & 0 & 0 \\ 0 & \lambda_m 1_A & 0 \\ 0 & 0 & c \end{pmatrix} v^*,$$

where c is a selfadjoint matrix over $\mathbb{C}1_A$. Let δ be such that $0 < \delta < \epsilon - |\lambda_m|$ and $c + \delta$ is invertible. Then

$$b = b^* = v \begin{pmatrix} x & \delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & c+\delta \end{pmatrix} v^* \in GL(A_m^1)$$

and $\|\phi_{m,n}^1(a) - b\| \leq |\lambda_m| + \delta < \epsilon$.

Next we show that $sr(A) = 1$. Reasoning as in the first part of the proof, it suffices to show that for every $n \geq 1$, every $\epsilon > 0$ and every element $a = (a_1, \ldots, a_{t_n}) \in A_n$, whose only noninvertible component is $x = a_1$, there is $m > n$ and $b \in GL(A_m^1)$ such that $\|\phi_{m,n}^1(a) - b\| < \epsilon$. Let σ_m and σ be as in the first part of the proof. Using the polar decomposition of $\sigma_m(x)$, we can write $\sigma_m(x) = w_m x_m$, where w_m is a unitary and $x_m \geq 0$. Since $\sigma(x)$ is not invertible, the sequence $(x_m)_{m>n}$ must verify either (i) or (ii). Reasoning as above, we find $m > n$ and unitaries $u, v \in A_m^1$ such that

$$\phi_{m,n}^1(a) = u \begin{pmatrix} x & 0 & 0 \\ 0 & \lambda_m 1_A & 0 \\ 0 & 0 & c \end{pmatrix} v$$

where $|\lambda_m| < \epsilon$ and c is some matrix over $\mathbb{C}1_A$. We complete the proof by taking

$$b = u \begin{pmatrix} x & \delta & 0 \\ \delta & 0 & 0 \\ 0 & 0 & c + \delta \end{pmatrix} v \in GL(A_m^1)$$

with $0 < \delta < \epsilon - |\lambda_m|$ and $c + \delta$ invertible. $\square$

PROPOSITION 6. *Let $(A, \underline{\pi})$ be a restricted Bratteli system.*
(i) *$A(\underline{\pi})$ is exact if and only if A is exact.*
(ii) *$A(\underline{\pi})$ is nuclear if and only if A is nuclear.*

Proof. (i) If A is exact, then $A(\underline{\pi})$ is exact being an inductive limit of the exact C*-algebras $A_n = \oplus_i M_{[n,i]}(A)$ [W, 2.5.5]. Conversely if $A(\underline{\pi})$ is exact, then A is exact as it embeds in $A(\underline{\pi})$ [W, Proposition 2.6].

(ii) If A is nuclear, then $A(\underline{\pi})$ is nuclear being an inductive limit of nuclear C*-algebras. Conversely, suppose that $A(\underline{\pi})$ is nuclear. Let $F \subset A_1 = A$ be a finite subset and let $\epsilon > 0$. Let $\phi_{\infty,n}: A_n \to A(\underline{\pi})$ be the induced map. Since $A(\underline{\pi})$ is nuclear, the map $\phi_{\infty,1}$ is nuclear, hence there exist a finite dimensional C*-algebra C and unital completely positive maps $\alpha: A_1 \to C$ and $\beta: C \to A(\underline{\pi})$ such that

(1) $\|\phi_{\infty,1}(a) - \beta\alpha(a)\| < \epsilon/2$

for all $a \in F$. By [DL, Lemma 4.2] (an application of the Choi-Effros lifting Theorem [ChE]), we find $n \geq 1$ and a unital completely positive map $\beta_n: C \to A_n$ such that

(2) $\|\beta(x) - \phi_{\infty,n}\beta_n(x)\| < \epsilon/2$

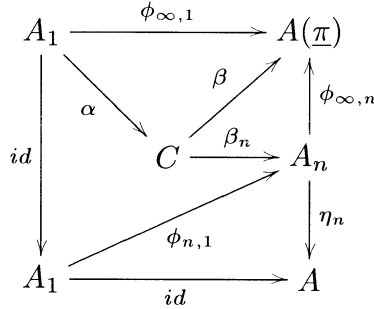for all $x \in \alpha(F)$. From (1) and (2) we obtain

$$\|\phi_{\infty,n}(\phi_{n,1}(a) - \beta_n\alpha(a))\| \leq \|\phi_{\infty,1}(a) - \beta\alpha(a)\| + \|\beta\alpha(a) - \phi_{\infty,n}\beta_n\alpha(a)\| < \epsilon$$

for all $a \in F$. Since $\phi_{\infty,n}$ is isometric, it follows that

(3) $\|\phi_{n,1}(a) - \beta_n\alpha(a)\| < \epsilon$

for all $a \in F$. Let $e = 1_A \oplus 0_{[n,1]-1}$ be the projection in $A_n^1 = M_{[n,1]}(A)$ corresponding to the (1,1) corner. We have $e\phi_{n,1}(a)e = a$, hence if $\eta_n = e(-)e: A_n \to A$, then from (3), we obtain $\|a - \eta_n\beta_n\alpha(a)\| < \epsilon$ for all $a \in F$. This proves that A is

nuclear. The flow of the proof is illustrated by the following diagram.

