JOURNAL OF FUNCTIONAL ANALYSIS **151**, 213–233 (1997)
ARTICLE NO. FU973145

Quasidiagonal Morphisms and Homotopy

Marius Dadarlat

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Received February 4, 1997; accepted March 31, 1997

Adapting certain methods of Voiculescu we show that the quasidiagonality of C*-algebras is invariant under completely positive deformations coming from asymptotic morphisms. We define a notion of quasidiagonal morphism. The quasidiagonality of morphisms from exact separable C*-algebras is shown to be invariant under completely positive deformations. © 1997 Academic Press

1. INTRODUCTION

A linear operator acting on a separable Hilbert space is called quasidiagonal if it is a compact perturbation of a block-diagonal operator (see [H]). This notion extends to C*-algebras in the form of a local approximation property. Quasidiagonality has important applications to the extension theory of C*-algebras. It was shown by Voiculescu [V_2] that quasidiagonality is a topological invariant. Specifically, he has proven that a C*-algebra which is homotopically dominated by a quasidiagonal C*-algebra is quasidiagonal.

In the first half of the paper we show that the methods of [V_2] can be adapted to extend Voiculescu's result to a setting where the $*$-homomorphisms (and their homotopies) are being replaced by completely positive linear contractive asymptotic morphisms (CP-asymptotic morphisms). For separable C*-algebras A, B let $[[A, B]]_{CP}$ denote the homotopy classes of CP-asymptotic morphisms from A to B. The category with objects separable C*-algebras and morphisms $[[A, B]]_{CP}$ is a variant of the asymptotic homotopy category $\mathscr{A}$ of Connes and Higson [CH]. It will be denoted by $\mathscr{A}_{CP}$. We have:

THEOREM 1.1. *Let A, B be separable C*-algebras. Suppose that A is dominated by B in the category $\mathscr{A}_{CP}$, i.e., there are CP-asymptotic*

0022-1236/97 $25.00

morphisms $(\varphi_t)\colon A \to B$ *and* $(\psi_t)\colon B \to A$ *such that* $[[\psi_t]][[\varphi_t]] = [[id_A]]$ *in* $\mathscr{A}_{CP}$. *If B is quasidiagonal, then A is quasidiagonal.*

Very recently Houghton-Larsen and Thomsen [LT] have proven that for stable separable C*-algebras $KK(A, B) \cong [[SA, SB]]_{CP}$. Their result reinforces the suggestion that $[[A, B]]_{CP}$ plays the role of positive morphisms in KK-theory. Theorem 1.1 shows that quasidiagonality is invariant under "positive KK-equivalence". Thus one may argue that quasidiagonality is related to K-theoretical phenomena. For more on the connections of quasidiagonal to K-theory see [Br, $\mathrm{Sa}_{1,2}$, Zh, BrD, Li, Sc].

In the second half of the paper we introduce a notion of quasidiagonality for $*$-homomorphsisms and CP-asymptotic morphisms. This corresponds to the property of factorization through quasidiagonal C*-algebras. For morphisms out of exact C*-algebras we show that quasidiagonality is a homotopy invariant. We have:

THEOREM 1.2. *Let* $(\varphi_t), (\psi_t)\colon A \to B$ *be two homotopic CP-asymptotic morphisms from a separable, exact C*-algebra A to a C*-algebra B. If* (ψ_t) *is quasidiagonal, then* (φ_t) *is quasidiagonal.*

The composition of two homotopy classes of CP-asymptotic morphisms between exact C*-algebras is quasidiagonal provided that at least one of them is quasidiagonal. The nuclear quasidiagonal $*$-homorphisms from an exact C*-algebra are characterized by a certain approximation property which appears in [BKi] (see Definition 4.6 and Theorem 4.8).

As a consequence of the above we show that a separable nuclear C*-algebra which is dominated in the category $\mathscr{A}$ of [CH] by a separable quasidiagonal C*-algebra is quasidiagonal. Another corollary shows that a nuclear separable C*-algebra which is shape equivalent [EK, B] to a separable quasidiagonal C*-algebra is quasidiagonal. Let us mention that the class of separable nuclear quasidiagonal C*-algebras has been recently identified with a certain class of generalized inductive limits of finite dimensional C*-algebras [BKi]. Their study goes back to [Sa_2]. Our arguments make use of a stabilization result for asymptotic morphisms to $L(H)$ (Proposition 2.3) which was inspired by [Ki]. For the convenience of the reader, a series of useful properties of exact C*-algebras were collected in Section 3.

For a discussion of various aspects of quasidiagonality we refer the reader to the survey paper [V_3]. It is expected that quasidiagonality has to play a role in the classification of nuclear C*-algebras of real rank zero, see [Po].

The notion of quasidiagonal morphism employed in this paper is not directly related to the notion of quasidiagonal KK-classes studied in [Sa_2] and [Sc] where the emphasis is on relative quasidiagonality. For a quasidiagonal C*-algebra A, the KK-class of id_A is not quasidiagonal in the sense of [Sc] if $K_0(A) \neq 0$.

We are indebted to E. Kirchberg for pointing to us Propositions 3.2 and 3.3. Part of this work was done while the author was visiting the University of Marseille at Luminy. He thanks R. Zekri, E. Blanchard and G. Kasparov for their support and hospitality. Thanks are due to L. Brown for a number of helpful suggestions.

The present paper replaces a preliminary version entitled "A remark on a paper of Voiculescu on quasidiagonality and homotopy" whose content is now part of Section 2.

2. QUASIDIAGONAL C*-ALGEBRAS

Let H be a separable infinite dimensional Hilbert space, $L(H)$ the linear bounded operators on H and $K(H)$ the compact operators. Recall that a set $A \subset L(H)$ is called quasidiagonal if there is a sequence (p_k) of finite dimensional orthogonal projections converging strongly to 1_H such that $\|p_k a - a p_k\| \to 0$ for all $a \in A$. A separable C*-algebra A is called quasidiagonal if there is a faithful representation $\rho\colon A \to L(H)$ such that the set $\rho(A)$ is quasidiagonal.

For C*-algebras A, B we denote by $CP(A, B)$ the set of all completely positive linear contractions from A to B endowed with the point-norm topology. The elements of $CP(A, B)$ will be called CP-maps. For a locally compact space X, we let $C_b(X, B)$ denote the continuous bounded functions from X to B. Recall that the multiplier algebra $M(C_0, (X, B))$ is isomorphic to the C*-algebra $C_{b,s}(X, M(B))$ of strictly continuous, bounded functions from X to the multiplier algebra of B (see [APT]). The cone and the suspension of a C*-algebra A are defined as $CA = C_0(\mathbb{R}_+, A)$ and $SA = C_0(\mathbb{R}, A)$.

The following characterization of quasidiagonality is given in [V_2, Theorem 1, Remark 2].

THEOREM 2.1. *Let A be a C*-algebra. Then A is quasidiagonal if and only if for any $\delta > 0$, $F \subset A$ a finite subset and $x \in F$, there is a finite dimensional* (*or just quasidiagonal*) *C*-algebra B and there is $\varphi \in CP(A, B)$ such that $\|\varphi(x)\| \geqslant \|x\| - \delta$ and $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for all $a, b \in F$.*

Let us note that by taking finite direct sums of CP-maps one can replace $\|\varphi(x)\| \geqslant \|x\| - \delta$ by the stronger condition $\|\varphi(a)\| \geqslant \|a\| - \delta$ for all $a \in F$, which appears in the original form of [V_2, Theorem 1].

Theorem 2.1 suggests that the asymptotic morphisms of [CH] should be useful in the study of quasidiagonality. Recall that an asymptotic morphism is a family of maps $(\varphi_t)\colon A \to B$, $t \in \mathbb{R}_+$, such that $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$ and

$$\|\varphi_t(a+\lambda b) - \varphi_t(a) - \lambda\varphi_t(b)\| \to 0$$

$$\|\varphi_t(a^*) - \varphi_t(a)^*\| \to 0$$

$$\|\varphi_t(ab) - \varphi_t(a)\,\varphi_t(b)\| \to 0$$

as $t \to \infty$, for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Two asymptotic morphisms (φ_t) and (ψ_t) are *equivalent* if $\|\varphi_t(a) - \psi_t(a)\| \to 0$ for all $a \in A$. They are *homotopic* if there is an asymptotic morphism $(\Phi_t)\colon A \to B[0,1] \cong B \otimes C[0,1]$ such that $\Phi_t^{(0)} = \varphi_t$ and $\Phi_t^{(1)} = \psi$. Up to an equivalence any asymptotic morphisms is given by a continuous map $\varphi\colon A \to C_b(\mathbb{R}_+, B)$ which satisfies the axioms of a $*$-homomorphism modulo the ideal $C_0(\mathbb{R}_+, B)$ or equivalently, by a $*$-homomorphism $\dot{\varphi}\colon A \to B_\infty \cong C_b(\mathbb{R}_+, B)/C_0(\mathbb{R}_+, B)$. The homotopy classes of asymptotic morphisms are denoted by $[[A, B]]$. The asymptotic morphisms of separable C*-algebras can be composed at the level of homotopy classes. The composition is associative. We denote by $\mathscr{A}$ the category of separable C*-algebras with morphisms from A to B given by $[[A, B]]$ and refer to it as the asymptotic homotopy category of Connes and Higson.

