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$\mathcal{Z}$-stability and infinite tensor powers of C*-algebras

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Abstract

We prove that under a mild hypothesis, the infinite tensor power of a unital separable C*-algebra absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially. Combining this result with a recent theorem of Winter, we complete Elliott's classification program for strongly self-absorbing ASH algebras. We also give a succinct universal property for $\mathcal{Z}$ in an ambient category so large that there are no unital separable C*-algebras without characters that are known to lie outside it. This category contains the vast majority of our stock-in-trade separable amenable C*-algebras, and is closed under passage to quotients and separable superalgebras. In particular, the category is closed under the formation of unital direct limits, unital tensor products, and crossed products by countable discrete groups. Finally, we take a significant step toward the confirmation of Elliott's classification conjecture for the C*-algebras of minimal diffeomorphisms.

MSC: primary 46L35; secondary 46L80

Keywords: Jiang–Su algebra; Strongly self-absorbing C*-algebras

1. Introduction

The Jiang–Su algebra, denoted by $\mathcal{Z}$, occupies a central position in the structure theory of separable amenable C*-algebras. Most striking is its role in Elliott's classification program, the goal

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of which is to prove that simple separable amenable C*-algebras are classified up to isomorphism by their Banach algebra K-theory and traces. Examples due to Rørdam and the second named author show that the property of absorbing the Jiang–Su tensorially—being *$\mathcal{Z}$-stable*—is a necessary condition for the completion of Elliott's program [10,14], and recent results of Winter and Lin show $\mathcal{Z}$-stability is, in considerable generality, also sufficient. In particular, the C*-algebras of minimal uniquely ergodic diffeomorphisms satisfy Elliott's classification conjecture, modulo the assumption of $\mathcal{Z}$-stability [17].

The importance of the Jiang–Su algebra manifests itself in other ways, too. The examples of [14] demonstrate the extreme sensitivity of the Cuntz semigroup in the matter of distinguishing simple separable amenable C*-algebras. Assuming $\mathcal{Z}$-stability, however, one recovers this semigroup from Banach algebra K-theory and traces in a functorial manner [1]. This structure theorem for the Cuntz semigroup leads to the solution of conjectures of Blackadar and Handelman concerning the basic structure of dimension functions on C*-algebras, and offers insight into the remarkable power of Elliott's invariant (K-theory and traces) for simple separable amenable C*-algebras which absorb $\mathcal{Z}$. Finally, $\mathcal{Z}$-stability generalises $\mathcal{O}_\infty$-stability for separable amenable C*-algebras; these two properties are equivalent in the traceless case.

These results invite two questions:

(i) Which natural classes of (separable amenable) C*-algebras are $\mathcal{Z}$-stable?
(ii) To what extent is the Jiang–Su algebra unique?

Concerning (i), there are several classes of inductive limit C*-algebras whose members are known to be $\mathcal{Z}$-stable, but only because they are already known to satisfy Elliott's classification conjecture. Instead, one wants to prove $\mathcal{Z}$-stability directly, in order to apply classification theorems such as those of Winter and Lin–Winter [16,17]. This paper contains the first significant results in this direction (see Sections 1.1 and 1.3), leading to a classification of strongly self-absorbing C*-algebras. As for (ii), we know that $\mathcal{Z}$ is determined up to $*$-isomorphism inside a somewhat small and artificial class of inductive limit C*-algebras. Here we establish a succinct universal property for $\mathcal{Z}$ valid in an ambient category which, potentially, contains all unital separable C*-algebras without characters (see Section 1.2).

Recall that a C*-algebra is *subhomogeneous* if all of its irreducible representations are of bounded finite dimension. We let $A^{\otimes\infty}$ denote the minimal tensor product of countably many copies of a unital C*-algebra A. Our main results are applications of the following theorem.

Theorem 1.1. *Let A be a unital separable C*-algebra, and suppose that $A^{\otimes\infty}$ contains, unitally, a subhomogeneous algebra without characters. It follows that $A^{\otimes\infty}$ is $\mathcal{Z}$-stable.*

Some comments on Theorem 1.1 are in order. It is necessary to take the infinite (as opposed to some finite) tensor power of A in Theorem 1.1: [13] contains an example of a simple unital separable infinite-dimensional C*-algebra A with the property that $A^{\otimes n} \otimes \mathcal{Z} \ncong A^{\otimes n}$ for each $n \in \mathbb{N}$ and yet $A^{\otimes\infty} \otimes \mathcal{Z} \cong A^{\otimes\infty}$. The subalgebra hypothesis of Theorem 1.1 is not only necessary ($\mathcal{Z}$ contains, unitally, a subhomogeneous algebra without characters), but also extremely weak. Indeed, it is potentially vacuous for unital separable C*-algebras without characters. Examples of unital separable C*-algebras known to lie in the class $\mathcal{C}$ of C*-algebras covered by Theorem 1.1 include the following (we explain why in Section 6.4):

(a) simple exact C*-algebras containing an infinite projection;
(b) inductive limits of subhomogeneous algebras (ASH algebras) without characters;

(c) properly infinite C*-algebras;
(d) real rank zero C*-algebras without characters;
(e) C*-algebras arising from minimal dynamics on a compact infinite Hausdorff space;
(f) algebras considered pathological with respect to the strong form of Elliott's classification conjecture for separable amenable C*-algebras [10,12,14].

The question of whether (a) and (b) together encompass the class of simple unital separable amenable C*-algebras entirely is an outstanding open problem. Note that the hypotheses of Theorem 1.1 imply immediately that $\mathcal{C}$ is closed under passage to unital separable superalgebras and quotients. In particular, $\mathcal{C}$ is closed under arbitrary unital tensor products, unital direct limits and crossed products by countable discrete groups.

The bulk of the difficulty in establishing Theorem 1.1 is overcome by Theorem 5.3, which states that the space of unital $*$-homomorphisms

$$\mathrm{Hom}_1\big(\mathrm{I}_{p,q};\mathrm{M}_k\big(\mathrm{C}(X)\big)\equiv \mathrm{C}\big(X;\mathrm{Hom}_1(\mathrm{I}_{p,q};\mathrm{M}_k)\big)\big)$$

is path connected whenever k is large relative to $\dim(X)$. (Here $\mathrm{I}_{p,q}$ denotes the *prime dimension drop* algebra associated to the relatively prime integers p and q—see Section 2 for its definition—and M_k denotes the set of $k\times k$ matrices with complex entries. The connection with $\mathcal{Z}$ is the fact that $\mathcal{Z}$ is approximated locally by prime dimension drop algebras.) In particular, the homotopy groups of $F^k := \mathrm{Hom}_1(\mathrm{I}_{p,q};\mathrm{M}_k)$ vanish in dimensions $< ck$ where $c\in(0,1)$ depends only on p,q. We prove this result by filtering the non-manifold F^k so that the successive differences in the filtration are smooth manifolds, applying Thom's transversality theorem to perturb maps into the said differences, applying a continuous selection argument to semicontinuous fields of representations of $\mathrm{I}_{p,q}$, and finally appealing to Kasparov's KK-theory.

We turn now to the applications of Theorem 1.1.

1.1. A classification of strongly self-absorbing C-algebras*

A unital separable C*-algebra $\mathcal{D}\not\cong\mathbb{C}$ is *strongly self-absorbing* if the factor inclusion

$$\mathcal{D}\otimes 1_{\mathcal{D}}\hookrightarrow\mathcal{D}\otimes\mathcal{D}$$

is approximately unitarily equivalent to a $*$-isomorphism [15]. Results of Haagerup, Kirchberg, and Rosenberg show that these algebras are always simple, amenable, and either purely infinite or stably finite. They are also rare, and connected deeply to the classification theory of separable amenable C*-algebras. Among Kirchberg algebras—simple separable amenable purely infinite C*-algebras satisfying the UCT—there are only $\mathcal{O}_2$, $\mathcal{O}_\infty$, and tensor products of $\mathcal{O}_\infty$ with UHF algebras of "infinite type," i.e., tensor products of countably many copies of a single UHF algebra; this list is a result of the Kirchberg–Phillips classification of purely infinite C*-algebras. Among ASH algebras the only known examples are the UHF algebras of infinite type—these are the only examples which contain a non-trivial projection—and the projectionless algebra $\mathcal{Z}$. Here we combine Theorem 1.1 with a recent theorem of Winter (with appendix by Lin) to complete the classification of strongly self-absorbing ASH C*-algebras. Note that there are no simple unital separable amenable C*-algebras with a trace which are known not to be ASH.

Theorem 1.2. *Let A be a unital simple separable ASH algebra with unique tracial state. Suppose that A and $\mathcal{Z}$ have isomorphic scaled ordered K-groups. It follows that $A^{\otimes\infty} \cong \mathcal{Z}$. In particular, $\mathcal{Z}$ is the only projectionless strongly self-absorbing ASH algebra.*

Proof. By Theorem 1.1, $A^{\otimes\infty}$ is $\mathcal{Z}$-stable. Now $A^{\otimes\infty}$ and $\mathcal{Z}$ are unital simple separable ASH algebras with unique trace which moreover absorb $\mathcal{Z}$ tensorially. It follows from the main result of [17] that $A^{\otimes\infty} \cong \mathcal{Z}$. The last statement of the theorem follows from the fact that $\mathcal{Z}^{\otimes\infty} \cong \mathcal{Z}$ [4]. □

1.2. A universal property for the Jiang–Su algebra

We now address our second question: "In what sense is the Jiang–Su algebra unique?" Let $\mathcal{C}$ be a class of unital C*-algebras. The following pair of conditions on a C*-algebra $B \in \mathcal{C}$ constitute a universal property, which we will denote by (UP) (cf. [13]):

(i) $B^{\otimes\infty} \cong B$;
(ii) $B \otimes A^{\otimes\infty} \cong A^{\otimes\infty}$, $\forall A \in \mathcal{C}$.

It is easy to see that if $B_1, B_2 \in \mathcal{C}$ satisfy properties (i) and (ii), then they are isomorphic. To wit,

$$B_1 \overset{\text{(i)}}{\cong} B_1^{\otimes\infty} \overset{\text{(ii)}}{\cong} B_1^{\otimes\infty} \otimes B_2 \overset{\text{(i)}}{\cong} B_1 \otimes B_2^{\otimes\infty} \overset{\text{(ii)}}{\cong} B_2^{\otimes\infty} \overset{\text{(i)}}{\cong} B_2.$$

The question, of course, is whether a given class $\mathcal{C}$ contains an algebra B satisfying (i) and (ii) at all.