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ \phi_{\infty,1}\ } & A(\underline{\pi}) \\
\end{array}
$$

$$
\alpha \qquad \beta \qquad \phi_{\infty,n}
$$

$$
C \xrightarrow{\ \beta_n\ } A_n
$$

$$
id \qquad \qquad \eta_n
$$

$$
A_1 \xrightarrow[id]{\ \phi_{n,1}\ } A
$$

□

3. Nonexact models of AF algebras.

LEMMA 7. *Let B be a non-type I unital AF algebra. Let* $(k(n))_{n=1}^{\infty}$ *be a sequence of integers. Then there is an inductive system* $(B_n, \psi_{n+1,n})$ *with limit isomorphic to B, where* B_n *are finite dimensional C*-algebras,* $B_1 = \mathbb{C}$, $\psi_{n+1,n}$ *are unital *-homomorphisms given by incidence matrices* $(k_n^{i,j})$, $1 \leq i \leq t_{n+1}$, $1 \leq j \leq t_n$, *and* $k_n^{1,1} > k(n)$ *for all* $n \geq 1$.

Proof. By [Pe, p. 191], if B is not type I, then B has a quotient D without nonzero abelian elements, i.e., D is antiliminary. By Theorem 3.3 of [Br], any nonzero quotient of B admits a Bratteli diagram given by a full subgraph of some Bratteli diagram of B. Therefore it suffices to prove the statement for some nonzero quotient of B. Thus, without any loss of generality, we may assume that B itself is antiliminary. In particular, $\overline{eBe}$ is not abelian if $e \in B$ is any nonzero projection. Write $B = \varinjlim (D_p, \phi_{p+1,p})$, with unital connecting maps, and with D_p finite dimensional C*-algebras, $D_1 = \mathbb{C}$. A system like the one in the statement will be obtained by a suitable contraction of the system $(D_p, \phi_{p+1,p})$, i.e by taking $B_n = D_{p(n)}$ and $\psi_{n+1,n} = \phi_{p(n+1),p(n)}$ for some increasing sequence $(p(n))_{n=1}^{\infty}$. Actually it suffices to find a sequence $(i_n)_{n=1}^{\infty}$ such that $1 \leq i_n \leq t_n$, $k_n^{i_{n+1},i_n} \geq 2$. Indeed after a suitable contraction, one may arrange that $k_n^{i_{n+1},i_n} > k(n)$ and then relabel the blocks of $D_{p(n)}$. The construction of $p(n)$ and i_n is done by induction. Let $p(1) = i_1 = 1$ and suppose that $p(r)$ and i_r were constructed for $r \leq n$ such that $k_r^{i_r,i_{r-1}} \geq 2$ and the map $\phi_{\infty,p(n)} \colon D_{p(n)}^{i_n} \to B$ is injective. Then $p(n+1)$ and i_{n+1} are obtained as follows. Let e be a rank-one projection in $D_{p(n)}^{i_n}$. Since $\phi_{\infty,p(n)}(e)B\phi_{\infty,p(n)}(e)$ is nonzero, it cannot be abelian. Hence there is $m > p(n)$ such that the finite dimensional C*-algebra $\phi_{m,p(n)}(e)D_m\phi_{m,p(n)}(e)$ is not an abelian algebra. Therefore there is i such that rank $\phi_{m,p(n)}^i(e) \geq 2$ in D_m^i and the map $\phi_{\infty,m} \colon D_m^i \to B$ is injective. Thus we may take $p(n+1) = m$ and $i_{n+1} = i$. □

If A is a C*-algebra, let $CA \cong C_0(0,1] \otimes A$ denote the cone algebra of A and let $\widetilde{CA}$ be its unitization. Note that if A is RFD, then so is $\widetilde{CA}$.