It is convenient to consider a related category $\mathscr{A}_{CP}$ having the same objects as $\mathscr{A}$ and whose morphisms are homotopy classes of CP-asymptotic morphisms. By a CP-asymptotic morphism we mean an asymptotic morphism $(\varphi_t)\colon A \to B$ such that each individual map of the family is an element of $CP(A, B)$. Equivalently, the map defined by $\varphi(a)(t) = \varphi_t(a)$ is an element of $CP(A, C_b(\mathbb{R}_+, B))$. The homotopy classes of CP-asymptotic morphisms, denoted by $[[A, B]]_{CP}$, are defined by allowing only homotopies that are CP-asymptotic morphisms. As in [CH] there is a well defined associative product

$$[[A, B]]_{CP} \times [[B, C]]_{CP} \to [[A, C]]_{CP}$$

with the simplification that $[[\psi_t]][[\varphi_t]] = [[h_t]]$, where $h_t = \psi_{r(t)}\varphi_t$ and $r(t) \geqslant s(t)$ for a suitable rescaling $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ (see [L_2, Theorems 13–13′]). As opposed to that, recall that for arbitrary asymptotic morphisms we only know that h_t can be taken such that $\|h_t(a) - \psi_{r(t)}\varphi_t(a)\| \to 0$ for a in some dense $*$-subalgebra of A. Using the Choi–Effros Theorem [ChE] it is easily seen that if A is nuclear, then the natural map $[[A, B]]_{CP} \to [[A, B]]$ is a bijection. The notation $[A, B]$ is reserved for the usual homotopy classes of $*$-homomorphisms. We have a natural map $[A, B] \to [[A, B]]_{CP}$.

Let A be a quasidiagonal separable C*-algebra. As a consequence of Voiculescu's Theorem [V_1], one knows that if $\rho\colon A \to L(H)$ is any *essential* representation, i.e., ρ is a faithful representation such that $\rho(A) \cap K(H) = \{0\}$, then $\rho(A)$ is quasidiagonal (see [ER]). We need a related result for CP-asymptotic morphisms, which asserts, roughly speaking, that $\varphi_t(A)$ tends to be quasidiagonal, at least when φ_t is replaced by $\varphi_t \oplus \rho$, for some representation ρ.

The following proposition about the vanishing of $KK(A, L(H))$ is proved in a similar manner to the vanishing of $K_0(M(A \otimes K(H)))$ (cf. [B_1]) by using the isomorphism $KK(A, L(H)) \cong [qA, L(H) \otimes K(H)]$ of Cuntz [Cu]. A related argument appears in [G, Proposition 6.2].

Proposition 2.2. *If A is a separable C*-algebra and B is a C*-algebra with a countable approximate unit, then $KK(A, L(H) \otimes B) = 0$.*

Proof. We work with the minimal tensor product. We may assume that B is stable and $B \subset L(E)$ for some Hilbert space E. Since $L(H) \otimes B$ has a countable approximate unit, $KK(A, L(H) \otimes B) \cong [qA, L(H) \otimes B]$ by [Cu]. The addition on the latter group is given by $[\varphi] + [\psi] = [\theta_{\varphi,\psi}]$, $\theta_{\varphi,\psi}(a) = w_1 \varphi(a) w_1^* + w_2 \psi(a) w_2^*$ where $w_i \in M(L(H) \otimes B)$ are isometries with $w_1 w_1^* + w_2 w_2^* = 1$. This definition does not depend on the particular choice of the pair w_i (see [JT, 1.3]). Let $s_1, s_2\colon H \to H$ be isometries with $s_1 s_1^* + s_2 s_2^* = 1_H$ and set $w_i = s_i \otimes 1_E\colon H \otimes E \to H \otimes E$. Let $s\colon H \otimes H \to H$ be defined by $s(x_1 \oplus x_2) = s_1 x_1 + s_2 x_2$. Then $s \otimes 1_E\colon H \otimes E \oplus H \otimes E \to H \otimes E$ is a unitary such that $\theta_{\varphi,\psi}(a) = s \otimes 1_E(\varphi(a) \oplus \psi(a))\, s^* \otimes 1_E$.

If $\varphi\colon qA \to L(H) \otimes B$ is an arbitrary $*$-homomorphism, define $\eta\colon qA \to L(H) \otimes B$ by $\eta(a) = (v \otimes 1_E)\, 1_H \otimes \varphi(a)(v^* \otimes 1_E)$, where $v\colon H \otimes H \to H$ is a fixed unitary. We will show that $[\varphi] + [\eta] = [\eta]$, hence $[\varphi] = 0$. Since φ was arbitrarily chosen that will imply the vanishing of $[qA, L(H) \otimes B]$. To that purpose it suffices to exhibit a unitary $w \in M(L(H) \otimes B)$ such that $w\theta_{\varphi,\eta} w^* = \eta$ as the unitary group of $M(L(H) \otimes B)$ is path-connected in the strict topology [JT, 1.37]. Let (h_i), $i = 0, 1, \ldots$ be an orthonormal basis of H and define a unitary $u\colon H \oplus H \otimes H \to H \otimes H$ by $u(h) = h_0 \otimes h$, $u(h_i \otimes h) = h_{i+1} \otimes h$, $h \in H$. It is easy to see that $1_H \otimes y = u(y \oplus 1_H \otimes y)\, u^*$ for all $y \in L(H)$. It follows that, for all $a \in qA$,

$$1_H \otimes \varphi(a) = u \otimes 1_E(\varphi(a) \oplus 1_H \otimes \varphi(a))\, u^* \otimes 1_E.$$

Finally one checks that $w = (vu(1_H \oplus v^*)\, s^*) \otimes 1_E \in L(H) \otimes \mathbb{C}1_E \subset M(L(H) \otimes B)$ is a unitary satisfying $w\theta_{\varphi,\eta} w^* = \eta$. ∎

The following proposition and its proof were inspired by [Ki].

Proposition 2.3. *Let A be a separable C^*-algebra and let $(\varphi_t)\colon A \to L(H)$ be a CP-asymptotic morphism. Then there are strongly continuous families of representations $(\rho_t)\colon A \to L(H)$ and $(\pi_t)\colon A \to L(H \oplus H)$ such that $\|\varphi_t(a) \oplus \rho_t(a) - \pi_t(a)\| \to 0$ for all $a \in A$.*

Proof. Set $B = L(H) \otimes K(H)$ and identify CB with $C_0(\mathbb{R}_+, B)$. Following [Ki] we associate with (φ_t) an invertible element of $Ext(A, CB)$. Specifically, let

$$\chi\colon A \to C_b(\mathbb{R}_+, B) \subset C_{b,s}(\mathbb{R}_+, M(B)) \cong M(C_0(\mathbb{R}_+, B))$$

be defined by $\chi_t(a) = \varphi_t(a) \otimes p$, where $p \in K(H)$ is a rank one orthogonal projection. One sees immediately that $\chi\colon A \to M(C_0(\mathbb{R}_+, B))$ is a CP-map and $\chi(ab) - \chi(a)\,\chi(b) \in C_0(\mathbb{R}_+, B)$ for all $a, b \in A$. It follows from [Ka_1] that χ defines an element of $Ext(A, CB)^{-1} \cong KK_1(A, CB)$. Since the latter group vanishes by the contractibility of CB (see [Ka_1]), or by Proposition 2.2, it follows from the very definition of the Ext-group that there are $*$-homomorphisms $\rho\colon A \to M(CB)$ and $\pi\colon A \to M_2(M(CB))$ such that

$$\chi(a) \oplus \rho(a) - \pi(a) \in M_2(CB) \cong C_0(\mathbb{R}_+, M_2(B)). \tag{1}$$

Via the isomorphism $M(CB) \cong C_{b,s}(\mathbb{R}_+, M(B))$ we can identify ρ and π with strictly continuous families of $*$-homomorphisms $(\rho_t)\colon A \to M(B)$ and $(\pi_t)\colon A \to M_2(M(B))$. Since the inclusion of $M(B) = M(L(H) \otimes K(H))$ in $M(K(H) \otimes K(H)) \cong L(H \otimes H)$ is strictly continuous, $(\rho_t)\colon A \to L(H \otimes H)$ and $(\pi_t)\colon A \to M_2(L(H \otimes H))$ are strongly continuous families of representations. Note that (χ_t) acts on the subspace $H \cong H \otimes pH$ of $H \otimes H \oplus H \otimes H$. Let u be a unitary operator identifying the orthogonal complement of $H \otimes pH$ with H and let $v = 1_{H \otimes pH} \oplus u$. Then it follows from (1) that

$$\lim_{t \to \infty} \|\varphi_t(a) \oplus u\rho_t(a)\,u^* - v\pi_t(a)\,v^*\| = 0$$

for all $a \in A$. ∎

Let $P(H)$ denote the set of all finite dimensional orthogonal projections acting on H. $P(H)$ is filtrated by the order $p \leqslant q$ if and only if $pq = p$. Recall from [PPV] that the modulus of quasidiagonality of a set $F \subset L(H)$ is defined by

$$qd(F) = \liminf_{p \in P(H)} \sup_{a \in F} \|[a, p]\|.$$

If the set F is finite, then F is quasidiagonal if and only if $qd(F) = 0$.