Theorem 1.3. *Let $\mathcal{C}$ denote the class of unital separable C*-algebras which satisfy the hypotheses of Theorem* 1.1. *It follows that $\mathcal{Z}$ satisfies (UP) in $\mathcal{C}$.*

Proof. Combine Theorem 1.1 with the fact that $\mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$. □

Note the considerable scope of $\mathcal{C}$, as discussed following the statement of Theorem 1.1.

1.3. $\mathcal{Z}$-stability and the C-algebras of minimal diffeomorphisms*

Let M be a compact smooth manifold, and $f : M \to M$ a minimal diffeomorphism. The crossed product $C^*(M, \mathbb{Z}, f)$ is a simple separable amenable C*-algebra, and it is conjectured that the class of all such algebras, denoted by $\mathcal{M}$, will satisfy Elliott's classification conjecture. Winter has shown that this is the case provided that f is uniquely ergodic and $C^*(M, \mathbb{Z}, f)$ is $\mathcal{Z}$-stable, and has moreover outlined a program to remove the unique ergodicity hypothesis. The completion of Elliott's program for $\mathcal{M}$ thus depends on proving $\mathcal{Z}$-stability for each $A \in \mathcal{M}$.

To prove $\mathcal{Z}$-stability for A, it suffices to prove that for each pair of relatively prime positive integers p and q there is an approximately central sequence of unital $*$-homomorphisms $\gamma_n : \mathrm{I}_{p,q} \to A$ (see Section 2.2 for the definition of $\mathrm{I}_{p,q}$). We therefore require a picture of the commutant of a finite set in A. Q. Lin and Phillips have shown that each $A \in \mathcal{M}$ is the inductive limit of a sequence of *recursive subhomogeneous algebras* (RSH algebras; see Definition 6.1) with slow dimension growth. Given such a sequence, say (A_i, γ_i) with A_i an RSH algebra, one knows that the commutant of the image of A_i in A_j over each point in the spectrum of A_j is well approximated by a finite-dimensional C*-algebra F, all of whose simple summands have large dimension compared with the dimension of the spectrum of A_j. It stands to reason that one should be able to map an RSH algebra, all of whose matrix fibres are large in comparison with its topological dimension, into the approximate commutant of the image of a finite set $F \subseteq A_i$ in A_j. (Indeed, the proof of this fact is a work in progress.) The final piece of the puzzle, then, is to prove that $\mathrm{I}_{p,q}$ admits a unital $*$-homomorphism into any RSH algebra whose matrix fibres have simple summands which are large relative to the dimension of the spectrum of the algebra. Our Theorem 6.2 proves that such a $*$-homomorphism always exists, and that it is even unique up to homotopy. This brings us significantly closer to the confirmation of Elliott's conjecture for $\mathcal{M}$.

1.4. Organisation

The sequel is organised as follows: Section 2 introduces some notation pertaining to finite-dimensional representations of dimension drop algebras; Section 3 examines the homotopy groups of finite-dimensional representations of dimension drop algebras; Section 4 provides a continuous selection theorem for subrepresentations of a semicontinuous field of representations of a dimension drop algebra over a compact Hausdorff space; in Section 5 we prove an extension theorem for certain maps out of dimension drop algebras; Section 6 combines our technical results to prove Theorem 1.1.

2. Preliminaries and notation

2.1. Basic assumptions

Unless otherwise noted, all morphisms in this paper are $*$-preserving algebra homomorphisms. We use M_k to denote the C*-algebra of $k \times k$ matrices with complex entries. If X is a compact Hausdorff space, then $\mathrm{C}(X)$ denotes the C*-algebra of continuous complex-valued functions on X. If A is a unital C*-algebra, then $\mathcal{U}(A)$ is its unitary group.

2.2. Finite-dimensional representations of dimension drop algebras

We assume throughout that p and q are relatively prime integers strictly greater than one. The prime dimension drop algebra $\mathrm{I}_{p,q}$ is defined as follows:

$$\mathrm{I}_{p,q} = \bigl\{ f \in \mathrm{C}\bigl([0,1]; \mathrm{M}_p \otimes \mathrm{M}_q\bigr) \bigm| f(0) \in \mathrm{M}_p \otimes 1_q,\ f(1) \in 1_p \otimes \mathrm{M}_q \bigr\},$$

with the usual pointwise operations.

For any $t \in [0,1]$, we define a $*$-homomorphism $ev_t : \mathrm{I}_{p,q} \to \mathrm{M}_{pq} \cong \mathrm{M}_p \otimes \mathrm{M}_q$ by setting $ev_t(f) = f(t)$. If $t \in (0,1)$, then we will refer to ev_t as a *generic evaluation*. Now suppose that $f(0) = a \otimes 1_q$ and $f(1) = 1_p \otimes b$. Define maps $e_0 : \mathrm{I}_{p,q} \to \mathrm{M}_p$ and $e_1 : \mathrm{I}_{p,q} \to \mathrm{M}_q$ by $e_0(f) = a$ and $e_1(f) = b$. We will refer to e_0 and e_1 as *endpoint evaluations*. Set $F^k = \mathrm{Hom}_1(\mathrm{I}_{p,q}; \mathrm{M}_k)$. It is known that every integer $k \geqslant pq - p - q$ can be written as a non-negative integral linear combination of p and q, whence F^k is not empty in that case. Each $\phi \in F^k$ is unitarily equivalent to

$$\tilde{\phi} = \left(\bigoplus_{i=1}^{a_\phi} e_0\right) \oplus \left(\bigoplus_{k=1}^{b_\phi} e_1\right) \oplus \left(\bigoplus_{j=1}^{c_\phi} ev_{x_j}\right),$$

where $x_j \in (0,1)$. Let us denote by $\mathrm{sp}(\phi)$ the multiset consisting of the x_js.

2.3. Spectral multiplicity

Let X be a compact Hausdorff space, and let

$$\phi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$$

be a unital $*$-homomorphism. If $Y \subseteq X$, then $\phi|_Y$ will denote the restriction of ϕ to Y. This restriction is actually a $*$-homomorphism into $\mathrm{C}(Y) \otimes \mathrm{M}_k$ whenever Y is closed, and is always at least a $*$-preserving algebra homomorphism into the set of continuous M_k-valued functions on Y.

We define $N_0^\phi : X \to \mathbb{Z}^+$ (resp. $N_1^\phi : X \to \mathbb{Z}^+$) to be the upper semicontinuous function which, at $x \in X$, returns the number of e_0s (resp. e_1s) occurring as direct summands of $\phi|_{\{x\}}$. Similarly, we define $N_g^\phi : X \to \mathbb{Z}^+$ to be the lower semicontinuous function which, at $x \in X$, returns the number of generic evaluations occurring as direct summands of $\phi|_{\{x\}}$. These functions are related as follows:

$$k = pN_0^\phi(x) + qN_1^\phi(x) + pqN_g^\phi(x), \quad \forall x \in X. \tag{1}$$

3. Reducing the number of generic representations

Let F_l^k denote the subset of F^k consisting of those ϕ for which $\mathrm{sp}(\phi)$ contains at most l points, counted with multiplicity. It follows that the difference $F_l^k \backslash F_{l-1}^k$ consists of those ϕ for which $\mathrm{sp}(\phi)$ contains exactly l points, counted with multiplicity. Let a and b be non-negative integers satisfying

$$ap + bq + lpq = k.$$

Let $F^k(a,b,l)$ denote the set of $\phi \in F_l^k \backslash F_{l-1}^k$ which, up to unitary equivalence, contain exactly a direct summands of the form e_0 and b direct summands of the form e_1. For a fixed l, the various $F^k(a,b,l)$ are clopen subsets of $F_l^k \backslash F_{l-1}^k$. Finally, let $S^k(a,b,l)$ denote the subset of $F^k(a,b,l)$ consisting of those ϕ which contain exactly l summands of the form $ev_{1/2}$.

Lemma 3.1. *The inclusion*

$$F_{l-1}^k \xrightarrow{\iota} F_{l-1}^k \cup F^k(a,b,l) \backslash S^k(a,b,l)$$

is a deformation retract.

Proof. Define a family $\{h_t\}_{t\in[0,1/2)}$ of continuous self-maps of $[0,1]$ as follows:

$$h_t(x)=\begin{cases}0, & x\in[0,t],\\ (x-t)/(1-2t), & x\in(t,1-t),\\ 1, & x\in[1-t,1].\end{cases}$$

Note that $(x,t)\mapsto h_t(x)$ is continuous in both variables. Since h_t fixes 0 and 1, it induces an endomorphism η_t of $\mathrm{I}_{p,q}$ by acting on $\mathrm{Spec}(\mathrm{I}_{p,q})\cong[0,1]$.

Let us now define a continuous map

$$d: F^k_{l-1}\cup F^k(a,b,l)\backslash S^k(a,b,l)\to[0,1/2).$$

On F^k_{l-1}, set $d=0$. On $F^k(a,b,l)\backslash S^k(a,b,l)$, d is the Hausdorff distance between the following two subsets of $[0,1]$: first, the (non-empty) set of points in $\mathrm{sp}(\phi)$ which also lie in $(0,1/2)\cup(1/2,1)$; second, the set $\{0,1\}$.

Define a homotopy $H(t)$ of self-maps of $F^k_{l-1}\cup F^k(a,b,l)\backslash S^k(a,b,l)$ by

$$H(t)(\phi)=\phi\circ\eta_{td(\phi)}.$$

Since $d(\phi)\equiv 0$ on F^k_{l-1} and $\eta_0=\mathbf{id}_{\mathrm{I}_{p,q}}$, we have

$$H(t)|_{F^k_{l-1}}=\mathbf{id}_{F^k_{l-1}}$$

and

$$H(0)=\mathbf{id}_{F^k_{l-1}\cup F^k(a,b,l)\backslash S^k(a,b,l)}.$$

Since $\phi\circ\eta_{d(\phi)}\in F^k_{l-1}$ whenever $\phi\in F^k(a,b,l)\backslash S^k(a,b,l)$, ι is a deformation retract. □

Proposition 3.2. *The topological space $F^k(a,b,l)$ can be endowed with the structure of a smooth manifold; the subspace $S^k(a,b,l)$ is then a compact submanifold of codimension l^2.*

Proof. Note that $M=\{\varphi\colon \mathrm{Hom}(C[0,1],M_l)\colon Sp(\varphi)\subset(0,1)\}$ can be naturally identified with the set of all self-adjoint matrices in M_l having all their eigenvalues in $(0,1)$ and hence M is homeomorphic to $\mathbb{R}^{l^2}$.