PROPOSITION 8. *Let B be a non-type I unital AF algebra. Let A be a RFD algebra, and let $(\gamma_n')_{n=1}^{\infty}$ be a strongly separating sequence of unital finite dimensional representations of $\widetilde{CA}$. Then there is a restricted Bratteli system $(\widetilde{CA}, \underline{\pi})$ such that $AF(\underline{\pi}) \cong B$ and γ_n' is a subrepresentation of γ_n, $n \geq 1$, where $(\gamma_n)_{n=1}^{\infty}$ is the distinguished component of $\underline{\pi}$.*

Proof. Let $\gamma_n' \colon \widetilde{CA} \to M_{k(n)}(\mathbb{C})$ be as above. Let $(B_n, \psi_{n+1,n})$ be given by Lemma 7 for the input sequence $(k(n))$. Define $\pi_n = (\pi_n^{i,j})$, $\pi_n^{i,j} \colon \widetilde{CA} \to M_{k_n^{i,j}}(\mathbb{C}) \subset M_{k_n^{i,j}}(\widetilde{CA})$, as follows. If $(i,j) \neq (1,1)$ then $\pi_n^{i,j}(a) = a(0) \otimes 1_{k_n^{i,j}}$. If $(i,j) = (1,1)$, then

$$\pi_n^{1,1}(a) = \begin{pmatrix} a & 0 \\ 0 & \gamma_n(a) \end{pmatrix},$$

where, since $k_n^{1,1} > k(n)$, we can define

$$\gamma_n(a) = \begin{pmatrix} \gamma_n'(a) & 0 \\ 0 & a(0) \otimes 1_{\ell(n)} \end{pmatrix},$$

$\ell(n) = k_n^{1,1} - k(n) - 1 \geq 0$.   □

The next result answers a question of [Po] as it was explained in the introduction.

PROPOSITION 9. *Let B be a non-type I unital separable AF algebra. Then there exists a unital quasidiagonal nonexact separable C*-algebra C with $RR(C) = 0$, $sr(C) = 1$, such that C contains B and is asymptotically homotopy equivalent to B. If B has a unique unital trace, then so does C. Moreover B and C have isomorphic ideal lattices. In particular, if B is simple, then C is simple.*

Proof. Let A be a nonexact RFD C*-algebra. For instance we take A to be the C*-algebra of the free group on two generators. This is an RFD C*-algebra by [Ch], and it is not exact [W, Corollary 3.7]. Then $\widetilde{CA}$ is not exact, since it has a nonexact quotient [Ki]. Let $(\gamma_n')_{n=1}^{\infty}$ be a strongly separating sequence of unital finite dimensional representations of $\widetilde{CA}$. If $(\widetilde{CA}, \underline{\pi})$ is the restricted Bratteli system given by Proposition 8, then $AF(\underline{\pi}) \cong B$, and $(\gamma_n)_{n=1}^{\infty}$ is a strongly separating sequence of $\widetilde{CA}$, since γ_n' is a subrepresentation of γ_n. Let $\kappa \colon \mathbb{C} \to \widetilde{CA}$ be the (unital) inclusion map. If we set $C = \widetilde{CA}(\underline{\pi})$, then by Remark 4, $\mathbb{C}(\underline{\pi} \circ \kappa) \hookrightarrow \widetilde{CA}(\underline{\pi})$, hence

$$B \cong AF(\underline{\pi}) \cong \mathbb{C}(\underline{\pi} \circ \kappa) \hookrightarrow \widetilde{CA}(\underline{\pi}) = C.$$

Note that $RR(C) = 0$ and $sr(C) = 1$ by Proposition 5. Moreover C is quasidiagonal

since it is an inductive limit of quasidiagonal C*-algebras. By Proposition 6, C is not exact since $\widetilde{CA}$ is not exact. Note that C is shape equivalent to B since $\kappa\colon \mathbb{C} \to \widetilde{CA}$ is a homotopy equivalence [B$_2$]. By [D$_1$, Theorem 3.9] it follows that C is asymptotically homotopy equivalent to B. Since $RR(C) = 0$, the traces of C are separated by the projections of C, hence they are determined uniquely by their action on $K_0(C) \cong K_0(B)$. Thus if B has a unique unital trace, it follows that C has at most one unital trace. On the other hand, C is unital and quasidiagonal hence it has a unital trace by [V, 2.5]. Finally, if E is a unital C*-algebra with $RR(E) = 0$ and $sr(E) = 1$, then its two-sided closed ideals are parameterized by the order ideals of $K_0(E)$ (see [Z]). That proves the last part of the statement, since $K_0(B) \cong K_0(C)$ as ordered groups. $\square$

4. Nonnuclear models of AF algebras.

PROPOSITION 10. *Let A be an RFD AF algebra. Let $(\widetilde{CA}, \underline{\pi})$ be a restricted Bratteli system such that $(\gamma_n)_{n=1}^{\infty}$ is strongly separating for $\widetilde{CA}$. Then $\widetilde{CA}(\underline{\pi})$ is an AF algebra isomorphic to $AF(\underline{\pi})$.*