A representation $\rho: A \to L(H)$ will be called *essentially degenerated* if the orthogonal complement of $\rho(A) H$ in H is infinite dimensional.

COROLLARY 2.4. *Let A be a separable C^*-algebra and let $F \subset A$ be a finite set. Let $(\varphi_t): A \to L(H)$ be a CP-asymptotic morphism and let $\rho: A \to L(H)$ be an essential, essentially degenerated representation. If A is quasidiagonal, then* $\lim_{t\to\infty} qd((\varphi_t \oplus \rho)(F)) = 0$.

Proof. Let ρ_t and π_t be as in Proposition 2.3. By taking direct sum with a suitable representation, we may assume that both ρ_t and π_t are essential and essentially degenerated. Since A is quasidiagonal, $qd(\pi_t(F)) = 0$ for all t. It follows from Proposition 2.3 that $\lim_{t\to\infty} qd((\varphi_t \oplus \rho_t)(F)) = 0$. By Voiculescu's theorem [V_1] $\varphi_t \oplus \rho_t$ is approximately unitarily equivalent to $\varphi_t \oplus \rho$ for all $t \geqslant 0$.

This easily implies that

$$qd((\varphi_t \oplus \rho_t)(F)) = qd((\varphi_t \oplus \rho)(F)).$$

We conclude that $\lim_{t\to\infty} qd((\varphi_t \oplus \rho)(F)) = 0$. ∎

The first part of following result is implicitly contained in the proof of [V_2, Proposition 3]. We reproduce the complete argument for it is the most important single piece that explains the homotopy invariance of quasidiagonality.

PROPOSITION 2.5 (Cf. [V_2, Proposition 3]). *Let A, B be a C^*-algebras and let $\chi: A \to B$ be a $*$-homomorphism. Suppose that for any $\delta > 0$, any $F \subset A$ a finite subset and any $x \in F$, there is a continuous path $s \mapsto \Phi^{(s)}$ in $CP(A, L(H))$, $s \in [0, 1]$ such that*

(i) $\|\Phi^{(s)}(ab) - \Phi^{(s)}(a)\,\Phi^{(s)}(b)\| < \delta$ *for all $a, b \in F$ and $s \in [0, 1]$.*

(ii) $\|\Phi^{(0)}(x)\| > \|\chi(x)\| - \delta$

(iii) $qd((\Phi^{(1)}(F)) < \delta$.

Then we have the following:

(a) *If χ is isometric, then A is quasidiagonal.*

(b) *If A is separable and exact, then χ factors through a quasidiagonal C^*-algebra.*

Proof. Fix ε_1, $F \subset A$ a finite subset and $x \in F$. The first segment of the proof is common to (a) and (b) and it is almost identical to the original argument of Voiculescu. It consists of finding a finite dimensional C^*-algebra C and a map $\psi \in CP(A, C)$ such that $\|\psi(ab) - \psi(a)\,\psi(b)\| < \varepsilon_1$ for $a, b \in F$ and $\|\psi(x)\| \geqslant \|\chi(x)\| - \varepsilon_1$. Assuming that ψ has been found, we prove (a) and (b) as follows. If χ is isometric then $\|\psi(x)\| \geqslant \|x\| - \varepsilon_1$, hence A is quasidiagonal by Theorem 2.1. The proof of part (b) depends on a subsequent result. If A is exact, then χ factors through a quasidiagonal C*-algebra by Proposition 4.4.

We now proceed to constructing ψ. Set $M = \sup\{\|a\| : a \in F\}$ and $\varepsilon = \varepsilon_1/(M+1)$. As in [Ar] one can find $\delta > 0$ such that for every pair $y, z \in L(H)$ with $0 \leqslant y \leqslant 1$ and $\|z\| \leqslant M$ one has that $\|[y, z]\| < 4\delta$ implies $\|y^{1/2}, z]]\| < \varepsilon/10$. We may assume that $\delta < \varepsilon/10$. For this particular δ, let $\Phi^{(s)}$ be as in the statement of the proposition. By uniform continuity there is an integer n such that if $|s_1 - s_2| < 1/n$ then $\|\Phi^{(s_1)}(a) - \Phi^{(s_2)}(a)\| < \delta$ for all $a \in F$. Let H^{n+1} be the direct sum of $n+1$ copies of H and let $\varphi \in CP(A, L(H^{n+1}))$ be defined by $\varphi(a) = diag(\varphi_0(a), \varphi_1(a), ..., \varphi_n(a))$ where $\varphi_i(a) = \Phi^{(i/n)}(a)$, $a \in A$. It is clear that $\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$ for $a, b \in F$. Using (ii) we find $q \in P(H)$ such that

$$\|q\Phi^{(0)}(x)\, q\| > \|\chi(x)\| - \delta > \|\chi(x)\| - \varepsilon_1.$$

By general facts on central approximate units [Ar, P] there are finite rank positive operators $x_j \in L(H)$ with

$$q \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1_H$$

such that $x_{j+1}x_j = x_j$ for $0 \leqslant j \leqslant n-1$ and $\|[x_j, \varphi_j(a)]\| < \delta$ for $0 \leqslant j \leqslant n$, $a \in F$. Since $qd(\Phi^{(1)}(F)) < \delta$, we can choose x_n to be a finite rank orthogonal projection. Let $v_n: H \to H^{n+1}$ be given by $v_n = x_0^{1/2} \oplus (x_1 - x_0)^{1/2} \oplus \cdots \oplus (x_n - x_{n-1})^{1/2}$. Then $v_n^* v_n = x_n$ hence v_n is a partial isometry. We have for $a \in F$

$$\begin{aligned}\|[x_{j+1} - x_j, \varphi_{j+1}(a)]\| &\leqslant \|[x_{j+1}, \varphi_{j+1}(a)]\| + \|[x_j, \varphi_j(a)]\| \\ &\quad + 2\,\|\varphi_{j+1}(a) - \varphi_j(a)\| < 4\delta;\end{aligned}$$

hence, by our choice of δ, $\|[(x_{j+1} - x_j)^{1/2}, \varphi_{j+1}(a)]\| < \varepsilon/10$. Letting $y_0 = x_0^{1/2}$ and $y_{j+1} = (x_{j+1} - x_j)^{1/2}$, $0 \leqslant j \leqslant n-1$ note that $y_k y_j = 0$ if $|j-k| > 1$, so that the projection $p = v_n v_n^* = (y_k y_j) \in L(H^{n+1})$ is tridiagonal. We are going to show that p approximately commutes with $\varphi(a)$ for $a \in F$. Indeed

$$\begin{aligned}\|[p, \varphi(a)]\| &\leqslant 3 \sup_{|j-k| \leqslant 1} \|\varphi_j(a)\, y_j y_k - y_j y_k \varphi_k(a)\| \\ &\leqslant 6 \sup_{0 \leqslant j \leqslant n} \|[\varphi_j(a), y_j]\| + 3 \sup_{|j-k| \leqslant 1} \|\varphi_j(a) - \varphi_k(a)\| \\ &\leqslant 6\varepsilon/10 + 3\delta < \varepsilon\end{aligned}$$

since $\delta < \varepsilon/10$. Define $\psi \in CP(A, pL(H^{n+1})\,p)$ by $\psi(a) = p\varphi(a)\,p$. Note that $p \geqslant x_0 \geqslant q$. Then $\|\psi(x)\| = \|p\varphi(x)\,p\| \geqslant \|q\Phi^{(0)}(x)\,q\| > \|\chi(x)\| - \varepsilon_1$ and

$$\begin{aligned}\|\psi(ab) - \psi(a)\,\psi(b)\| &\\ &\leqslant \|p(\varphi(ab) - \varphi(a)\,\varphi(b))\,p\| + \|p\varphi(a)\,\varphi(b)\,p - p\varphi(a)\,p\varphi(b)\,p\| \\ &< \delta + \|\varphi(a)\|\,\|[\varphi(b), p]\| \leqslant \delta + M\varepsilon < (M+1)\,\varepsilon = \varepsilon_1\end{aligned}$$

for all $a, b \in F$. ∎

Let $(\varphi_t)\colon A \to B$ be an asymptotic morphism and let $\dot{\varphi}\colon A \to B_\infty$ be the associated $*$-homomorphism. Then $\|\dot{\varphi}(a)\| = \limsup_{t\to\infty} \|\varphi_t(a)\|$; hence

$$\ker(\varphi_t) := \ker \dot{\varphi} = \{a \in A\colon \lim_{t\to\infty} \|\varphi_t(a)\| = 0\}$$

is a closed two-sided ideal of A. Let μ denote the canonical map $A \to A/\ker(\varphi_t)$. Then (φ_t) is equivalent to $(\hat{\varphi}_t\mu)$, where $(\hat{\varphi}_t)\colon A/\ker(\varphi_t) \to B$ is an asymptotic morphism with trivial kernel. Since μ is surjective, $(\hat{\varphi}_t)$ is unique up to an equivalence. An asymptotic morphism $(\varphi_t)\colon A \to B$ is called *weakly injective* if $\ker(\varphi_t) = 0$, or equivalently if $\limsup_{t\to\infty} \|\varphi_t(a)\| = \|a\|$, for all $a \in A$. We call (φ_t) *injective* if $\lim_{t\to\infty} \|\varphi_t(a)\| = \|a\|$, for all $a \in A$ (see [L_1]).