We are going to exhibit a free, proper and smooth right action of the compact Lie group $G=U(a)\times U(b)\times U(l)$ on the manifold $X=M\times U(k)\cong\mathbb{R}^{l^2}\times U(k)$, such that the quotient space is homeomorphic to $F^k(a,b,l)$. Then we invoke a result from [11] to conclude that X/G and hence $F^k(a,b,l)$ admits a unique smooth structure for which the quotient map is a submersion. The uniqueness part of the same result shows that if Y is a G-invariant submanifold of X, then Y/G is a submanifold of X/G.

If $\varphi\in M$, then $\varphi\otimes\mathrm{id}_{pq}: C[0,1]\otimes M_{pq}\to M_l\otimes M_{pq}$ defines by restriction a morphism on $I_{p,q}\subset C[0,1]\otimes M_{pq}$. Let us define a continuous map $P: M\times U(k)\to F^k(a,b,l)$ by

$$P(\varphi,u)(f)=u\big[\big(e_0(f)\otimes 1_a\big)\oplus\big(e_1(f)\otimes 1_b\big)\oplus(\varphi\otimes\mathrm{id}_{pq})(f)\big]u^*,$$

where for $f\in I_{p,q}$, $f(0)=e_0(f)\otimes 1_q\in M_p\otimes M_q$ and $f(1)=1_p\otimes e_1(f)\in M_p\otimes M_q$.

One verifies that the map P is surjective and that $P(\varphi,u)=P(\psi,v)$ if and only if there is $W=(w_0,w_1,w)\in G$ such that $\psi=w^*\varphi w$ and $v=uj(W)$ where $j:G\to U(k)$ is the injective morphism

$$j(w_0,w_1,w)=(1_p\otimes w_0)\oplus(1_q\otimes w_1)\oplus(w\otimes 1_{pq}),$$

induced by the embedding of C*-algebras

$$B=(1_p\otimes M_a)\oplus(1_q\otimes M_b)\oplus(M_l\otimes 1_{pq})\subset M_k.$$

This shows that if we define a right action of G on $X=M\times U(k)$ by

$$(\varphi,u)W=\big(w^*\varphi w, vj(W)\big),$$

where $W=(w_0,w_1,w)\in U(a)\times U(b)\times U(l)=G$, then P induces a continuous bijection $X/G\to F^k(a,b,l)$. This induced map is actually a homeomorphism since one can verify that P is an open map as follows. Fix (φ,u) in X and let V be a neighbourhood of (φ,u). We need to show that if $P(\psi,v)$ is sufficiently close to $P(\varphi,u)$, then there is (ψ_1,v_1) in V such that $P(\psi_1,v_1)=P(\psi,v)$. Fix a metric d for the point-norm topology of F^k. Suppose that $d(P(\varphi,u),P(\psi,v))<\delta$ for some $\delta>0$ to be specified later. Then v^*u approximately commutes with the unit ball of the subalgebra $A=(M_p\otimes 1_a)\oplus(M_q\otimes 1_b)\oplus(M_l\otimes 1_{pq})$ of M_k. By a classical perturbation result for finite-dimensional C*-algebras, there is a unitary z in the relative commutant of A in M_k, $z\in U(A'\cap M_k)=U(B)$ such that $\|v^*u-z\|<g(\delta)$, where g is a universal positive map with converges to 0 when $\delta\to 0$. We can write $z=j(W)$ where $W=(w_0,w_1,w)\in U(a)\times U(b)\times U(l)$, as above. Let us set $\psi_1=w^*\psi w$ and $v_1=vj(W)$. Then $P(\psi_1,v_1)=P(\psi,v)$ and if δ is chosen sufficiently small, then (ψ_1,v_1) is in V since $\|u-vj(W)\|<g(\delta)$ and since $d(w^*\psi w,\varphi)\to 0$ as $\delta\to 0$, because $d(P(\varphi,1),P(\psi,u^*v))<\delta$.

Having established the homeomorphism $F^k(a,b,l)\cong X/G$, we need to show that X/G is a smooth manifold. To this purpose we apply Proposition 5.2 on page 38 of [11], according to which X/G is a manifold provided that X is a manifold and the action of the Lie group G on X is proper, free and smooth. Recall that a free right action $G\times X\to X$ is proper if

(1) The set $C=\{(x,xg)\colon x\in X, g\in G\}$ is closed in $X\times X$ and
(2) The map $\iota:C\to G$, $\iota(x,xg)=g$ is continuous.

The first condition is easily verified if G is compact as shown in the last part of the proof of Proposition 3.1 on page 23 of [11]. To verify the second condition suppose that (x_n,x_ng_n) converges to (x,xg) in $X\times X$. Then x_n converges to x and hence $\mathrm{dist}(xg_n,x_ng_n)=d(x,x_n)$ converges to zero. It follows that xg_n converges to xg. If u is the component of x in $U(k)$, then $uj(g_n)$ converges to $uj(g)$ in $U(k)$. Therefore g_n must converge to g.

In conclusion $F^k(a,b,l)$ is a manifold of dimension equal to $\dim(X)-\dim(G)=l^2+k^2-(a^2+b^2+l^2)$.

On the other hand, $S^k(a,b,l)$ is the image in the quotient space X/G of the G-invariant submanifold $\{\mu\}\times U(k)$ of $X=M\times U(k)$, where $\mu(f)=f(1/2)\otimes 1_l$. By applying [11, Proposition 5.2] once more we deduce that $S^k(a,b,l)$ is a submanifold of $F^k(a,b,l)$ of dimension $k^2-(a^2+b^2+l^2)$ and hence of codimension l^2. □

Proposition 3.3. *The inclusion $F^k_{l-1}\hookrightarrow F^k_l$ is an (l^2-1)-equivalence.*

Proof. All manifolds in this proof are assumed to be metrisable and separable. Let S_l^k be the union of all $S^k(a,b,l)$. Let X be a smooth manifold of dimension $\leqslant l^2-1$ and let Y be a compact subset of X. Let $f: X \to F_l^k$ be a continuous map such that $f(Y) \cap S_l^k = \emptyset$ and let $\varepsilon > 0$. We are going to show that there is a continuous map $g: X \to F_l^k$ such that

(i) $d(f(x), g(x)) < \varepsilon$, for all $x \in X$,
(ii) $g = f$ on Y,
(iii) $g(X) \cap S_l^k = \emptyset$.

We shall apply this perturbation result in the realm of cellular maps from a pair of CW-complexes, $(X,Y) = (S^n, *)$ or $(X,Y) = (S^n \times [0,1], \{*\} \times [0,1])$ to the CW-complex F_l^k. This implies that f is homotopic to g via a homotopy that is constant on Y, assuming that ε was chosen sufficiently small. Consequently, the inclusion

$$F_l^k \setminus S_l^k \hookrightarrow F_l^k$$

is an l^2-1 equivalence. One concludes the proof by combining this fact with Lemma 3.1. Let us turn to the proof of the perturbation result. Note that S_l^k is a compact submanifold of $F_l^k \setminus F_{l-1}^k$ and the latter is an open subset of F_l^k. We may assume that $f^{-1}(S_l^k) \neq \emptyset$, for otherwise there is nothing to prove. By a classical approximation result, we may assume that f is smooth on the pre-image $f^{-1}(U)$ of some open neighbourhood U of S_l^k in $F_l^k \setminus F_{l-1}^k$. After shrinking U, if necessary, we can arrange that $U \cap f(Y) = \emptyset$ since $f(Y) \cap S_l^k = \emptyset$. Choose a smaller open neighbourhood V of S_l^k such that $\overline{V} \subset U$.

Let $f_1 : f^{-1}(U) \to U$ denote the restriction of f to the manifold $f^{-1}(U)$. Note that $f_1(f^{-1}(U \setminus V))$ is disjoint from S_l^k by construction. By Thom's transversality theorem [3, p. 557], since $\dim(f^{-1}(U)) \leqslant l^2-1$ and S_l^k has codimension l^2 in U, there is a smooth map $g_1 : f^{-1}(U) \to U$ such that the image of g_1 is disjoint from S_l^k and $g_1 = f_1$ on the closed subset $f^{-1}(U \setminus V)$ of $f^{-1}(U)$ and $d(f_1(x), g_1(x)) < \varepsilon$ for all $x \in f^{-1}(U)$. The map g obtained by gluing g_1 with the restriction of f to $f^{-1}(F_l^k \setminus \overline{V})$ along $f^{-1}(U \setminus \overline{V})$ then satisfies the conditions (i)–(iii) from above. □

4. Selections up to homotopy

Let $\psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ be a unital $*$-homomorphism. Let us abuse notation slightly and write ψ_x for $\psi|_{\{x\}}$. Set

$$V_i = \{x \in X \mid N_1^{\psi}(x) = i\}$$

and

$$O_i = \{x \in X \mid N_1^{\psi}(x) < i+1\} = \bigcup_{j=1}^{i} V_j.$$

(We will write $V_{i,\psi}$ and $O_{i,\psi}$ for V_i and O_i, respectively, whenever it is not clear that ψ is the $*$-homomorphism with respect to which V_i and O_i have been defined.) Note that O_i is open for each i.