Proof. Let $A = \overline{\cup_{k=1}^{\infty} A_k}$, where (A_k) is an increasing sequence of finite dimensional C*-algebras. Let $j_{k+1,k}\colon \widetilde{CA_k} \to \widetilde{CA_{k+1}}$ be the inclusion map. Let $(\widetilde{CA_k}, \underline{\pi}^{(k)})$ be the Bratteli system obtained be restricting $\pi_n^{i,j}$ to $\widetilde{CA_k}$ (see Remark 4). Let $J_{k+1,k}\colon \widetilde{CA_k}(\underline{\pi}^{(k)}) \to \widetilde{CA_{k+1}}(\underline{\pi}^{(k+1)})$ be the homomorphism induced by $j_{k+1,k}$. Then we have

$$\widetilde{CA}(\underline{\pi}) = \varinjlim (\widetilde{CA_k}(\underline{\pi}^{(k)}), J_{k+1,k}).$$

Each algebra $\widetilde{CA_k}(\underline{\pi}^{(k)})$ has real rank zero, trivial K_1-group, and it is an inductive limit of one dimensional noncommutative CW complexes as defined in [ELP]. Then $\widetilde{CA_k}(\underline{\pi}^{(k)})$ is AF by [ELP, Theorem 4.2]. It follows that $\widetilde{CA}(\underline{\pi})$ is AF since the class of AF algebras in closed to inductive limits. Let $\kappa\colon \mathbb{C} \to \widetilde{CA}$ denote the inclusion map. As in Proposition 9, κ induces a unital $*$-monomorphisms $\mathbb{C}(\underline{\pi} \circ \kappa) \to \widetilde{CA}(\underline{\pi})$ that is an isomorphism of scaled ordered K-theory groups since κ is a homotopy equivalence. Therefore $AF(\underline{\pi}) \cong \mathbb{C}(\underline{\pi} \circ \kappa) \cong \widetilde{CA}(\underline{\pi})$ by Elliott's Theorem [Ell$_1$]. $\square$

Proposition 10 is actually true for any nuclear RFD C*-algebra A. A proof of this fact is a straightforward consequence of Theorem 5 of [D$_3$].

THEOREM 11. *Let B be a non-type I unital AF algebra. Then B contains a unital nonnuclear C*-subalgebra A with the following properties.*
   (a) $RR(A) = 0$ *and* $sr(A) = 1$.
   (b) *If $\iota\colon A \to B$ is the inclusion map, then there is a unital $*$-monomorphism $\kappa\colon B \to A$ such that $\iota \circ \kappa$ is homotopic to id_B and $\kappa \circ \iota$ is asymptotically homotopic to id_A.*

In particular $K_*(A) \cong K_*(B)$ as scaled ordered groups. Therefore A and B have isomorphic lattices of ideals and (affinely) homeomorphic trace spaces.

Proof. First we construct a non-type I RFD AF algebra. Let L be a non-type I AF algebra, for instance the CAR algebra. Write $L = \overline{\cup_{k=1}^{\infty} L_k}$, where (L_k) is an increasing sequence of finite dimensional C*-algebras. Let $\mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and let D be the C*-algebra of all continuous functions $f \colon \mathbb{N} \cup \{\infty\} \to L$ such that $f(k) \in L_k$ for all $k \in \mathbb{N}$. Then D is an RFD AF algebra. It is not type I as it has L as a quotient. Therefore by [B$_1$, Theorem 1], D contains a nonnuclear C*-subalgebra E. Using Proposition 8 for B and the RFD algebra D, we find a restricted Bratteli system $(\widetilde{CD}, \underline{\pi})$ such that its distinguished component $(\gamma_n)_{n=1}^{\infty}$ is a strongly separating sequence of unital finite dimensional representations of $\widetilde{CD}$, and $B \cong AF(\underline{\pi})$. Let $\iota_0 \colon \widetilde{CE} \to \widetilde{CD}$ be the (unital) inclusion map. By Remark 4, ι_0 induces a $*$-monomorphism $\iota \colon \widetilde{CE}(\underline{\pi} \circ \iota_0) \to \widetilde{CD}(\underline{\pi})$. We let $A = \widetilde{CE}(\underline{\pi} \circ \iota_0)$ and note that A is not nuclear by Proposition 6. By Proposition 10, $\widetilde{CD}(\underline{\pi})$ is isomorphic to $AF(\underline{\pi}) \cong B$, so that we may identify these algebras. Note that now (a) is a consequence of Proposition 5. The proof of (b) is more involved. One can see that since ι_0 is a homotopy equivalence, A is shape equivalent to B, hence A is asymptotically homotopy equivalent to B by [D$_1$, Theorem 3.14]. While this proves the assertion about K-theory, it does not prove part (b). One needs to show that the maps implementing the asymptotic homotopy are genuine $*$-homomorphisms. This is achieved by using the following two claims whose proofs are given later.