The following is a version of [V_2, Theorem 4] for CP-asymptotic morphisms.

THEOREM 2.6. *Let A, D, M be separable C^*-algebras. Let $(\psi_t)\colon A \to M$, $(\mu_t)\colon A \to D$ and $(\varphi_t)\colon D \to M$ be CP-asymptotic morphisms. Suppose that $[[\psi_t]] = [[\varphi_t]][[\mu_t]]$ in the CP-asymptotic homotopy category $\mathscr{A}_{CP}$. If (ψ_t) is weakly injective and D is quasidiagonal then A is quasidiagonal.*

Proof. We may assume that $M \subset L(H)$. Fix $\delta > 0$, $F \subset A$ a finite subset and $x \in F$. Let $\rho\colon D \to L(H)$ be an essential, essentially degenerated representation. Since D is quasidiagonal, it follows from Corollary 2.4, that for any $t > 0$

$$\lim_{r\to\infty} qd((\varphi_r \oplus \rho)(\mu_t(F))) = 0.$$

Let (r_n) be a sequence in $\mathbb{R}_+$ such that $qd((\varphi_r \oplus \rho)(\mu_n(F))) < 1/n$, for all $r > r_n$ and $n \in \mathbb{N}$. Let $r\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism such that $r(n) > r_n$. By replacing $r(t)$ by a bigger function we can arrange that $[[\varphi_t]][[\mu_t]] = [[\varphi_{r(t)}\mu_t]]$. Since ψ_t is weakly injective, there is an increasing homeomorphism $\ell\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\psi_{\ell(n)}(x)\| > \|x\| - 1/n$ for all $n \in \mathbb{N}$. It is clear that $(\psi_{\ell(t)})$ is CP-homotopic to (ψ_t). Let

$(\Gamma_t^{(s)})\colon A \to M$ be a CP-asymptotic homotopy with $\Gamma_t^{(0)} = \psi_{\ell(t)}$ and $\Gamma_t^{(1)} = \varphi_{r(t)}\mu_t$. Define $\Phi_t^{(s)}\colon A \to L(H \oplus H)$ by $\Phi_t^{(s)} = \Gamma_t^{(s)} \oplus \rho\mu_t$. Then $(\Phi_t^{(s)})$ is a CP-asymptotic homotopy from $(\psi_{\ell(t)} \oplus \rho\mu_t)$ to $(\varphi_{r(t)}\mu_t \oplus \rho\mu_t)$. We have

$$\lim_{n\to\infty} qd(\Phi_n^{(1)}(F)) = \lim_{n\to\infty} qd((\varphi_{r(n)} \oplus \rho)(\mu_n(F))) = 0$$

since $r(n) > r_n$. On the other hand,

$$\lim_{n\to\infty} \|\Phi_n^{(0)}(x)\| \geqslant \lim_{n\to\infty} \|\psi_{\ell(n)}(x)\| = \|x\|.$$

It is then clear that there exists big enough $n \in \mathbb{N}$, such that $\Phi_n^{(s)}$ satisfies the conditions (i), (ii) and (iii) of Proposition 2.5, with $\chi = id_A$. The desired conclusion follows now from the part (a) of that Proposition. ∎

Note that Theorem 1.1 is a direct corollary of Theorem 2.6. Theorem 1.1 shows that quasidiagonality is an invariant of isomorphism classes in $\mathscr{A}_{CP}$. We now turn to the nuclear case.

THEOREM 2.7. *Let A, B be separable C^*-algebras. Suppose that A is dominated by B in the asymptotic homotopy category $\mathscr{A}$; i.e., there are asymptotic morphisms $(f_t)\colon A \to B$ and $(g_t)\colon B \to A$ such that $[[g_t]][[f_t]] = [[id_A]]$. If A is nuclear and B is quasidiagonal, then A is quasidiagonal.*

Proof. Note that we cannot directly apply Theorem 2.6, for we do not know that (g_t) is homotopic to a CP-asymptotic morphism. However, one can proceed as follows. Let J denote the kernel of (f_t) and let $\mu\colon A \to A/J$ be the canonical map. A/J is nuclear as a quotient of a nuclear C*-algebra [ChE]. Therefore we can find a weakly injective CP-asymptotic morphism $(\hat{f}_t)\colon A/J \to B$ such that $(\hat{f}_t\mu)$ is equivalent to (f_t). Let $\delta > 0$, let $F \subset A/J$ be a finite set and let $x \in F$. Since $(\hat{f}_t)$ is weakly injective $\limsup_{t\to\infty} \|\hat{f}_t(x)\| = \|x\|$, hence we can find a t large enough such that $\|\hat{f}_t(ab) - \hat{f}_t(a)\,\hat{f}_t(b)\| < \delta$ for all $a, b \in F$ and $\|\hat{f}_t(x)\| > \|x\| - \delta$. Since B is quasidiagonal, it follows that A/J is quasidiagonal by Theorem 2.1.

Since A/J is nuclear, we can find a CP-asymptotic morphism $(h_t)\colon A/J \to A$ such that $[[g_t]][[\hat{f}_t]] = [[h_t]]$. Then

$$[[h_t\mu]] = [[g_t]][[\hat{f}_t]][[\mu]] = [[g_t]][[f_t]] = [[id_A]].$$

Since A is nuclear it follows that $[[h_t]][[\mu]] = [[id_A]]$ in $\mathscr{A}_{CP}$. We conclude that A is quasidiagonal by Theorem 2.6. ∎

COROLLARY 2.8. *Let A, B be separable nuclear C^*-algebras. Suppose that A is shape equivalent to B. If B is quasidiagonal then A is quasidiagonal.*

Proof. This is a consequence of Theorem 2.7 since by [D] any shape equivalence lifts to an isomorphism in $\mathscr{A}$. ∎

A C*-algebra A is called homotopy symmetric if $[[id_A]]$ is invertible in the semi-group $[[A, A \otimes K(H)]]$, the neutral element being represented by the class of the null morphism. Our interest in such algebras comes from the fact that the suspension map $[[A, B \otimes K(H)]] \to [[SA, SB \otimes K(H)]]$ is a bijection if and only if A is homotopy symmetric (see [DL]).

Corollary 2.9. *Any separable nuclear homotopy symmetric C*-algebra is quasidiagonal.*

Proof. Apply Theorem 2.6 with $M = A \otimes K(H)$, $D = 0$, and $(\psi_t) = id_A \oplus (\gamma_t)$, where (γ_t) is a CP-asymptotic morphism such that $[[id_A]] + [[\gamma_t]] = 0$ in $[[A, A \otimes K(H)]]_{CP}$. ∎

Remark 2.10. A similar argument shows that if A is separable, nuclear, homotopy symmetric and if B is a separable C*-algebra, then $A \otimes B$ is quasidiagonal. More generally, if A is any separable C*-algebra such that $[[A, A \otimes K(H)]]_{CP}$ is a group, then A is quasidiagonal.

3. EXACT C*-ALGEBRAS

In this section we collect a number of useful facts about exact C*-algebras. We will freely use two results of Kirchberg [Ki] asserting that an exact C*-algebra is nuclearly embeddable and locally reflexible. (see also [W]). The notions of local reflexibility and nuclear embeddability were introduced in [EH] and respectively [V_4].

Proposition 3.1 [EH]. *Suppose that A is a locally reflexible (or exact) C*-algebra. Then for any closed two sided ideal J in A, every unital CP-map of a finite dimensional operator system E into A/J is liftable to a unital CP-map from E to A.*

Proof. The result follows from Theorem 3.2 and Proposition 5.3 of [EH]. ∎

We are indebted to E. Kirchberg for pointing out to us the following two propositions.