Each ψ_x has the form

$$u\left(\gamma' \oplus \bigoplus_{i=1}^{N_1^{\psi}(x)} e_1\right)u^*$$

for some $u \in \mathcal{U}(\mathrm{M}_k)$. Set

$$\psi_x^1 = u\left(\bigoplus_{i=1}^{N_1^{\psi}(x)} e_1\right)u^*.$$

In words, ψ_x^1 is the largest direct summand of ψ_x which factors through e_1. We may view ψ_x^1 as a unital $*$-homomorphism from M_q to $\psi_x^1(1)\mathrm{M}_k\psi_x^1(1)$. If $Y \subset X$ is closed, then we define $\psi_Y^1 : \mathrm{I}_{p,q} \to \mathrm{Map}(Y; \mathrm{M}_k)$ by

$$\psi_Y^1(a)(x) = \psi_x^1(a), \quad \forall x \in Y.$$

Lemma 4.1. *Let $\psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ be a unital $*$-homomorphism, and let $Y \subset X$ be closed. If N_1^{ψ} is constant on Y, then ψ_Y^1 is a $*$-homomorphism from $\mathrm{I}_{p,q}$ into $\mathrm{C}(Y) \otimes \mathrm{M}_k \cong \mathrm{C}(Y; \mathrm{M}_k)$ which factors through e_1.*

Proof. Since ϕ_Y^1 preserves pointwise operations by definition, we need only check that $\psi_Y^1(a) \in \mathrm{C}(Y) \otimes \mathrm{M}_k$ for each $a \in \mathrm{I}_{p,q}$. Using the fact that N_1^{ψ} is constant on Y and a compactness argument, we can find some $0 < \delta < 1/2$ such that if ev_t is, up to unitary equivalence, a summand of ψ_x and $x \in Y$, then $t \in (0, 1-\delta)$. Let $\tilde{a} \in \mathrm{I}_{p,q}$ be an element which is equal to a at $1 \in \mathrm{Spec}(\mathrm{I}_{p,q})$ and which vanishes on $[0, 1-\delta) \subseteq \mathrm{Spec}(\mathrm{I}_{p,q})$. Since both $\psi_Y^1(a)$ and $\psi(\tilde{a})|_Y$ depend only on the value of a at 1, we conclude that they are equal. Since $\psi(\tilde{a})|_Y \in \mathrm{C}(Y) \otimes \mathrm{M}_k$, this completes the proof. $\square$

Let $\{e_{ij}\}_{i,j=1}^q$ be a set of matrix units for M_q, so that $\{\psi_x^1(e_{ij})\}_{i,j=1}^q$ is a set of matrix units for the image of ψ_x^1. Given a subprojection r of $\psi_x^1(e_{11})$, we can generate a direct summand of ψ_x^1 as follows: for each $i, j \in \{1, \dots, q\}$, set $f_{ij} = \psi_x^1(e_{i1})r\psi_x^1(e_{1j})$; notice that the f_{ij} form a set of matrix units, so that

$$\psi_x^1 = \psi_x^{\bar{r}} \oplus \psi_x^r$$

with $\psi_x^r(e_{ij}) := f_{ij}$. We will refer to the map ψ_x^r as the *direct summand of ψ_x^1 generated by r*. If $Y \subset X$ is closed and $r : Y \to \mathrm{M}_k$ is a projection-valued function such that $r(x) \leqslant \psi_x^1(e_{11})$ for every $x \in Y$, then we define $\psi_Y^r : \mathrm{M}_q \to \mathrm{Map}(Y; \mathrm{M}_k)$ by

$$\psi_Y^r(a)(x) = \psi_x^{r(x)}(a).$$

If Y satisfies the hypotheses of Lemma 4.1 and r is continuous, then ψ_Y^r defines a unital $*$-homomorphism from M_q into a corner of $(\mathrm{C}(Y) \otimes \mathrm{M}_k)$.

Let h_t be the continuous self-map of $[0,1]$ introduced in the proof of Lemma 3.1, and let η_t be the induced endomorphism of $\mathrm{I}_{p,q}$. Observe that $(\psi \circ \eta_t)_x$ has fewer generic representations than ψ_x. Alternatively,

$$N_i^{\psi\circ\eta_t}(x) \geqslant N_i^{\psi}(x), \quad \forall x \in X,\ t \in [0,1/2),\ i \in \{0,1\}.$$

Also note that $\psi \circ \eta_t$ is homotopic to ψ for each $t \in [0,1/2)$. Since h_t fixes 1, we have that

$$(\psi \circ \eta_t)_x^1 = \psi_{x,t}^1 \oplus \psi_x^1, \tag{2}$$

for some suitable $*$-homomorphism $\psi_{x,t}^1$ which factors through e_1. Also, for any $0 \leqslant t < s < 1/2$ we have that

$$\overline{O_{i,(\psi\circ\eta_s)}} \subseteq O_{i,(\psi\circ\eta_t)}. \tag{3}$$

Inspection of the definition of h_t shows that $h_t \circ h_s = h_{s'}$ for some $s' \in [0,1/2)$.

Lemma 4.2. *Let $\psi : \mathrm{I_{p,q}} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ be a unital $*$-homomorphism, and let $i \in \{1,\dots,\lfloor k/q \rfloor - 1\}$ be given. Suppose that the following statements hold:*

(i) *there is a continuous and constant rank projection-valued map*

$$Q : \overline{O_{i,\psi}} \to \mathrm{M}_k$$

corresponding to a trivial vector bundle ($O_{i,\psi}$ is assumed to be non-empty);

(ii) *for each $x \in \overline{O_{i,\psi}}$,*

$$Q(x) \leqslant \psi_x^1(e_{11})$$

and

$$\operatorname{rank}\big(Q(x)\big) + 2\dim(X) \leqslant \operatorname{rank}\big(\psi_x^1(e_{11})\big);$$

(iii) *the map $\psi\frac{Q(x)}{O_{i,\psi}}$ defines a $*$-homomorphism from M_q to $\mathrm{C}(\overline{O_{i,\psi}}) \otimes \mathrm{M}_k$.*

Then, there is a unital $*$-homomorphism $\gamma : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ homotopic to ψ such that the following statements hold:

(i) *there is a continuous and constant rank projection-valued map*

$$\tilde{Q} : \overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}} \to \mathrm{M}_k$$

which corresponds to a trivial vector bundle and extends the function Q above;

(ii) *for each $x \in \overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}}$,*

$$\tilde{Q}(x) \leqslant \gamma_x^1(e_{11})$$

and

$$\operatorname{rank}\big(\tilde{Q}(x)\big) + 2\dim(X) \leqslant \operatorname{rank}\big(\gamma_x^1(e_{11})\big);$$

(iii) *the map* $\tilde{\phi} : \mathrm{M}_q \to \mathrm{Map}(\overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}}; \mathrm{M}_k)$ *which agrees with* $\psi^{Q(x)}_{\overline{O_{i,\psi}}}$ *on* $\overline{O_{i,\psi}}$ *and is equal to* $\gamma_x^{\tilde{Q}(x)}$ *otherwise is in fact equal to* $\gamma^{\tilde{Q}(x)}_{\overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}}}$, *and is a* $*$*-homomorphism into* $\mathrm{C}(\overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}}) \otimes \mathrm{M}_k$.

Proof. Choose $t \in (0, 1/2)$, and set $\gamma = \psi \circ \eta_t$. This ensures that ψ and γ are homotopic. Note that $A := \overline{O_{i+1,\gamma}} \backslash O_{i,\psi}$ is closed in X, and that Q is already defined on $O_{i,\psi}$. The task of extending Q to $\tilde{Q}$ is thus reduced to the problem of extending $Q|_{A \cap \overline{O_{i,\psi}}}$ to all of A and proving that our extension satisfies (i), (ii), and (iii) in the conclusion of the lemma.

Any constant rank extension of $Q|_{A \cap \overline{O_{i,\psi}}}$ to all of A will automatically satisfy the rank requirement in conclusion (ii). So, to begin, let us extend $Q|_{A \cap \overline{O_{i,\psi}}}$ to a trivial constant rank projection $\tilde{Q}$ defined on all of A and subordinate to $\gamma_x^1(e_{11})$ at each $x \in A$. For each $x \in A$, we have that

$$\operatorname{rank}\big(\gamma_x^1(e_{11})\big) \geqslant \operatorname{rank}\big(\psi_x^1(e_{11})\big) \geqslant 2\dim(X) + \operatorname{rank}(Q). \tag{4}$$

By (3), A is a closed subset of $V_{i+1} = O_{i+1,\psi} \backslash O_{i,\psi}$. In particular, the rank of $\psi_x^1(e_{11})$ is constant on A, and $x \mapsto \psi_x^1(e_{11})$ is a continuous and constant rank projection valued function on A, call it $R(x)$. Moreover, $Q(x) \leqslant R(x) \leqslant \gamma_x^1(e_{11})$ for each $x \in A \cap \overline{O_{i,\psi}}$. It is a general fact that $Q|_{A \cap \overline{O_{i,\psi}}}$ can be extended to a trivial projection defined on A and subordinate to R, as desired—all that is required for this is the rank inequality of (4). Our $\tilde{Q}$ thus satisfies parts (i) and (ii) in the conclusion of the lemma.

Let us now establish part (iii) in the conclusion of the lemma. Observe that at each point in X, γ_x^1 decomposes as the direct sum of ψ_x^1 and a second morphism, say λ (cf. (2) above). Also recall that $Q(x) = \tilde{Q}(x)$ for each $x \in A \cap \overline{O_{i,\psi}}$. It follows that for any $x \in A \cap \overline{O_{i,\psi}}$ and $i, j \in \{1, \dots, q\}$, we have

$$\begin{aligned}
\gamma_x^{\tilde{Q}(x)}(e_{ij}) &= \gamma_x^1(e_{i1})\tilde{Q}(x)\gamma_x^1(e_{1j}) \\
&= \big(\psi_x^1(e_{i1}) \oplus \lambda(e_{i1})\big)Q(x)\big(\psi_x^1(e_{1j}) \oplus \lambda(e_{1j})\big) \\
&= \psi_x^1(e_{i1})Q(x)\psi_x^1(e_{1j}) \\
&= \psi_x^{Q(x)}(e_{ij}),
\end{aligned}$$

and so

$$\gamma_x^{\tilde{Q}(x)} = \gamma_x^{Q(x)} = \psi_x^{Q(x)}, \quad \forall x \in A \cap \overline{O_{i,\psi}}.$$

Our problem is thus reduced to proving that $\gamma_A^{\tilde{Q}}$ is a $*$-homomorphism from M_q into $\mathrm{C}(A) \otimes \mathrm{M}_k$. In a near repeat of our calculation above we have

$$\begin{aligned}\gamma_x^{\tilde{Q}(x)}(e_{ij}) &= \gamma_x^1(e_{i1})\tilde{Q}(x)\gamma_x^1(e_{1j}) \\ &= \big(\psi_x^1(e_{i1}) \oplus \lambda(e_{i1})\big)\tilde{Q}(x)\big(\psi_x^1(e_{1j}) \oplus \lambda(e_{1j})\big) \\ &= \psi_x^1(e_{i1})\tilde{Q}(x)\psi_x^1(e_{1j}) \\ &= \psi_x^{\tilde{Q}(x)}(e_{ij});\end{aligned}$$

$\psi_x^{\tilde{Q}(x)}(e_{1j})$ is defined because $\tilde{Q}(x) \leqslant R(x) \equiv \psi_x^1(e_{11})$ by construction. We conclude that $\psi_A^{\tilde{Q}} = \gamma_A^{\tilde{Q}}$. Since $A \subseteq V_{i+1}$, the function N_1^{ψ} is constant on A. It now follows from Lemma 4.1 that $\psi_A^{\tilde{Q}}$ is a $*$-homomorphism, completing the proof. □

Proposition 4.3. *Let X be a compact metric space of covering dimension $d < \infty$, and let $\psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ be a unital $*$-homomorphism. Suppose that $N_1^{\psi}(x) > 2d$ for each $x \in X$, and let m be the miminum value taken by N_1^{ψ} on X. Then, ψ is homotopic to a unital $*$-homomorphism $\gamma : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ with the following property: γ can be decomposed as a direct sum $\eta \oplus \phi$, where ϕ is unitarily equivalent to*

$$\bigoplus_{j=1}^{m-2d} e_1$$

inside $\mathrm{C}(X) \otimes \mathrm{M}_k$.