Claim 1. There is a unit preserving asymptotic morphism $(\kappa_t) \colon B \to A$ such that $(\kappa_t) \circ \iota$ is asymptotically homotopic to id_A.

Claim 2. Any unital asymptotic morphism from B to A is (asymptotically) homotopic to a unital $*$-homomorphism.

If (φ_t), (ψ_t) are asymptotic morphisms we write $(\varphi_t) \sim (\psi_t)$ if the two asymptotic morphisms are homotopic. Let (κ_t) be given by Claim 1. Then $(\kappa_t) \circ \iota \sim id_A$. By Claim 2, there is a unital $*$-homomorphism $\kappa \colon B \to A$ such that $(\kappa_t) \sim \kappa$. Thus $\kappa \circ \iota \sim id_A$. It remains to prove that $\iota \circ \kappa \sim id_B$ and that κ is injective. To that purpose, note that $\iota_* \colon K_*(A) \to K_*(B)$ is an isomorphism since ι_0 is a homotopy equivalence. Therefore $\kappa_* = (\iota_*)^{-1}$, since $\kappa_* \circ \iota_* = (id_A)_*$, so that κ is injective and $\iota_* \circ \kappa_* = (id_B)_*$. That implies that $\iota \circ \kappa$ is homotopic to id_B, since B is an AF algebra, by an argument similar to one in [B$_3$, 7.7.5].

Next we discuss the last part of the Theorem. If C is a unital C*-algebra, let $T(C)$ denote the space of unital traces of C and let $SK_0(C)$ denote the set of all positive group homomorphisms $f \colon K_0(C) \to \mathbb{R}$ with $f[1_C] = 1$. If $RR(C) = 0$, then the natural map $T(C) \to SK_0(C)$ is injective. It is also surjective if C is

AF [B_3]. It follows that the restriction map $T(B) \to T(A)$ is bijective (hence a homeomorphism in the weak* topology), since $K_0(A) \cong K_0(B)$ as ordered scaled groups. As in the proof of Proposition 9, the two-sided closed ideals are in bijection with the order ideals of the K_0-group. □

The proofs of the two claims above require some preparation. Let $\mathcal{A}$ be the asymptotic homotopy category of Connes and Higson [CH]. Its objects are separable C*-algebras and its morphisms are homotopy classes of asymptotic morphisms. One can construct asymptotic morphisms using the homotopy inductive limit functor L of [D_1]. L is defined on the strong shape category with values in $\mathcal{A}$. It takes an inductive system of C*-algebras $(C_n, p_{n+1,n})$ to its limit $\varinjlim (C_n, p_{n+1,n})$. To describe the action of L on morphisms, recall that a *strong map of (inductive) systems*

$$(\underline{f}, \underline{h}): \ (C_n, p_{n+1,n}) \to (D_n, q_{n+1,n})$$

can be given by a sequence of $*$-homomorphisms $f_n: C_n \to D_n$ and by a sequence of homotopies $h_n = (h_n^t): \ C_n \to D_{n+1}[0,1]$ such that $h_n^0 = q_{n+1,n}f_n$ and $h_n^1 = f_{n+1}p_{n+1,n}$. Let $p_{k,n}: \ C_n \to C_k$ denote the connecting map. Let $D = \varinjlim (D_n, q_{n+1,n})$ and let $q_n: D_n \to D$ be the canonical map. Then $L(\underline{f}, \underline{h})$ is an asymptotic morphism (φ_t) whose construction goes as follows. Let $T_n = [n, \infty)$ and for every n define $*$-homomorphisms $\varphi_n: \ C_n \to C_b(T_n, D)$ by

$$\varphi_n^t = q_{k+1}h_k^{t-k}p_{k,n}, \quad n \le k \le t \le k+1.$$

Let $r_{n+1,n}: \ C_b(T_n, D) \to C_b(T_{n+1}, D)$ be the restriction map. The $*$-homomorphisms φ_n fit together in a commutative diagram.

$$
\begin{array}{ccccccccc}
C_b(T_n, D) & \longrightarrow & \cdots & \longrightarrow & C_b(T_k, D) & \xrightarrow{r_{k+1,k}} & C_b(T_{k+1}, D) & \longrightarrow & \cdots \\
\varphi_n \uparrow & & & & \varphi_k \uparrow & & \uparrow \varphi_{k+1} & & \\
C_n & \longrightarrow & \cdots & \longrightarrow & C_k & \xrightarrow{p_{k+1,k}} & C_{k+1} & \longrightarrow & \cdots
\end{array}
$$

Passing to inductive limit we obtain a $*$-homomorphism

$$\hat{\varphi} = \varinjlim \varphi_n: \ C = \varinjlim (C_n, p_{n+1,n}) \to \varinjlim C_b(T_n, D) = C_b(\mathbb{R}_+, D)/C_0(\mathbb{R}_+, D).$$