Proposition 3.2. *Let A, B be C*-algebras and let $\varphi: A \to B$ be a nuclear CP-map. Suppose that $J \subset \ker \varphi$ is a closed two sided ideal of A. If A is locally reflexible (or exact), then the induced map $\hat{\varphi}: A/J \to B$ is a nuclear CP-map.*

Proof. By a standard reduction it is enough to consider the case when both A and B are unital and φ is unit preserving. Given $\{x_1, ..., x_r\} \subset A/J$ and $\varepsilon > 0$ we will find unital CP-maps $\alpha: A/J \to M_n$ and $\beta: M_n \to B$ such that $\|\hat{\varphi}(x_i) - \beta\alpha(x_i)\| < \varepsilon$ for $1 \leqslant i \leqslant r$. Let E be the finite dimensional operator system

$$E = \mathbb{C}1 + \sum_{i=1}^{r} \mathbb{C}x_i + \sum_{i=1}^{r} \mathbb{C}x_i^* \subset A/J.$$

Let $\mu: A \to A/J$ be the canonical map. Since A is locally reflexive, by Proposition 3.1 we find a unital CP-map $\sigma: E \to A$ such that $\mu\sigma(x_i) = x_i$ for $1 \leqslant i \leqslant r$. Since φ is a unital nuclear CP-map, there exist unital CP-maps $\alpha_1: A \to M_n$ and $\beta: M_n \to B$ such that $\|\varphi\sigma(x_i) - \beta\alpha_1\sigma(x_i)\| < \varepsilon$ for $1 \leqslant i \leqslant r$. By Arveson's extension theorem [W, 1.8], $\alpha_1\sigma$ extends to a unital CP-map $\alpha: A/J \to M_n$. Then

$$\|\hat{\varphi}(x_i) - \beta\alpha(x_i)\| = \|\varphi\sigma(x_i) - \beta\alpha_1\sigma(x_i)\| < \varepsilon$$

for $1 \leqslant i \leqslant r$. ∎

PROPOSITION 3.3. *Let A be a C^*-algebra and let $\varphi_n: A \to B_n$ be a sequence of nuclear CP-maps to C^*-algebras B_n. If A is nuclearly embeddable (or exact), then $\varphi = (\varphi_n) : A \to \prod B_n$ is a nuclear CP-map.*

Proof. Let $\xi: A \to C$ be a nuclear embedding of A into some C*-algebra C. Let F be a finite subset of A and let $\varepsilon > 0$. We will find a CP-map $\eta: C \to \prod B_n$ such that $\|\varphi(a) - \eta\xi(a)\| < \varepsilon$ for all $a \in F$. Since ξ is nuclear, that will imply the nuclearity of φ.

Since each φ_n is nuclear, we find a sequence $(k(n))$ of positive integers and CP-maps $\alpha_n: A \to M_{k(n)}$, $\beta_n: M_{k(n)} \to B_n$ such that $\|\varphi_n(a) - \beta_n\alpha_n(a)\| < \varepsilon$ for all $a \in F$. By Averson's extension theorem, α_n extends to a CP-map $\tilde{\alpha}_n: C \to M_{k(n)}$ with $\tilde{\alpha}_n\xi = \alpha_n$. Define $\eta: C \to \prod B_n$ by $\eta(z) = (\beta_n\tilde{\alpha}_n(z))$. Then

$$\|\varphi(a) - \eta\xi(a)\| = \sup_n \|\varphi_n(a) - \beta_n\tilde{\alpha}_n\xi(a)\| = \sup_n \|\varphi_n(a) - \beta_n\alpha_n(a)\| < \varepsilon$$

for all $a \in F$. ∎

PROPOSITION 3.4. *Let A, B be C^*-algebras and let T be a locally compact, σ-compact space. Suppose that A is nuclearly embeddable (or exact). Then a CP-map $\varphi: A \to C_b(T, B)$ is nuclear if and only if each $\varphi_t: A \to B$, (where $\varphi_t(a) = \varphi(a)(t)$, $t \in T$) is nuclear.*

Proof. We discuss only the nontrivial implication. Fix $F \subset A$ a finite subset and $\varepsilon > 0$. Find a sequence $(t(n))$ in T and a locally finite open cover

(V_n) of T such that for all n, $\|\varphi_t(a)-\varphi_{t(n)}(a)\|<\varepsilon$ for all $a\in F$ and $t\in V_n$. Let (χ_n) be a continuous partition of unity subordinated to (V_n). Define $\xi\colon A\to\prod B_n$ $(B_n=B)$ by $\xi(a)=(\varphi_{t(n)}(a))$ and $\eta\colon\prod B_n\to C_b(T,B)$ by $\eta((b_n))=\sum_n\chi_n b_n$. One checks that ξ, η are CP-maps and $\|\varphi(a)-\eta\xi(a)\|<\varepsilon$ for all $a\in F$. The map ξ is nuclear by Proposition 3.3. It follows that φ is a nuclear map. ∎

Proposition 3.5 [V_4]. *If A is a nuclearly embeddable (or exact) C^*-algebra, then any CP-map $\varphi\colon A\to L(H)$ is nuclear.*

Proof. We include a short alternative argument. Let $\gamma\colon A\to C$ be a nuclear embedding into some C*-algebra C. Using Arveson's extension theorem we extend $\varphi\gamma^{-1}\colon\gamma(A)\to L(H)$ to a CP-map $\psi\colon C\to L(H)$. Then $\varphi=\psi\gamma$ hence φ is nuclear. ∎

4. QUASIDIAGONAL $*$-HOMOMORPHISMS

Definition 4.1. Let A, B be C*-algebras. A $*$-homomorphism $\varphi\colon A\to B$ is called quasidiagonal if it factors through a quasidiagonal C*-algebra; i.e., there is a quasidiagonal C*-algebra D and there are $*$-homomorphisms $\mu\colon A\to D$ and $\nu\colon D\to B$ such that $\varphi=\nu\mu$.

Remarks 4.2. (i) φ is quasidiagonal if and only if the quotient map $A\to A/\ker\varphi$ is quasidiagonal, if and only if there is a closed two sided ideal J in A such that $J\subset\ker\varphi$ and A/J is quasidiagonal.

(ii) If φ is quasidiagonal, then there is a factorization of φ as in Definition 4.1, such that μ is surjective and $\nu=\hat{\varphi}\nu_1$ where ν_1 is the quotient map $D\cong A/\ker\mu\to A/\ker\varphi$ and $\hat{\varphi}\colon A/\ker\varphi\to B$ is the canonical map induced by φ.

(iii) Any $*$-homomorphism which factors through a quasidiagonal $*$-homomorphism is quasidiagonal. If $\varphi\oplus\psi$ is quasidiagonal, then both φ and ψ are quasidiagonal.

(iv) If $\psi\colon B\to C$ is a $*$-monomorphism, then $\psi\varphi$ is quasidiagonal if and only if φ is quasidiagonal. A is quasidiagonal if and only if id_A is quasidiagonal.

Proposition 4.3. *Let A be a separable C^*-algebra and let $\rho\colon A\to L(H)$ be a $*$-representation. Then the following conditions are equivalent.*

(i) *ρ is quasidiagonal.*

(ii) *There is a $*$-representation $\sigma\colon A\to L(H)$ such that $(\rho\oplus\sigma)(A)$ is a quasidiagonal set of operators.*

Proof. (i) $\Rightarrow$ (ii) Write $\rho = \nu\mu$ where $\mu: A \to D$ and $\nu: D \to L(H)$ are $*$-homomorphisms and D is quasidiagonal. Let $\sigma_1: D \to L(H)$ be an essential representation of D and set $\sigma = \sigma_1 \mu$. Then $(\rho \oplus \sigma)(A) \subset (\nu \oplus \sigma_1)(D)$ and the latter is a quasidiagonal set of operators in $L(H \oplus H)$.

(ii) $\Rightarrow$ (i) If $J = \ker(\rho \oplus \sigma)$, then $A/J \cong (\rho \oplus \sigma)(A)$, hence A/J is quasidiagonal. Since $J \subset \ker \rho$, it follows that ρ factors through A/J. ∎

As a consequence of the above, we see that a $*$-homomorphism $\varphi: A \to B$ is quasidiagonal if and only if for every (or for some) faithful $*$-representation $\rho: B \to L(H)$, there is a $*$-representation $\sigma: A \to L(H)$ such that $(\rho\varphi \oplus \sigma)(A)$ is a quasidiagonal set of operators.