Proof. We will proceed by iterated application of Lemma 4.2. First, we require some initial data satisfying the hypotheses of the said lemma. Suppose that $O_{j,\psi}$ is empty for $0 \leqslant j \leqslant i$, and that $O_{i+1,\psi}$ is not empty. It follows that for some choice of $t \in (0, 1/2)$, the morphism $\Delta := \psi \circ \eta_t$ has $O_{i+1,\Delta}$ non-empty. Since $O_{i,\psi}$ is empty, we have that

$$V_{i+1,\psi} = O_{i+1,\psi} \supseteq \overline{O_{i+1,\Delta}}.$$

In particular, N_1^{ψ} is constant on $\overline{O_{i+1,\Delta}}$, and so $\psi^1_{\overline{O_{i+1,\Delta}}}$ is a $*$-homomorphism. The map $x \mapsto \psi_x^1(e_{11})$ is continuous and projection-valued on $\overline{O_{i+1,\Delta}}$. Using the stability properties of vector bundles, we may find a continuous and projection-valued map $Q : \overline{O_{i+1,\Delta}} \to \mathrm{M}_k$ which is subordinate to $\phi_x^1(e_{11})$ at each $x \in \overline{O_{i+1,\Delta}}$, has constant rank equal to $m - 2d$, and corresponds to a trivial vector bundle. The maps Δ, Q, and $\Delta^{Q}_{\overline{O_{i+1,\Delta}}}$ thus constitute acceptable initial data for Lemma 4.2.

Notice that in part (iii) of the conclusion of Lemma 4.2, we may replace the set $\overline{O_{i+1,\gamma}} \cup \overline{O_{i,\psi}}$ with the smaller set $\overline{O_{i+1,\gamma}}$. Beginning with the initial data constructed above, we iterate this modified version of Lemma 4.2 as many times as is necessary in order to arrive at a $*$-homomorphism γ, homotopic to ψ by construction, which has the property that $O_{l+1,\gamma} = X$. This map has the desired direct summand ϕ upon restriction to $\overline{O_{l,\gamma}}$, as provided by part (iii) in the conclusion of Lemma 4.2. In order to extend ϕ to all of X, simply follow the proof of Lemma 4.2 with $A := O_{l+1,\gamma} \backslash O_{l,\gamma} = V_{l+1,\gamma}$. The proof works because this choice of A is closed.

The unitary equivalence of ϕ with

$$\bigoplus_{j=1}^{m-2d} e_1$$

follows from two facts: first, the images of the projections $e_{11}, e_{22}, \ldots$ under ϕ all correspond to trivial vector bundles by construction; second, the complement of the sum of these images has rank larger than d and the same K_0-class as a trivial vector bundle, whence it is unitarily equivalent to any projection corresponding to a trivial vector bundle of the same rank. $\square$

Remark 4.4. By replacing 1 with 0 and p with q in this subsection, Proposition 4.3 can be restated with N_0^{ψ} substituted for N_1^{ψ} and e_0 substituted for e_1. This fact will be used in the proof of Lemma 4.5 below.

Lemma 4.5. *Let X be a compact metric space of covering dimension $d < \infty$. Let there be given a unital $*$-homomorphism $\phi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ with the property that $N_g^{\phi} < L$ on X for some $L > 2d$. Assume that $k > (2L + 2d + 1)pq$. It follows that ϕ is homotopic to a second morphism ψ with the property that*

$$N_1^{\psi} \geqslant (k/q) - 2p(L + d).$$

Proof. Since N_0^{ϕ} is upper semicontinuous, the set

$$F_z^{\phi} := \{x \in X \mid N_0^{\phi}(x) \geqslant z\}$$

is closed for every $z \in \mathbb{Z}^+$. We also have that $F_z^{\phi} \subseteq F_{z'}^{\phi}$ whenever $z \geqslant z'$.

Claim 1. *F_{z+qL}^{ϕ} is contained in the interior of F_z^{ϕ}.*

Proof. Let $x \in F_{z+qL}$. By the upper semicontinuity of N_0^{ϕ} and N_1^{ϕ}, there is a neighbourhood V of x such that $N_0^{\phi}(x) - N_0^{\phi}(v) \geqslant 0$ and $N_1^{\phi}(x) - N_1^{\phi}(v) \geqslant 0$ for all $v \in V$. Using (1) and the assumption that $N_g^{\phi} < L$ we have that

$$\begin{aligned} p\big(N_0^{\phi}(x) - N_0^{\phi}(v)\big) &\leqslant p\big(N_0^{\phi}(x) - N_0^{\phi}(v)\big) + q\big(N_1^{\phi}(x) - N_1^{\phi}(v)\big) \\ &= pq\big(N_g^{\phi}(v) - N_g^{\phi}(x)\big) < pqL \end{aligned}$$

for all $v \in V$. It follows that $N_0^{\phi}(v) > N_0^{\phi}(x) - qL \geqslant z + qL - qL = z$ and hence $V \subset F_z^{\phi}$. $\square$

From the claim above we conclude that for each $n \in \mathbb{N}$, there is an open set $U_{nqL}^{\phi} \subseteq X$ such that $F_{nqL}^{\phi} \subseteq U_{nqL}^{\phi} \subseteq F_{(n-1)qL}^{\phi}$.

Consider the following chain of inclusions:

$$F_{qL}^{\phi} \supseteq U_{2qL}^{\phi} \supseteq F_{2qL}^{\phi} \supseteq U_{3qL}^{\phi} \supseteq F_{3qL}^{\phi} \supseteq \cdots.$$

It follows from the definition of η_s that each $x \in F_z^{\phi}$ is an interior point of $F_z^{\phi\circ\eta_s}$ whenever $s > 0$. We may thus assume, by modifying U_{nqL}^{ϕ} as necessary, that for some $s_0 \in [0, 1/2)$ and each $n \in \mathbb{N}$ we have

$$F_{nqL}^{\phi\circ\eta_{s_0}} \supseteq \overline{U_{nqL}^{\phi}} \supseteq U_{nqL}^{\phi} \supseteq F_{nqL}^{\phi}.$$

Set $\phi_0 = \phi \circ \eta_{s_0}$. Let us consider the finite set S consisting of those positive integers n such that $nL > 2d$ and $F_{nqL}^{\phi} \neq \emptyset$. Since $L > 2d$ by hypothesis, we have $1 \in S$ whenever S is non-empty.

Claim 2. *There is a $t_0 \in [0, 1/2)$ such that the following statements hold for $\phi' := \phi_0 \circ \eta_{t_0}$:*

(i) *for every $n \in S$, the restriction $\phi'|_{\overline{U_{nqL}^{\phi}}}$ has a direct summand of the form*

$$\gamma_n := \bigoplus_{i=1}^{(nL-2d)q} e_0 = \bigoplus_{j=1}^{nL-2d} ev_0;$$

(ii) *$\gamma_n|_{\overline{U_{mqL}^{\phi}}}$ is a direct summand of γ_m whenever $m > n$.*

Proof. We proceed by induction on n. Let $n = 1$. Upon adjusting Proposition 4.3 according to Remark 4.4, we may apply it to $\phi_0|_{\overline{U_{qL}^{\phi}}}$ and find a unital $*$-homomorphism

$$\phi_1 : \mathrm{I}_{p,q} \to \mathrm{C}\big(\overline{U_{qL}^{\phi}}\big) \otimes \mathrm{M}_k$$

homotopic to $\phi_0|_{\overline{U_{qL}^{\phi}}}$ such that ϕ_1 has a direct summand of the form

$$\gamma_1 := \bigoplus_{i=1}^{(L-2d)q} e_0 = \bigoplus_{j=1}^{L-2d} ev_0.$$

The homotopy which arises in the proof of Proposition 4.3 comes from composing $\phi_0|_{\overline{U_{qL}^{\phi}}}$ with a path of η_ts. This homotopy may be extended to all of X by composing ϕ_0 with the same path of η_ts, and so we may assume that ϕ_1 is defined on all of X.

Now let us suppose that we have found a unital $*$-homomorphism $\phi_n : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ homotopic to ϕ_0 via composition with η_ts such that the following statements hold:

(i) for every $k \in \{1, \ldots, n\} \subseteq S$, the restriction $\phi_n|_{\overline{U_{kqL}^{\phi}}}$ has a direct summand of the form

$$\gamma_k := \bigoplus_{i=1}^{(kL-2d)q} e_0 = \bigoplus_{j=1}^{kL-2d} ev_0;$$

(ii) $\gamma_k|_{\overline{U_{mqL}^{\phi}}}$ is a direct summand of γ_m whenever $m > k$.

Assuming that $n+1 \in S$, we will construct ϕ_{n+1}, homotopic to ϕ_n via composition with η_ts, such that the statements (i) and (ii) above hold with n replaced by $n+1$. Through successive applications of this inductive step we will arrive at the map ϕ' required by the claim.