Let $\varphi = (\varphi_t): C \to C_b(\mathbb{R}_+, D)$ be a continuous lifting of $\hat{\varphi}$. Then $L(\underline{f}, \underline{h}) = (\varphi_t)$ is defined up to an equivalence of asymptotic morphisms. There is an appropriate notion of homotopy for strong maps of systems such that L descends to a well-defined map on homotopy classes. A rather general example of a homotopy of strong maps $(\underline{f}, \underline{h}), (\underline{f}', \underline{h}'): (C_n, p_{n+1,n}) \to (D_n, q_{n+1,n})$ is given by a strong map

of systems $(\underline{\nu}, \underline{\mu})$: $(C_n, p_{n+1,n}) \to (D_n[0,1], q_{n+1,n} \otimes id_{C[0,1]})$ where $\underline{\nu} = (\nu^s)_{s \in [0,1]}$ and $\underline{\mu} = (\mu^s)_{s \in [0,1]}$ are such that $(\underline{\nu}^0, \underline{\mu}^0) = (\underline{f}, \underline{h})$ and $(\underline{\nu}^1, \underline{\mu}^1) = (\underline{f}', \underline{h}')$. More explicitly, for every n, (ν_n^s): $C_n \to D_n$ is a homotopy of $*$-homomorphisms with $\nu_n^0 = f_n$ and $\nu_n^1 = f_n'$ while $\mu_n^{s,t}$: $C_n \to D_{n+1}$ is a two-parameter homotopy of $*$-homomorphisms such that for all $0 \le s, t \le 1$

$$(4) \qquad \begin{aligned} \mu_n^{s,0} &= q_{n+1,n}\, \nu_n^s \\ \mu_n^{s,1} &= \nu_{n+1}^s\, p_{n+1,n} \\ \mu_n^{0,t} &= h_n^t \\ \mu_n^{1,t} &= h_n'^t. \end{aligned}$$

If $(\underline{\nu}, \underline{\mu})$ is as above, then $L(\underline{\nu}, \underline{\mu})$ is a homotopy of asymptotic morphisms from $L((\underline{f}, \underline{h}))$ to $L((\underline{f}', \underline{h}'))$ (see [D$_1$]).

Claim 2 is a consequence of the following Lemma.

LEMMA 12. *Let B be an AF algebra and let A be a unital C*-algebra. Then every asymptotic morphism (φ_t): $B \to A$ is homotopic to a $*$-homomorphism.*

Proof. Write $B = \varinjlim (B_n, p_{n+1,n})$ with B_n finite dimensional C*-algebras. The system $(B_n, p_{n+1,n})$ is semiprojective [B$_2$], [D$_1$, Def. 3.2]. Then as in [D$_1$, Corollary 3.14], we can lift (φ_t) to a strong map of systems $(\underline{f}', \underline{h}')$: $(B_n, p_{n+1,n}) \to (A, id)$

with h_n': $B_n \to A[0,1]$ satisfying $h_n'^0 = f_n'$ and $h_n'^1 = f_{n+1}' p_{n+1,n}$. We will construct inductively a sequence of unitaries (z_n) in A such that if we define $f_n = ad(z_n)f_n'$, then $f_{n+1}p_{n+1,n} = f_n$ and if we set $h_n^t = f_n$, then $(\underline{f}, \underline{h})$ is homotopic to $(\underline{f}', \underline{h}')$ as strong maps of systems via some homotopy $(\underline{\nu}, \underline{\mu})$. That will imply that the $*$-homomorphism $\varinjlim f_n = L(\underline{f}, \underline{h})$ is homotopic to $L(\underline{f}', \underline{h}') \sim (\varphi_t)$.

The homotopies ν_n^s and $\mu_n^{s,t}$ are constructed by induction and such that $\nu_n^s = ad(u_n^s)f_n'$ where (u_n^s) is a path of unitaries in A with $u_n^0 = z_n$ and $u_n^1 = 1$. The first step is trivial as we can assume that $B_1 = B_2 = \mathbb{C}$. Suppose now that z_k, u_k^s, ν_k^s and $\mu_{k-1}^{s,t}$ were constructed for $k \le n$. Then $z_{n+1}, u_{n+1}^s, \nu_{n+1}^s$ and $\mu_n^{s,t}$ are constructed as follows. We regard $h_n'^t$ as a continuous path into the path-connected component of f_n' in the base space of the fibration $U(A) \to Hom(B_n, A)$ (see [N]). Therefore we can find a continuous path of unitaries (w_n^t) in A such that $w_n^0 = 1$ and $h_n'^t = ad(w_n^t)f_n'$. Let (v_n^s) be a path given by the concatenation $(u_n^s) * (w_n^s)$. Then

there exists a continuous map $\omega\colon [0,1] \times [0,1] \to U(A)$ such that

$$\omega(s,0) = u_n^s,\ \omega(s,1) = v_n^s,\ \omega(0,t) = u_n^0,\ \omega(1,t) = w_n^t.$$