PROPOSITION 4.4. *Let A, B be C^*-algebras and let $\varphi: A \to B$ be a $*$-homomorphism. Suppose that for any $\delta > 0$, $F \subset A$ a finite subset and $x \in F$, there is a finite dimensional (or just quasidiagonal) C^*-algebra C and there is $\psi \in CP(A, C)$ such that $\|\psi(x)\| > \|\varphi(x)\| - \delta$ and $\|\psi(ab) - \psi(a)\,\psi(b)\| < \delta$ for all $a, b \in F$. If A is separable and exact, then φ is quasidiagonal.*

Proof. We give the proof only in the case when C is finite dimensional. The proof for quasidiagonal C is reduced to that case by composing ψ with suitable CP-maps into finite dimensional C*-algebras. Let F_n be an increasing sequence of finite subsets of A whose union is dense in A. Using the assumption we find a sequence $\psi_n: A \to C_n$ of CP-maps to finite dimensional C*-algebras such that $\|\psi_n(ab) - \psi_n(a)\,\psi_n(b)\| < 1/n$ and $\|\psi_n(a)\| > \|\varphi(a)\| - 1/n$ for all $a, b \in F_n$. This immediately implies that $\lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\,\psi_n(b)\| = 0$ and $\liminf_{n\to\infty} \|\psi_n(a)\| \geqslant \|\varphi(a)\|$ for all $a, b \in A$. Since A is exact, we obtain from Proposition 3.3 that $\psi: A \to \prod C_n$, $\psi(a) = (\psi_n(a))$ is a nuclear CP-map, hence the associated $*$-homomorphism $\dot{\psi}: A \to \prod C_n / \sum C_n$ is nuclear. Letting $J = \ker \dot{\psi}$ we see from Proposition 3.2 that the induced $*$-monomorphism $\dot{\hat{\psi}}: A/J \to \prod C_n / \sum C_n$ is nuclear. Let $\hat{\psi} = (\hat{\psi}_n) : A/J \to \prod C_n$ be a CP-lifting of $\dot{\hat{\psi}}$ given by the Choi–Effros theorem. Then $(\hat{\psi}_n \mu - \psi_n) \to 0$ in the point norm topology, where $\mu: A \to A/J$ is the quotient map. This implies that for any $a \in A$

$$\|\mu(a)\| = \|\dot{\hat{\psi}}\mu(a)\| = \limsup_{n\to\infty} \|\hat{\psi}_n \mu(a)\| = \limsup_{n\to\infty} \|\psi_n(a)\| \geqslant \|\varphi(a)\| \quad (2)$$

Therefore $\hat{\psi}_n: A/J \to C_n$ is a sequence of CP-maps such that $\|\hat{\psi}_n(yz) - \hat{\psi}_n(y)\,\hat{\psi}_n(z)\| \to 0$ and $\limsup_{n\to\infty} \|\hat{\psi}_n(y)\| = \|y\|$ for all $y, z \in A/J$ by (2). By Theorem 2.1 it follows that A/J is quasidiagonal. On the other hand $J = \ker \mu \subset \ker \varphi$ by (2). We conclude that φ factors through the quasidiagonal C*-algebra A/J. ∎

The following proposition is known to specialists. For nuclear C*-algebras the result is due to Davidson, Herrero, and Salinas [DHS].

We include it here only because it is a straightforward consequence of the proof of Proposition 4.4.

PROPOSITION 4.5. *Let $A \subset L(H)$ be a separable exact C*-algebra. If A is quasidiagonal as a set of operators, then its image in the Calkin algebra is a quasidiagonal C*-algebra.*

Proof. Let π denote the map of $L(H)$ onto the Calkin algebra. We want to show that $\pi(A)$ is quasidiagonal. By assumption there is a sequence (p_n) of finite dimensional orthogonal projections converging strongly to 1_H such that $\|p_n a - a p_n\| \to 0$ for all $a \in A$. Define $\psi_n \colon A \to A + K(H)$ by $\psi_n(a) = (1 - p_n)\, a(1 - p_n)$. Then (ψ_n) is an asymptotically multiplicative sequence of CP-maps into a quasidiagonal C*-algebra such that $\|\psi_n(a)\| \to \|\pi(a)\|$ as $n \to \infty$. As in the proof of Proposition 4.4, we see that if $J := \ker \dot{\psi}$, then A/J is quasidiagonal. On the other hand since $\ker \dot{\psi} = \ker \pi|_A$, we have $\pi(A) \cong A/J$, hence $\pi(A)$ is quasidiagonal. ∎

It was shown by S. Wasserman that Proposition 4.5 it is not true for arbitrary quasidiagonal separable C*-algebras $A \subset L(H)$ (see [W, 3.4]).

DEFINITION 4.6. A $*$-homomorphism $\varphi \colon A \to B$ has the multiplicative-CPAP (completely positive approximation property) if for any finite subset $F \subset A$ and any $\varepsilon > 0$, there is a finite dimensional C*-algebra C and there are $\alpha \in CP(A, C)$ and $\beta \in CP(C, B)$ such that $\|\alpha(ab) - \alpha(a)\,\alpha(b)\| < \varepsilon$ and $\|\varphi(a) - \beta\alpha(a)\| < \varepsilon$ for all $a, b \in F$.

Definition 4.6 was inspired by a result of Blackadar and Kirchberg [BKi] who proved that a C*-algebra A is quasidiagonal and nuclear if and only if id_A has the multiplicative-CPAP. Their result easily extends to $*$-homomorphisms from exact C*-algebras, see Theorem 4.8. The following Lemma can be derived from the proof of Theorem 5.2.2 of [BKi].

LEMMA 4.7. *Let A be a quasidiagonal C*-algebra and let $\varphi \colon A \to B$ be a nuclear $*$-homomorphism to a C*-algebra B. Then φ has the multiplicative*-CPAP.

Proof. Since A is quasidiagonal, by Theorem 2.1 we find a CP-map $\sigma = (\sigma_n) : A \to \prod M_{k(n)}$ with asymptotically multiplicative components and $\lim_{n \to \infty} \|\sigma_n(a)\| = \|a\|$ for all $a \in A$. Note that σ is a complete order embedding. From this point the proof mimics the proof of the implication (iii) $\Rightarrow$ (vi) of Theorem 5.2.2 of [BKi]. The only change is to replace id_A by φ. ∎

THEOREM 4.8. *Let $\varphi: A \to B$ be a $*$-homomorphism from a separable exact C^*-algebra A to a C^*-algebra B. Then the following conditions are equivalent.*

(i) *φ is nuclear and quasidiagonal.*

(ii) *φ has the multiplicative*-CPAP.

Proof. (ii) $\Rightarrow$ (i) Let F, ε, α, β and C be as in Definition 4.6. Then $\|\varphi(a)\| \leqslant \|\beta\alpha(a)\| + \varepsilon \leqslant \|\alpha(a)\| + \varepsilon$. Since A is exact and C is finite dimensional the conclusion follows by Proposition 4.4, applied for $\psi = \alpha$.

(i) $\Rightarrow$ (ii) By Remark 4.2(i) there is a closed two-sided ideal J in A such that $J \subset \ker \varphi$ and A/J is quasidiagonal. Let $\mu: A \to A/J$ be the canonical map and let $\nu: A/J \to B$ be the map induced by φ such that $\varphi = \nu\mu$. Since φ is nuclear it follows that ν is nuclear by Proposition 3.2. Since A/J is quasidiagonal, ν has the multiplicative-CPAP by Lemma 4.7. We conclude that $\varphi = \nu\mu$ has this property. ∎

COROLLARY 4.9. *Let A be a separable, exact C^*-algebras. A $*$-representation $\rho: A \to L(H)$ is quasidiagonal if and only if ρ has the multiplicative*-CPAP.

The following result is a consequence of Theorem 1.2 which is proved in Section 5. However, rather than deriving it later, we choose to include here a streamlined proof which does not require the notion of a quasidiagonal asymptotic morphism.

THEOREM 4.10. *Let $\varphi, \psi: A \to B$ be two $*$-homomorphisms from a separable, exact C^*-algebra A to a C^*-algebra B. Suppose that φ is CP-asymptotically homotopic to ψ. If ψ is quasidiagonal, then φ is quasidiagonal.*

Proof. Without loss of generality, by Remark 4.2(iv) we may assume that $B \subset L(H)$. Let $(\Gamma_t^{(s)}): A \to B$ be a CP-homotopy with $\Gamma_t^{(0)} = \varphi$ and $\Gamma_t^{(1)} = \psi$ for all $t \geqslant 0$. Write $\psi = \nu\mu$ with $\mu: A \to D$, $\nu: D \to B$, and D a quasidiagonal C^*-algebra. Let $\rho: D \to L(H)$ be an essential $*$-representation. Then $(\psi \oplus \rho\mu)(A) \subset (\nu \oplus \rho)(D)$ is a quasidiagonal set of operators; hence $qd(\psi \oplus \rho\mu)(F) = 0$ for any finite subset $F \subset A$. Fix such an F, $x \in F$, and $\varepsilon > 0$. Set $\Phi_t^{(s)} = \Gamma_t^{(s)} \oplus \rho\mu$. Then $\Phi_t^{(s)}$ is a CP-homotopy from $\varphi \oplus \rho\mu$ to $\psi \oplus \rho\mu$, and which for some large t will satisfy the conditions of Proposition 2.5(b) for $\chi = \varphi$. Indeed $\Phi_t^{(s)}$ is asymptotically multiplicative as $t \to \infty$, $qd(\Phi_t^{(1)}(F)) = 0$ and $\|\Phi_t^{(0)}(x)\| = \|\varphi(x) \oplus \rho\mu(x)\| \geqslant \|\varphi(x)\|$. We conclude that φ is quasidiagonal. ∎

Combining Theorem 4.8 and Theorem 4.10 we see that the multiplicative-CPAP is a homotopy invariant for nuclear $*$-homomorphisms from separable exact C*-algebras.