If $\overline{U^{\phi}_{(n+1)qL}} = \emptyset$ then we simply put $\phi' = \phi_n$; suppose that $\overline{U^{\phi}_{(n+1)qL}} \neq \emptyset$. Denote by p_n the image of the unit of $\mathrm{I}_{p,q}$ under γ_n, restricted to $\overline{U^{\phi}_{(n+1)qL}}$. Applying Proposition 4.3 to the cut-down map

$$(1-p_n)\big(\phi_n|_{\overline{U^{\phi}_{(n+1)qL}}}\big)(1-p_n),$$

we find a unital $*$-homomorphism

$$\phi_{n+1} : \mathrm{I}_{p,q} \to \mathrm{C}\big(\overline{U^{\phi}_{(n+1)qL}}\big) \otimes \mathrm{M}_k$$

(which, as in the establishment of the base case, arises from composing $\phi_n|_{\overline{U^{\phi}_{(n+1)qL}}}$ with some η_t) admitting a direct summand

$$\alpha_{n+1} = \bigoplus_{i=1}^{Lq} e_0 = \bigoplus_{j=1}^{L} ev_0.$$

Define

$$\gamma_{n+1} = \alpha_{n+1} \oplus \gamma_n|_{\overline{U^{\phi}_{(n+1)qL}}}.$$

As before, we may extend the definition of ϕ_{n+1} to all of X. These choices of ϕ_{n+1} and γ_{n+1} establish the induction step of our argument, proving the claim. $\square$

Set $\alpha_1 = \gamma_1$. Choose, for each $n \in S$, a continuous map $g_n : \overline{U^{\phi}_{nqL}} \to [0,1]$ with the property that g_n is identically zero off U^{ϕ}_{nqL} and identically one on F^{ϕ}_{nqL}. Notice that at any given $x \in X$, at most one of the g_ns defined at x can take a value other than one.

Define a homotopy $H : [0,1] \to \mathrm{Hom}_1(\mathrm{I}_{p,q}; \mathrm{C}(X) \otimes \mathrm{M}_k)$ as follows:

(i) $H(0) = \phi'$.

(ii) For $t \in (0,1]$, n such that $\overline{U^{\phi}_{nqL}} \backslash \overline{U^{\phi}_{(n+1)qL}} \neq \emptyset$, and $x \in \overline{U^{\phi}_{nqL}} \backslash \overline{U^{\phi}_{(n+1)qL}}$, $H(t)_x$ differs from ϕ'_x as follows: with $\gamma_n^{\perp}$ denoting the complement of γ_n inside $\phi'|_{\overline{U^{\phi}_{nqL}}}$ we have

$$\begin{aligned}\phi'_x &= \big(\gamma_n^{\perp}\big)_x \oplus \bigoplus_{i=1}^{n} (\alpha_i)_x \\ &= \big(\gamma_n^{\perp}\big)_x \oplus \left(\bigoplus_{j=1}^{L-2d} ev_0\right) \oplus \underbrace{\left(\bigoplus_{j=1}^{L} ev_0\right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{L} ev_0\right)}_{n-1 \text{ times}},\end{aligned}$$

where the summands of the form $\bigoplus_{j=1}^{L} ev_0$ correspond, in order, to the $(\alpha_i)_x$s with $i > 1$; on the other hand, modifying the second equation above, we set

$$H(t)_x = \left(\gamma_n^\perp\right)_x \oplus \left(\bigoplus_{j=1}^{L-2d} ev_{tg_1(x)}\right) \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_2(x)}\right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_n(x)}\right).$$

(iii) For $t \in (0, 1]$ and $x \notin \overline{U_{qL}^\phi}$, we set $H(t)_x = \phi'_x$.

To see that $H(t)$ is a homotopy, one need only check the continuity of $H(t)_x$ at the boundary of $\overline{U_{nqL}^\phi}$. This amounts to checking that if $x, y \in X$ are close and $x \in \partial\overline{U_{nqL}^\phi}$, then $H(t)_x$ and $H(t)_y$ are close. Each y sufficiently close to x is either in $\overline{U_{nqL}^\phi}\backslash\overline{U_{(n+1)qL}^\phi}$ or $\overline{U_{(n-1)qL}^\phi}\backslash\overline{U_{nqL}^\phi}$ by Claim 1, so we need only address this situation. If $y \in \overline{U_{nqL}^\phi}\backslash\overline{U_{(n+1)qL}^\phi}$, then $H(t)_x$ and $H(t)_y$ are close by part (ii) of the definition of $H(t)$ and the fact that $(\gamma_n^\perp)_x$ is continuous in x on $\overline{U_{nqL}^\phi}\backslash\overline{U_{(n+1)qL}^\phi}$ (it is the complement of $(\gamma_n)_x$, and the latter is continuous in x on $\overline{U_{nqL}^\phi}\backslash\overline{U_{(n+1)qL}^\phi}$ by construction). If, on the other hand, $y \in \overline{U_{(n-1)qL}^\phi}\backslash\overline{U_{nqL}^\phi}$, then we must check that

$$H(t)_x = \left(\gamma_n^\perp\right)_x \oplus \left(\bigoplus_{j=1}^{L-2d} ev_{tg_1(x)}\right) \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_2(x)}\right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_n(x)}\right)$$

is close to

$$H(t)_y = \left(\gamma_{n-1}^\perp\right)_y \oplus \left(\bigoplus_{j=1}^{L-2d} ev_{tg_1(y)}\right) \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_2(y)}\right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_{n-1}(y)}\right).$$

Since $\bigoplus_{j=1}^{L} ev_{tg_k(x)}$ is continuous on $\overline{U_{(n-1)qL}^\phi}$ for each $k \in \{1, \ldots, n-1\}$, we need only check that $(\gamma_n^\perp)_x \oplus (\bigoplus_{j=1}^{L} ev_{tg_n(x)})$ is close to $(\gamma_{n-1}^\perp)_y$. By the definition of g_n, we have $g_n(x) = 0$, so that

$$\left(\gamma_n^\perp\right)_x \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_n(x)}\right) = \left(\gamma_n^\perp\right)_x \oplus \left(\bigoplus_{j=1}^{L} ev_0\right) = \left(\gamma_{n-1}^\perp\right)_x.$$

Our desired conclusion now follows from the continuity of $\gamma_{n-1}^\perp$ on $\overline{U_{(n-1)qL}^\phi}$.

Put $\psi = H(1)$, so that ψ is homotopic to ϕ, as required. To complete the proof of the lemma, we analyse the function N_1^ψ. The homotopy H leaves untouched those direct summands of ϕ'_x of the form e_1, whence $N_1^\psi \geqslant N_1^{\phi'}$. By construction, we have that $N_1^{\phi'}(x) \geqslant N_1^\phi(x)$ for each $x \in X$.

First consider the case $S = \emptyset$. (Note that this assumption implies $F_{qL}^\phi = \emptyset$.) It follows that $N_0^\phi < qL$ on X. Using (1) we obtain the following for each $x \in X$:

$$\begin{aligned} qN_1^\phi(x) &= k - pN_0^\phi(x) - pqN_g^\phi(x) \\ &> k - pqL - pqL \geqslant k - 2pq(L + d). \end{aligned}$$

We conclude that

$$N_1^{\psi}(x) \geqslant N_1^{\phi'}(x) \geqslant N_1^{\phi}(x) > \big(k - 2pq(L+d)\big)/q,$$

as desired.

Now suppose that $S \neq \emptyset$ and write $S = \{1, \dots, n\}$. We have

$$X = X \backslash \overline{U_{qL}^{\phi}} \cup \left(\bigcup_{s=1}^{n-1} \overline{U_{sqL}^{\phi}} \backslash \overline{U_{(s+1)qL}^{\phi}} \right) \cup \overline{U_{nqL}^{\phi}}.$$

Fix $x \in X$. In light of the partition of X above, we consider two cases, depending on whether or not $x \in \overline{U_{qL}^{\phi}}$.

First suppose that $x \in X \backslash \overline{U_{qL}^{\phi}}$. In this case the definition of the homotopy H implies that $\psi_x = (\phi \circ \eta_t)_x$ for some $t \in [0,1)$. By the spectral properties of η_t we conclude that $N_1^{\psi}(x) \geqslant N_1^{\phi}(x)$. From here, one simply applies the argument from the $S = \emptyset$ case.

Second, suppose that $x \in \overline{U_{sqL}^{\phi}} \backslash \overline{U_{(s+1)qL}^{\phi}}$ for some $s \in \{1, \dots, n\}$, with the convention that $\overline{U_{(n+1)qL}^{\phi}} = \emptyset$. From the definition of the homotopy H we have

$$\begin{aligned}
\psi_x &= \left[\big(\gamma_s^{\perp}\big)_x \oplus \left(\bigoplus_{j=1}^{L-2d} ev_{tg_1(x)} \right) \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_2(x)} \right) \oplus \cdots \oplus \left(\bigoplus_{j=1}^{L} ev_{tg_s(x)} \right) \right]_{t=1} \\
&= \big(\gamma_s^{\perp}\big)_x \oplus \left(\bigoplus_{j=1}^{(s-1)L-2d} ev_1 \right) \oplus \left(\bigoplus_{j=1}^{L} ev_{g_s(x)} \right)
\end{aligned}$$

(the last line follows from the fact that g_k is by definition identically one on

$$\overline{U_{sqL}^{\phi}} \subseteq F_{(s-1)qL}^{\phi}$$

whenever $k \leqslant s-1$). By construction we have $(\gamma_s^{\perp})_x \oplus (\gamma_s)_x = (\phi \circ \eta_t)_x$ for some $t \in [0,1)$. This, by the spectral properties of η_t, implies that

$$N_1^{(\gamma_s^{\perp})_x \oplus (\gamma_s)_x}(x) \geqslant N_1^{\phi}(x).$$

All of the e_1 summands of $(\gamma_s^{\perp})_x \oplus (\gamma_s)_x$ are contained in $(\gamma_s^{\perp})_x$, so we conclude that $N_1^{(\gamma_s^{\perp})_x}(x) \geqslant N_1^{\phi}(x)$. Combining this with our formula for ψ_x above, we have

$$N_1^{\psi}(x) \geqslant N_1^{\phi}(x) + p\big[(s-1)L - 2d\big]. \tag{5}$$

Since $F_{(s+1)qL}^{\phi} \subseteq \overline{U_{(s+1)qL}^{\phi}}$, we have $x \in X \backslash F_{(s+1)qL}^{\phi}$. In particular, $N_0^{\phi}(x) < (s+1)qL$. Combining this last fact with (1) and the inequality $N_g^{\phi} < L$ yields

$$N_1^{\phi}(x) > (k/q) - (s+2)pL.$$

Combining the inequality above with (5) then yields

$$N_1^{\psi}(x) \geqslant (k/q) - (s+2)pL + p(sL - 2d) = (k/q) - 2p(L+d),$$

as desired. □

Corollary 4.6. *Let X be a compact metric space of covering dimension $d < \infty$. Let there be given a unital $*$-homomorphism $\phi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ with the property that $N_g^{\phi} < 2d + 1$ on X. Assume that $k > (8d+4)pq$. It follows that ϕ is homotopic to a second $*$-homomorphism ψ which has a direct summand of the form*

$$\bigoplus_{i=1}^{M} ev_{1/2},$$

where

$$M = \lfloor (k/pq) - (8d+4) \rfloor.$$

Proof. Apply Lemma 4.5 to ϕ with $L = 2d+1$. This yields a map ϕ' homotopic to ϕ with the property that

$$N_1^{\phi'} \geqslant \lfloor (k/q) - 2p(3d+1) \rfloor =: m.$$

It follows from our assumption on the size of k that $m > \lfloor (2d+2)p \rfloor$. Apply Proposition 4.3 to ϕ' with m as above. This yields a map ϕ'' homotopic to ϕ' which has a direct summand of the form $\bigoplus_{i=1}^{m-2d} e_1$. Straightforward calculation shows that $m - 2d \geqslant pM$. The summand $\bigoplus_{i=1}^{pM} e_1$ is clearly homotopic to $\bigoplus_{i=1}^{M} ev_{1/2}$, and the corollary follows. □