The map ω exists as its prescribed restriction to the boundary of the unit square gives a trivial element of the fundamental group of $U(A)$, by our choice of v_n^s. Define

$$u_{n+1}^s = v_n^s w_n^{1\,*},\ z_{n+1} = u_{n+1}^0$$

$$v_{n+1}^s = ad(u_{n+1}^s)f_{n+1}' \quad \mu_n^{s,t} = ad(\omega(s,t))f_n'.$$

It is then easy to check that $(\underline{v},\underline{\mu})$ gives the desired homotopy, as it satisfies (4). $\square$

To complete the proof of Theorem 11 it remains to prove Claim 1.

The asymptotic morphism $(\kappa_t)\colon \widetilde{CD}(\underline{\pi}) \to \widetilde{CE}(\underline{\pi})$ will be constructed by using the following strong map of systems:

$$(\underline{\kappa},\underline{q})\colon \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CD}), \phi_n \right) \to \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CE}), \phi_n \right),$$

where $\kappa_n(a) = a(0)$ and $q_n^t = (id_{[n,j]} \otimes q_n^{i,j,t})$, $q_n^{i,j,t}\colon \widetilde{CD} \to M_{k_n^{i,j}}(\mathbb{C}) \subset M_{k_n^{i,j}}(\widetilde{CE})$, $q_n^{i,j,t}(a) = \pi_n^{i,j}(a(t\cdot))$ if $(i,j) \neq (1,1)$, and

$$q_n^{1,1,t}(a) = \begin{pmatrix} a(0) & 0 \\ 0 & \gamma_n(a(t\cdot)) \end{pmatrix}.$$

The $*$-homomorphism $\iota\colon \widetilde{CE} \to \widetilde{CD}$ is induced by the following strong map of systems:

$$(\underline{\iota},\underline{g})\colon \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CE}), \phi_n \right) \to \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CD}), \phi_n \right),$$

where ι_n is induced by the inclusion $\widetilde{CE} \hookrightarrow \widetilde{CD}$, and $g_n^t = \phi_n \iota_n$ is a constant homotopy. We have that

$$(\underline{\kappa},\underline{q}) \circ (\underline{\iota},\underline{g}) = (\underline{f},\underline{h})$$

where

$$(\underline{f},\underline{h})\colon \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CE}), \phi_n \right) \to \left(\bigoplus_{i=1}^{t_n} M_{[n,i]}(\widetilde{CE}), \phi_n \right),$$

$f_n^t(a) = a(0)$ and $h_n^t = (id_{[n,j]} \otimes h_n^{i,j,t})$, $h_n^{i,j,t} \colon \widetilde{CE} \to M_{k_n^{i,j}}(\widetilde{CE})$, $h_n^{i,j,t}(a) = \pi_n^{i,j}(a(t\cdot))$
if $(i,j) \neq (1,1)$, and

$$h_n^{1,1,t}(a) = \begin{pmatrix} a(0) & 0 \\ 0 & \gamma_n(a(t\cdot)) \end{pmatrix}.$$

We are going to prove that $(\underline{f}, \underline{h})$ is homotopic to $(\underline{f'}, \underline{h'})$ as strong maps of systems, where $f_n' = id_{\oplus_i M_{[n,i]}(\widetilde{CE})}$ and $h_n'^t = \phi_n$ is a constant homotopy. To that purpose define a homotopy $(\underline{\nu}, \underline{\mu})$ by $\nu_n^s(a) = a(s\cdot)$ and $\mu_n^{s,t} = (id_{[n,j]} \otimes \mu_n^{i,j,s,t})$, $\mu^{i,j,s,t} \colon \widetilde{CE} \to M_{k_n^{i,j}}(\widetilde{CE})$, $\mu^{i,j,s,t}(a) = \pi_n^{i,j}(a(\max(s,t)\cdot))$ if $(i,j) \neq (1,1)$, and

$$\mu^{1,1,s,t}(a) = \begin{pmatrix} a(s\cdot) & 0 \\ 0 & \gamma_n(a(\max(s,t)\cdot)) \end{pmatrix}.$$

Then $(\underline{\nu}, \underline{\mu})$ will give a homotopy from $(\underline{f}, \underline{h})$ to $(\underline{f'}, \underline{h'})$, since it satisfies (4). Finally let us notice that if we set $(\kappa_t) = L(\underline{\kappa}, \underline{q})$, then since $L(\underline{\iota}, \underline{g}) \sim \iota$, and $L(\underline{f'}, \underline{h'}) = id_A$, we have

$$(\kappa_t) \circ \iota \sim L(\underline{\kappa}, \underline{q}) \circ L(\underline{\iota}, \underline{g}) \sim L(\underline{f}, \underline{h}) \sim L(\underline{f'}, \underline{h'}) \sim id_A.$$

That proves Claim 1 and completes the Proof of Theorem 11. $\qquad\square$

5. Nonisomorphic exact C*-algebras. Note that the C*-algebras A, B of Theorem 11 have the same Elliott invariants. They are not isomorphic since A is not nuclear whereas B is nuclear.