5. QUASIDIAGONAL ASYMPTOTIC MORPHISMS

DEFINITION 5.1. A CP-asymptotic representation $(\varphi_t)\colon A \to L(H)$ of a separable C*-algebra A is called quasidiagonal if there exists a $*$-representation $\sigma\colon A \to L(H)$ such that for any finite subset $F \subset A$, $\lim_{t\to\infty} qd(\varphi_t \oplus \sigma)(F) = 0$. A CP-asymptotic morphism $(\varphi_t)\colon A \to B$ is called quasidiagonal if there exists a faithful $*$-representation $\rho\colon B \to L(H)$ such that $(\rho\varphi_t)$ is quasidiagonal.

Note that by Corollary 2.4 any CP-asymptotic representation of a quasidiagonal C*-algebra is quasidiagonal. More generally, the composition of any CP-asymptotic morphism $(\varphi_t)\colon B \to C$ with a quasidiagonal $*$-homomorphism $\alpha\colon A \to B$ is quasidiagonal. Any quasidiagonal $*$-homomorphism is quasidiagonal when regarded as an asymptotic morphism (see Proposition 4.3.) A CP-asymptotic morphism $(\varphi_t)\colon A \to B$ will be called *nuclear* if the corresponding CP-map $\varphi\colon A \to C_b(\mathbb{R}_+, B)$ is nuclear. For exact C*-algebras A, by Proposition 3.4, this is the case if and only if each individual map φ_t is nuclear. In particular, by Proposition 3.5, every CP-asymptotic representation $(\varphi_t)\colon A \to L(H)$ is nuclear.

PROPOSITION 5.2. *Let A be a separable exact C*-algebra and let $(\varphi_t)\colon A \to B$ be a CP-asymptotic morphism. Then the following conditions are equivalent.*

(i) *(φ_t) is quasidiagonal.*

(ii) *The $*$-homomorphism $\dot\varphi\colon A \to B_\infty$ is quasidiagonal.*

If (φ_t) is nuclear then (i) *and* (ii) *are equivalent to*

(iii) *There is a quasidiagonal C*-algebra D, and there exist a $*$-homomorphism $\mu\colon A \to D$ and a nuclear CP-asymptotic morphism $(\nu_t)\colon D \to B$ such that $(\nu_t\mu)$ is equivalent to (φ_t).*

Proof. Let $\rho\colon B \to L(H)$ be a faithful $*$-representation. By Remark 4.2(iv) $\dot\varphi$ is quasidiagonal if and only if $\rho_\infty\dot\varphi$ is quasidiagonal. Thus by replacing (φ_t) by $(\rho\varphi_t)$, we may assume that $B = L(H)$. As explained in the paragraph preceding Proposition 5.2, in this case φ and hence $\dot\varphi$ are nuclear.

(ii) $\Rightarrow$ (i) Write $\dot{\varphi} = \nu\mu$ with $\mu: A \to D$, $\nu: D \to B_\infty$ and D a quasidiagonal C*-algebra as in Remark 4.2(ii). Then ν is nuclear since it factors through $\hat{\dot{\varphi}}: A/\ker\dot{\varphi} \to B_\infty$ which is nuclear by Proposition 3.2. Let $\gamma: D \to C_b(\mathbb{R}_+, B)$ be a CP-lifting of ν given by the Choi–Effros theorem. Let $\sigma: D \to B = L(H)$ be an essential, essentially degenerated $*$-representation. Since D is quasidiagonal, by Corollary 2.4 $\lim_{t\to\infty} qd(\gamma_t \oplus \sigma)(\mu(F)) = 0$ for any finite subset F of A. On the other hand $\varphi(a) - \gamma\mu(a) \in C_0(\mathbb{R}_+, B)$ for all $a \in A$ as $\dot{\varphi} = \dot{\gamma}\mu = \nu\mu$. Therefore $\lim_{t\to\infty} qd(\varphi_t \oplus \sigma\mu)(F) = 0$, hence (φ_t) is quasidiagonal.

(i) $\Rightarrow$ (ii) Let $\sigma: A \to B = L(H)$ be a $*$-representation (necessarily nuclear by Proposition 3.5) such that

$$\lim_{t\to\infty} qd(\varphi_t \oplus \sigma)(F) = 0 \tag{3}$$

for any finite subset F of A. If we set $\psi = \varphi \oplus \sigma$, then $\dot{\psi}: A \to L(H \oplus H)_\infty$ is nuclear since both φ and σ are nuclear. Since A is exact, the induced $*$-monomorphism $\hat{\dot{\psi}}: A/J \to L(H \oplus H)_\infty$, $J = \ker\dot{\psi} = \ker\dot{\varphi} \cap \ker\sigma$, is nuclear by Proposition 3.2. Let $\hat{\psi}: A/J \to C_b(\mathbb{R}_+, L(H \oplus H))$ be a CP-lifting of $\hat{\dot{\psi}}$ given by the Choi–Effros theorem. Let $\mu: A \to A/J$ be the quotient map. Then $\hat{\psi}\mu(a) - \psi(a) \in C_0(\mathbb{R}_+, L(H \oplus H))$ for all $a \in A$, hence by using (3), we have $\lim_{t\to\infty} qd(\hat{\psi}_t(F)) = 0$ for any finite subset F of A/J. Moreover, for all $z \in A/J$, $\limsup_{t\to\infty} \|\hat{\psi}_t(z)\| = \|\hat{\dot{\psi}}(z)\| = \|z\|$ since $\hat{\dot{\psi}}$ is isometric. By Proposition 2.5(a) we conclude that A/J is quasidiagonal. Since $J \subset \ker\dot{\varphi}$ it follows that $\dot{\varphi}$ is quasidiagonal by Remark 4.2(i).

For the last part of the proof B is an arbitrary C*-algebra and (φ_t) is nuclear.

(ii) $\Leftrightarrow$ (iii) Write $\dot{\varphi} = \nu_1\mu$ with $\mu: A \to D$, $\nu_1: D \to B_\infty$ nuclear and D a quasidiagonal C*-algebra. Then let (ν_t) be given by any nuclear CP-lifting of ν_1. Conversely, if (ν_t) is given then set $\nu_1 = \dot{\nu}$. ∎

Proof of Theorem 1.2. Without loss of generality we may assume that $B \subset L(H)$. Let $(\Gamma_t^{(s)}): A \to B$ be a CP-homotopy with $\Gamma_t^{(0)} = \varphi_t$ and $\Gamma_t^{(1)} = \psi_t$ for all $t \geq 0$. Let $\sigma: A \to L(H)$ be a nuclear $*$-representation such that $\lim_{t\to\infty} qd(\psi_t \oplus \sigma)(F) = 0$ for all the finite subsets F of A. Set $\Phi_t^{(s)} = \Gamma_t^{(s)} = \Gamma_t^{(s)} \oplus \sigma$. Then $\Phi_t^{(s)}$ is a CP-homotopy from $\varphi_t \oplus \sigma$ to $\psi_t \oplus \sigma$, such that $\lim_{t\to\infty} qd(\Phi_t^{(1)}(F)) = 0$. Let $L = \ker(\dot{\varphi} \oplus \sigma)$ and let $\mu: A \to A/L$ denote the quotient map. Then

$$\|\mu(a)\| = \limsup_{t\to\infty} \|\varphi_t \oplus \sigma(a)\| = \limsup_{t\to\infty} \|\Phi_t^{(0)}(a)\|.$$

It is now clear that we can find t such that $\Phi_t^{(s)}$ satisfies all the conditions of Proposition 2.5(b) with $\chi=\mu$. Thus μ is quasidiagonal. This implies that $\dot{\varphi}$ is quasidiagonal since $\dot{\varphi}\oplus\sigma$ factors through μ. We conclude the proof by applying Proposition 5.2. ▮

As an immediate consequence of Proposition 5.2 and Theorem 1.2 we show that the composition of two CP-asymptotic morphisms (which is defined only up to homotopy) between exact C*-algebras is quasidiagonal provided that at least one of the morphisms is quasidiagonal. Recall that if A is separable and if $(\varphi_t)\colon A\to B$ and $(\psi_t)\colon B\to C$ are CP-asymptotic morphisms, then there is a suitable rescaling $r_0\colon \mathbb{R}_+\to\mathbb{R}_+$ such that for any other homeomorphism $r\colon \mathbb{R}_+\to\mathbb{R}_+$ with $r(t)\geqslant r_0(t)$, $(h_t):=(\psi_{r(t)}\varphi_t)$ is an asymptotic morphism and $[[\psi_t]][[\varphi_t]]=[[h_t]]$.