5. An extension result

We need the following corollary of [2, Corollary 4.6]

Corollary 5.1. *Let B be a separable, nuclear, unital, residually finite-dimensional C^*-algebra. Let $(\pi_k)_{k=1}^{\infty}$ be a sequence of unital finite-dimensional representations of B which separates the points of B and such that each representation occurs infinitely many times. For any unital C^*-algebra A and any two unital $*$-homomorphisms $\alpha, \beta : B \to M_k(A)$ with $KK(\alpha) = KK(\beta)$, there is a sequence of unitaries $u_n \in M_{k+r(n)}(A)$, where $r(n)$ is the rank of the projection $(\pi_1 \oplus \cdots \oplus \pi_n)(1_B)$, such that for all $b \in B$*

$$\lim_{n\to\infty} \big\| u_n\big(\alpha(b) \oplus \pi_1(b) \oplus \cdots \oplus \pi_n(b)\big)u_n^* - \beta(b) \oplus \pi_1(b) \oplus \cdots \oplus \pi_n(b) \big\| = 0.$$

For a finite-dimensional representation $\pi : B \to M_k$, the induced $*$-homomorphism $\pi \otimes 1_A : B \to M_k(A)$, $b \mapsto \pi(b) \otimes 1_A$ was also denoted by π in the statement above.

Proposition 5.2. *Let p and q be relatively prime integers strictly greater than one. There exists a constant $K \in \mathbb{N}$ such that the following holds: for any unital C^*-algebra A and unital*

$*$-homomorphisms $\psi, \gamma : \mathrm{I}_{p,q} \to A$, the $*$-homomorphisms $\psi \oplus K ev_{x_0}, \gamma \oplus K ev_{x_0} : \mathrm{I}_{p,q} \to \mathrm{M}_{Kpq+1}(A)$ are homotopic for any point $x_0 \in (0,1)$.

Proof. Let $a, b, m > 0$ be integers such that $ap + bq = mpq + 1$. Throughout the proof $\mathrm{I}_{p,q}$ will be denoted by B. Define $*$-homomorphisms $\alpha, \beta : B \to M_{mpq+1}(B)$ by $\alpha = \mathrm{id}_B \oplus m(ev_{x_0} \otimes 1_B)$ and $\beta = a(e_0 \otimes 1_B) \oplus b(e_1 \otimes 1_B)$. Since B is semiprojective, there exist a finite subset $\mathcal{F} \subset B$ and $\delta > 0$ such that if $\mu, \nu : B \to A$ are two unital $*$-homomorphisms such that $\|\mu(f) - \nu(f)\| < \delta$ for all $f \in \mathcal{F}$, then μ is homotopic to ν. By Corollary 5.1 there are points $t_1, \ldots, t_r \in (0,1)$ such that if $\eta = (ev_{t_1} \otimes 1_B) \oplus (ev_{t_2} \otimes 1_B) \oplus \cdots \oplus (ev_{t_r} \otimes 1_B)$, then there is a unitary $u \in M_{(m+r)pq+1}(B)$ such that

$$\left\| u\big(\alpha(f) \oplus \eta(f)\big)u^* - \beta(f) \oplus \eta(f) \right\| < \delta$$

for all $f \in \mathcal{F}$. By our choice of $\mathcal{F}$ and δ, it follows that $u(\alpha \oplus \eta)u^*$ is homotopic to $\beta \oplus \eta$. Since the unitary group of $M_{(m+r)pq+1}(B)$ is path connected [4] and since η is homotopic to $r(ev_{x_0} \otimes 1_B)$ we deduce that $\alpha \oplus r(ev_{x_0} \otimes 1_B)$ and $\beta \oplus r(ev_{x_0} \otimes 1_B)$ are homotopic as $*$-homomorphisms from $\mathrm{I}_{p,q}$ to $M_{(m+r)pq+1}(B)$. Consequently, for any unital $*$-homomorphism $\psi : B \to A$,

$$(\mathrm{id}_{(m+r)pq+1} \otimes \psi) \circ \big(\alpha \oplus r(ev_{x_0} \otimes 1_B)\big) = \psi \oplus (m+r)(ev_{x_0} \otimes 1_A)$$

is homotopic to

$$(\mathrm{id}_{(m+r)pq+1} \otimes \psi) \circ \big(\beta \oplus r(ev_{x_0} \otimes 1_B)\big) = a(e_0 \otimes 1_A) \oplus b(e_1 \otimes 1_A) \oplus r(ev_{x_0} \otimes 1_A).$$

It follows that if $\gamma : B \to A$ is any other unital $*$-homomorphism, and $K = m + r$, then $\psi \oplus K ev_{x_0}$ is homotopic to $\gamma \oplus K ev_{x_0}$ since they are both homotopic to $a(e_0 \otimes 1_A) \oplus b(e_1 \otimes 1_A) \oplus r(ev_{x_0} \otimes 1_A)$. $\square$

Theorem 5.3. *There is a constant $L > 0$ such that the following statement holds: Let X be a compact metric space of covering dimension $d < \infty$, and let $k \geqslant L(pq)^2(d+1)$ be a natural number. It follows that any two unital $*$-homomorphisms*

$$\phi, \psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$$

are homotopic.

Proof. Set $L = 16(K+1)$, where K is the constant of Proposition 5.2. We will first prove the theorem under the assumption that X is a finite CW-complex, and requiring only that

$$k \geqslant \frac{L}{2}(pq)^2(d+1) = 8(K+1)(pq)^2(d+1).$$

We will then use the semiprojectivity of $\mathrm{I}_{p,q}$ to deduce the general case.

Assume that X is a finite CW-complex. Using Proposition 3.3 and the fact that $\dim(X) = d$, we may assume that $N_g^\phi, N_g^\psi < d+1$. Since ϕ and ψ now satisfy the hypotheses of Corollary 4.6, we may simply assume that both ϕ and ψ have a direct summand of the form

$$\bigoplus_{i=1}^{M} ev_{1/2},$$

where

$$M = \lfloor (k/pq) - (8d+4) \rfloor.$$

Straightforward calculation shows that $k - Mpq \leqslant 8pq(d+1)$. Let us write

$$\phi = \phi' \oplus \left(\bigoplus_{i=1}^{M} ev_{1/2}\right); \qquad \psi = \psi' \oplus \left(\bigoplus_{i=1}^{M} ev_{1/2}\right).$$

Set $l = \mathrm{rank}(\phi'(1)) = \mathrm{rank}(\psi'(1)) \leqslant 8pq(d+1)$, so that $\phi', \psi' : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_l$ are unital $*$-homomorphisms. Set

$$\phi'' = \phi' \oplus K(1_{\mathrm{C}(X)\otimes \mathrm{M}_l} \otimes ev_{1/2}); \qquad \psi'' = \psi' \oplus K(1_{\mathrm{C}(X)\otimes \mathrm{M}_l} \otimes ev_{1/2}).$$

It follows from Proposition 5.2 that

$$\phi'', \psi'' : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_{(Kpq+1)l}$$

are homotopic. A short calculation shows that $Kl \leqslant M$. We may therefore view ϕ'' and ψ'' as direct summands of ϕ and ψ, respectively:

$$\phi = \phi'' \oplus \left(\bigoplus_{i=1}^{M-Kl} ev_{1/2}\right); \qquad \psi = \psi'' \oplus \left(\bigoplus_{i=1}^{M-Kl} ev_{1/2}\right).$$

The homotopy between ϕ'' and ψ'' now provides the desired homotopy between ϕ and ψ.

Now suppose that X is only a metric space of finite covering dimension d. We may embed X into a bounded subset of $\mathbb{R}^{2d+1}$, and so write $X = \bigcap_n X_n$, where (X_n) is a decreasing sequence of polyhedra. By the semiprojectivity of $\mathrm{I}_{p,q}$ we have

$$[\mathrm{I}_{p,q}, \mathrm{M}_k(\mathrm{C}(X))] = \lim_{n\to\infty} [\mathrm{I}_{p,q}, \mathrm{M}_k(\mathrm{C}(X_n))].$$

We may therefore assume that the homotopy classes of our given maps ϕ and ψ lie in some $[\mathrm{I}_{p,q}, \mathrm{M}_k(\mathrm{C}(X_n))]$. Having proved the theorem for finite CW-complexes, we conclude that ϕ and ψ are homotopic if

$$k \geqslant \frac{L}{2}(pq)^2\big(\dim(X_n)+1\big) \geqslant \frac{L}{2}(pq)^2(2d+2) = L(pq)^2(d+1).$$

This proves the theorem proper. $\square$

Corollary 5.4. *Let p and q be relatively prime positive integers strictly greater than one, and let X, L, and k be as in Theorem 5.3. Suppose that $Y \subseteq X$ is closed, and that we are given a unital $*$-homomorphism $\phi : \mathrm{I}_{p,q} \to \mathrm{C}(Y) \otimes \mathrm{M}_k$. It follows that there is a unital $*$-homomorphism $\psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ such that $\psi|_Y = \phi$.*

Proof. By the semiprojectivity of $\mathrm{I}_{p,q}$ we can extend ϕ to the closure of some open neighbourhood O of Y; that is, we may assume that $\phi : \mathrm{I}_{p,q} \to \mathrm{C}(\overline{O}) \otimes \mathrm{M}_k$ without changing the original definition of ϕ over Y. As explained in Section 2.2, F^k is non-empty. Choose a point $\gamma \in F^k$. By Theorem 5.3, ϕ and $1_{\mathrm{C}(\overline{O})} \otimes \gamma$ are homotopic as maps from $\mathrm{I}_{p,q}$ into $\mathrm{C}(\overline{O}) \otimes \mathrm{M}_k$. Let us denote this homotopy by

$$H : [0,1] \to \mathrm{Hom}_1\big(\mathrm{I}_{p,q}; \mathrm{C}(\overline{O}) \otimes \mathrm{M}_k\big),$$

where $H(0) = \phi$.