The following example is joint work with Chris Phillips.

Example 13. There are two *nonisomorphic* separable unital C*-algebras, E and F, such that

(i) E and F are simple, exact, nonnuclear, and have a unique unital trace.

(ii) $RR(E) = RR(F) = 0$ and $sr(E) = sr(F) = 1$.

(iii) $K_0(E) \cong K_0(F)$ as ordered scaled groups and $K_1(E) \cong K_1(F)$.

(iv) E and F have the same Haagerup invariant, equal to 1.

E is not isomorphic to F since E is quasidiagonal, whereas F is not. Actually $E \otimes K(H)$ is not asymptotically homotopy equivalent (via completely positive asymptotic morphisms) to $F \otimes K(H)$ since quasidiagonality is invariant under such an equivalence (see [D$_2$]).

Let B be the UHF algebra of type 2^∞. Let D be a simple real rank zero $A\mathbb{T}$ algebra, with $K_0(D) = \mathbb{Z}[1/2]$ (with natural order), $[1_D] = 1$, and $K_1(D) = \mathbb{Z}^2$ [Ell$_2$]. Let $A \subset B$ be given by Theorem 11. We set $E = B \otimes A \otimes D$ and $F = B \otimes A \otimes C_r^*(\mathbb{F}_2)$, where all tensor products are minimal. Recall that the

reduced C*-algebra of the free group on two generators, $C_r^*(\mathbb{F}_2)$, is simple and has a unique unital trace [Pow]. Next we verify the properties of E and F listed above.

(i) E and F are simple and exact since they are minimal tensor products of simple, exact C*-algebras [Ta$_2$, Ch IV, 4.21], [W]. They have a unique unital trace since each factor in the tensor product has a unital unique trace [CuP, Cor. 6.13]. Using a slice map, one shows that a (nonzero) minimal tensor product by a nonnuclear C*-algebra is nonnuclear. Thus both E and F are nonnuclear.

(ii) As in (i), $E_1 = A \otimes D$ and $F_1 = A \otimes C_r^*(\mathbb{F}_2)$ are simple with unique unital trace. It follows that $E = B \otimes E_1$ and $F_1 = B \otimes F_1$ have stable rank one by [Rø$_1$] and real rank zero by [Rø$_2$, Theorem 7.2] and [H$_3$, Theorem 5.11].

(iii) We have $B \otimes B \cong B$. Recall that shape equivalence is preserved under taking minimal tensor products by exact C^*-algebras [B$_2$], and that A is shape equivalent to B. Thus E is shape equivalent to $B \otimes D$ and F is shape equivalent to $B \otimes C_r^*(\mathbb{F}_2)$. By [PiV], and the Künneth formula [Sc], we see that $K_*(B \otimes D) \cong K_*(B \otimes C_r^*(\mathbb{F}_2))$ hence $K_*(E) \cong K_*(F)$. It remains to show that the positive cones $K_0(-)^+$ are isomorphic. First we see that $K_0(B \otimes D)^+ \cong \mathbb{Z}[1/2]^+$ by construction. The isomorphism $K_0(B \otimes C_r^*(\mathbb{F}_2))^+ \cong \mathbb{Z}[1/2]^+$ follows from the fact that any projection in $M_n(C_r^*(\mathbb{F}_2))$ is equivalent to 1_k, for some $k \leq n$, since $K_0(C_r^*(\mathbb{F}_2)) = \mathbb{Z}$ is generated by the class of 1, by [PiV], and since $C_r^*(\mathbb{F}_2)$ has stable rank one by [DyHRø].

(iv) Let Λ denote the Haagerup invariant. We have that $\Lambda = 1$ for nuclear C*-algebras and $\Lambda(C_r^*(\mathbb{F}_2)) = 1$ by [H$_1$], [H$_2$]. Therefore $\Lambda(E) = \Lambda(F)$ since $\Lambda(X \otimes Y) = \Lambda(X)\Lambda(Y)$, for (exact) C*-algebras X, Y and minimal tensor products [SiSm].

Examples of simple exact nonisomorphic C*-algebras (of real rank one) with the same Elliott and Haagerup invariants have recently been exhibited in [GePh].

Added in proof. The author has learned that a statement equivalent to Lemma 7 has appeared in [LT].

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