Proposition 5.3. *If A and B are separable exact C*-algebras and if either (φ_t) or (ψ_t) is quasidiagonal, then $(h_t)=(\psi_{r(t)}\varphi_t)$ is quasidiagonal.*

Proof. Suppose first that (φ_t) is quasidiagonal. Then $\dot{\varphi}$ is quasidiagonal by Proposition 5.2. By Remark 4.2(i) there is a closed two-sided ideal J in A such that $J\subset\ker\dot{\varphi}$ and A/J is quasidiagonal. Since $\|h_t(a)\|\leqslant\|\varphi_t(a)\|$ for $a\in A$ it is clear that $\ker\dot{\varphi}\subset\ker\dot{h}$, hence $J\subset\ker\dot{h}$. This implies that $\dot{h}$ and hence (h_t) is quasidiagonal.

For the second part of the proof suppose that (ψ_t) is quasidiagonal. As in the proof of Proposition 5.2 we may assume that $C=L(H)$. In that case (ψ_t) is nuclear and it is equivalent to a composite $(v_t\mu)$ as in Proposition 5.2(iii). Therefore $(h_t)=(\psi_{r(t)}\varphi_t)$ is equivalent to $(v_{r(t)}\mu\varphi_t)$. The latter composite is quasidiagonal by the first part of the proof since $(\mu\varphi_t)$ is quasidiagonal being a CP-asymptotic morphism into a quasidiagonal C*-algebra. ▮

In view of Theorem 1.2 and Proposition 5.3 it is natural to consider the category of stable separable exact C*-algebras with morphisms from A to B given by the semigroup $[[A,B]]_{CP}$ where the addition is induced by the direct sum. The homotopy classes of quasidiagonal CP-asymptotic morphisms form a subsemigroup $[[A,B]]^{qd}_{CP}\subset[[A,B]]_{CP}$. This subsemigroup is nonzero if and only if A has a nonzero quasidiagonal quotient. If $u,v\in[[A,B]]_{CP}$ then $u+v\in[[A,B]]^{qd}_{CP}$ if and only if $u,v\in[[A,B]]^{qd}_{CP}$. The product of two elements $u\in[[A,B]]_{CP}$, $v\in[[B,C]]_{CP}$ is quasidiagonal provided that either u or v is quasidiagonal.

REFERENCES

[Ar] W. B. Arveson, Notes on extensions of C*-algebras, *Duke Math. J.* **44** (1977), 329–355.

[APT] C. Akemann, G. Pedersen, and J. Tomiyama, Multipliers of C*-algebras, *J. Funct. Anal.* **13** (1973), 277–301.

[B_1] B. Blackadar, "*K*-theory for Operator Algebras," M.S.R.I. Monographs, No. 5, Springer-Verlag, Berlin/New York, 1986.

[B_2] B. Blackadar, Shape theory for C*-algebras, *Math. Scand.* **56** (1985), 249–275.

[BKi] B. Blackadar and E. Kirchberg, Generalized inductive limits of finite-dimensional C*-algebras, *Math. Ann.* **307**, No. 3 (1997), 343–380.

[Br] L. G. Brown, The universal coefficient theorem for ext and quasidiagonality, *in* "Operator Algebras and group Representations," Vol. 17, pp. 60–64, Pitman Press, Boston/London/Melbourne, 1983.

[BDF] L. Brown, R. Douglas, and P. Fillmore, Extensions of C*-algebras and *K*-homology, *Ann. of Math.* **105** (1977), 265–324.

[BrD] L. G. Brown and M. Dadarlat, Extensions of C*-algebras and quasidiagonality, *J. London Math. Soc.* **53** (1996), 582–600.

[ChE] M.-D. Choi and E. G. Effros, The completely positive lifting problem for C*-algebras, *Ann. of Math.* **104** (1976), 585–609.

[CH] A. Connes and N. Higson, Deformations, morphismes asymptotiques et *K*-théorie bivariante, *C. R. Acad. Sci. Paris* **313** (1990), 101–106.

[Cu] J. Cuntz, A new look at KK-theory, *K-Theory* **1** (1987), 35–51.

[D] M. Dadarlat, Shape theory and asymptotic morphisms for C*-algebras, *Duke Math. J.* **73** (1994), 687–711.

[DL] M. Dadarlat and T. A. Loring, *K*-homology, asymptotic representations and unsuspended *E*-theory, *J. Funct. Anal.* **126** (1994), 367–383.

[DHS] K. Davidson, D. Herrero, and N. Salinas, Quasidiagonal operators, approximation and C*-algebras, *Indiana Univ. Math. J.* **38** (1989), 973–998.

[EH] E. G. Effros and U. Haagerup, Lifting problems and locally reflexivity for C*-algebras, *Duke Math. J.* **52** (1985), 103–108.

[EK] E. G. Effros and J. Kaminker, Homotopy continuity and shape theory for C*-algebras, *in* "Geometric Methods in Operator Algebras," Pitnam Res. Notes Math., Vol. 123, pp. 152–180, Longman Sci. Tech., Harlow, 1986.

[ER] E. G. Effros and J. Rosenberg, C*-algebras with approximately inner flip, *Pacific J. Math.* **77** (1978), 417–443.

[G] E. Guentner, Relative *E*-theory, *K-theory*, to appear.

[H] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **76** (1970), 887–933.

[LT] T. G. Houghton-Larsen and K. Thomsen, Universal(co)homology theories, preprint, 1996.

[JT] K. K. Jensen and K. Thomsen, "Elements of KK-theory," Mathematics: Theory and Applications, Birkhäuser, Boston/Basel/Berlin, 1991.

[Ka_1] G. G. Kasparov, The operator *K*-functor and extensions of C*-algebras, *Math. URSS Izv.* **16** (1981), 513–572.

[Ka_2] G. G. Kasparov, Hilbert C*-modules: Theorems of Stinespring and Voiculescu, *J. Oper. Theory* **4** (1980), 133–150.

[Ki_1] E. Kirchberg, Exact C*-algebras, tensors products, and classification of purely infinite algebras, *in* "Proc. ICM Zurich 1994," pp. 943–954, Birkhäuser, Basel, 1995.

[Ki_2] E. Kirchberg, The classification of purely infinite C*-algebras using Kasparov's theory, preprint.

[Li] H. Lin, Extensions of $C(X)$ by simple C*-algebras, *Amer. J. Math.*, to appear.
[L_1] T. A. Loring, A test for injectivity for asymptotic morphisms, *in* "Algebraic Methods in Operator Theory," Birkhäuser, Basel, 1994.
[L_2] T. A. Loring, Almost multiplicative maps between C*-algebras, preprint, 1996.
[P] G. K. Pedersen, "C*-Algebras and Their Automorphism Groups," Academic Press, London, 1979.
[Po] S. Popa, On local finite dimensional approximation of C*-algebras, preprint, 1996.
[PPV] S. Popa, M. Pimsner, and D. Voiculescu, Homogeneous C*-extensions of $C(X) \otimes K(H)$, I, *J. Operator Theory* **1**, 55–108.
[Sa_1] N. Salinas, Homotopy invariance of Ext(A), *Duke J. Math.* **44** (1977), 777–794.
[Sa_2] N. Salinas, Relative quasidiagonality and KK-theory, *Houston J. Math.* **18** (1992), 97–116.
[Sc] C. Schochet, The fine structure of the Kasparov groups. II. Relative quasidiagonality, preprint, 1996.
[Va] G. Vaillant, On a class of stably finite nuclear C*-algebras, *Integral Equation Operator Theory* **22** (1995), 339–351.
[V_1] D. Voiculescu, A non-commutative Weyl–von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21** (1976), 97–113.
[V_2] D. Voiculescu, A note on quasidiagonal C*-algebras and homotopy, *Duke Math. J.* **62** (1991), 267–272.
[V_3] D. Voiculescu, Around quasidiagonality, *Integral Equations Operator Theory* **17** (1993), 137–148.
[V_4] D. Voiculescu, A note on quasidiagonal operators, *in* "Operator Theory: Advances and Applications," Vol. 32, pp. 265–274, Birkhäuser Verlag, Basel, 1988.
[W] S. Wassermann, "Exact C*-Algebras and Related Topics," Lecture Notes Series, No. 19, Seoul National University, Seoul, Korea.
[Zh] S. Zhang, K_1-groups, quasidiagonality and interpolation by multiplier projections, *Trans. Amer. Math. Soc.* **325** (1991), 793–818.