Find a continuous map $f : X \to [0,1]$ which is equal to zero on Y and equal to one off O. Define $\psi : \mathrm{I}_{p,q} \to \mathrm{C}(X) \otimes \mathrm{M}_k$ by the formula

$$\psi_x = \begin{cases} H(f(x))_x, & x \in \overline{O}, \\ \gamma, x \notin \overline{O}. \end{cases}$$

One checks that ψ so defined has the required property. □

6. Maps from $\mathbf{I}_{p,q}$ into recursive subhomogeneous algebras

6.1. Recursive subhomogeneous algebras

Let us recall some of the terminology and results from [9].

Definition 6.1.

(i) If X is a compact Hausdorff space and $m \in \mathbb{N}$, then $\mathrm{M}_m(\mathrm{C}(X))$ is a recursive subhomogeneous algebra (RSH algebra);

(ii) if A is a recursive subhomogeneous algebra, X is a compact Hausdorff space, $X^{(0)} \subseteq X$ is closed, $\phi : A \to \mathrm{M}_k(\mathrm{C}(X^{(0)}))$ is a unital $*$-homomorphism, and $\rho : \mathrm{M}_k(\mathrm{C}(X)) \to \mathrm{M}_k(\mathrm{C}(X^{(0)}))$ is the restriction homomorphism, then the pullback

$$A \oplus_{\mathrm{M}_k(\mathrm{C}(X^{(0)}))} \mathrm{M}_k\big(\mathrm{C}(X)\big) = \big\{(a,f) \in A \oplus \mathrm{M}_k\big(\mathrm{C}(X)\big) \bigm| \phi(a) = \rho(f)\big\}$$

is a recursive subhomogeneous algebra.

It is clear from the definition above that a C^*-algebra R is an RSH algebra if and only if it can be written in the form

$$R = \big[\cdots\big[[C_0 \oplus_{C_1^{(0)}} C_1] \oplus_{C_2^{(0)}} C_2\big]\cdots\big] \oplus_{C_l^{(0)}} C_l, \tag{6}$$

with $C_k = \mathrm{M}_{n(k)}(\mathrm{C}(X_k))$ for compact Hausdorff spaces X_k and integers $n(k)$, with $C_k^{(0)} = \mathrm{M}_{n(k)}(\mathrm{C}(X_k^{(0)}))$ for compact subsets $X_k^{(0)} \subseteq X$, and where the maps $C_k \to C_k^{(0)}$ are always the restriction maps. Let us call the C^*-algebra

$$R_k = \big[\cdots\big[[C_0 \oplus_{C_1^{(0)}} C_1] \oplus_{C_2^{(0)}} C_2\big]\cdots\big] \oplus_{C_k^{(0)}} C_k$$

the kth *stage algebra of* R. Let $\mathrm{Prim}_n(R)$ denote the space of irreducible representations of R of dimension n. We shall say that an RSH algebra has finite topological dimension if $\dim(\mathrm{Prim}_n(R))$ is finite for each n; if R has finite topological dimension, then let us call $d := \max_n \dim(\mathrm{Prim}_n(R))$ the *topological dimension of* R. If R is separable, then the X_k can be taken to be metrisable [9, Proposition 2.13]. Finally, if R has no irreducible representations of dimension less than or equal to N, then we may assume that $n(k) > N$. We shall refer to the smallest of the $n(k)$ as the *minimum matrix size of* R.

It is clear from the construction of R_{k+1} as a pullback of R_k and C_{k+1} that there is a canonical surjective $*$-homomorphism $\lambda_k : R_{k+1} \to R_k$. By composing several such, one has also a canonical surjective $*$-homomorphism from R_j to R_k for any $j > k$. Abusing notation slightly, we denote these maps by λ_k as well.

6.2. *An existence and uniqueness theorem*

Theorem 6.2. *Let R be a separable RSH algebra of finite topological dimension d and minimum matrix size n. Let p and q be relatively prime integers strictly greater than one. Suppose that $n \geqslant L(pq)^2(d+2)$, where L is the constant of Theorem* 5.3. *It follows that there is a unital $*$-homomorphism $\gamma : \mathrm{I}_{p,q} \to R$, and that any two such morphisms are homotopic.*

Proof. Let us first establish the existence of γ. This only requires

$$n \geqslant L(pq)^2(d+1).$$

We proceed by induction on the index k from Section 6.1. We have a decomposition

$$R = \big[\cdots\big[[C_0 \oplus_{C_1^{(0)}} C_1] \oplus_{C_2^{(0)}} C_2\big]\cdots\big] \oplus_{C_l^{(0)}} C_l$$

as in (6) above, where $C_0 = \mathrm{M}_{n(0)}(\mathrm{C}(X_0))$ and $n(0) \geqslant n > pq$. As explained in Section 2.2, $F^{n(0)}$ is not empty. Choose $\psi \in F^{n(0)}$. It follows that $\gamma_0 := 1_{\mathrm{C}(X_0)} \otimes \psi$ defines a unital $*$-homomorphism from $\mathrm{I}_{p,q}$ into C_0.

Suppose $k < l$, and that we have found a unital $*$-homomorphism $\gamma_k : \mathrm{I}_{p,q} \to R_k$. We will prove that γ_k can be extended to a unital $*$-homomorphism $\gamma_{k+1} : \mathrm{I}_{p,q} \to R_{k+1}$. Starting with γ_0 and applying this inductive result repeatedly will yield the map γ required by the theorem. We have

$$R_{k+1} = R_k \oplus_{C_{k+1}^{(0)}} C_{k+1}.$$

Notice that γ_k defines a unital $*$-homomorphism from $\mathrm{I}_{p,q}$ into $R_k \oplus C_{k+1}^{(0)}$ in a natural way—the map into the summand R_k is simply γ_k itself, while the map into the summand $C_{k+1}^{(0)}$ is the composition of γ_k with the clutching map $\phi : R_k \to C_{k+1}^{(0)}$ (cf. Definition 6.1). Viewing R_{k+1} as a subalgebra of $R_k \oplus C_{k+1}$, we see that our task is simply to extend the map $\phi \circ \gamma_k : \mathrm{I}_{p,q} \to C_{k+1}^{(0)}$ to all of $C_{k+1} = \mathrm{M}_{n(k+1)}(\mathrm{C}(X_{k+1}))$. Since $n(k+1) \geqslant n \geqslant L(pq)^2(d+1)$, the existence of the desired extension follows from Corollary 5.4.

Now assume that $\gamma, \psi : \mathrm{I}_{p,q} \to R$ are unital $*$-homomorphisms, and that

$$n \geqslant L(pq)^2(d+2).$$

We require a homotopy

$$H : [0,1] \to \mathrm{Hom}_1(\mathrm{I}_{p,q}; R)$$

such that $H(0) = \gamma$ and $H(1) = \psi$. We will again proceed by induction on the index k (see above). Recall the definition of the canonical surjection $\lambda_k : R \to R_k$ from the discussion following Definition 6.1. By Theorem 5.3, there is a homotopy

$$H_0 : [0,1] \to \mathrm{Hom}_1(\mathrm{I}_{p,q}; R_0)$$

such that $H_0(0) = \lambda_0 \circ \gamma$ and $H_0(1) = \lambda_0 \circ \psi$. This is the first step in our inductive construction of H.

Suppose that we have found a homotopy $H_k : [0,1] \to \mathrm{Hom}_1(\mathrm{I}_{p,q}; R_k)$ such that $H_k(0) = \lambda_k \circ \phi$ and $H_k(1) = \lambda_k \circ \psi$. Viewing H_k as a continuous map from $\mathrm{I}_{p,q}$ into $R_k \otimes \mathrm{C}([0,1])$, we will construct a homotopy $H_{k+1} : [0,1] \to \mathrm{Hom}_1(\mathrm{I}_{p,q}; R_{k+1})$ which extends H_k in the sense that $H_k = (\lambda_k \otimes \mathrm{id}) \circ H_{k+1}$, and which satisfies $H_{k+1}(0) = \lambda_{k+1} \circ \gamma$ and $H_{k+1}(1) = \lambda_{k+1} \circ \psi$. Starting with the homotopy H_0 and applying this construction inductively, we will arrive at the desired homotopy $H \equiv H_l$.

Let $\phi : R_k \to C_{k+1}^{(0)}$ be the clutching map. It will be enough for us to define H_{k+1} over

$$\mathrm{M}_{n(k+1)}\big(\mathrm{C}(X_{k+1})\big) \otimes \mathrm{C}[0,1] \cong \mathrm{M}_{n(k+1)}\big(\mathrm{C}\big(X_{k+1} \times [0,1]\big)\big)$$

subject to the following restrictions:

(i) $H_{k+1}|_{X_{k+1}^{(0)} \times \{t\}} = \phi \circ H_k(t)$, $\forall t \in [0,1]$;
(ii) $H_{k+1}|_{X_{k+1} \times \{0\}} = \gamma|_{X_{k+1}}$;
(iii) $H_{k+1}|_{X_{k+1} \times \{1\}} = \psi|_{X_{k+1}}$.

(The requirement that $H_k = (\lambda_k \otimes \mathrm{id}) \circ H_{k+1}$ takes care of the rest of the definition.) Setting

$$Y = X_{k+1}^{(0)} \times [0,1] \cup X_{k+1} \times \{0\} \cup X_{k+1} \times \{1\} \subseteq X_{k+1} \times [0,1],$$

we see that our problem is simply to extend a $*$-homomorphism

$$\alpha : \mathrm{I}_{p,q} \to \mathrm{M}_{n(k+1)}\big(\mathrm{C}(Y)\big)$$

to all of $\mathrm{M}_{n(k+1)}(\mathrm{C}(X_{k+1} \times [0,1]))$. The existence of this extension follows from Corollary 5.4, the fact that $\dim(X_{k+1} \times [0,1]) \leqslant d+1$, and our lower bound on n. $\square$

6.3. Proof of Theorem 1.1

Proof. Let A be a unital separable C*-algebra. By [13, Proposition 6.3], it will suffice to prove that for any $\mathrm{I}_{p,q}$, there is a unital $*$-homomorphism $\gamma : \mathrm{I}_{p,q} \to A^{\otimes\infty}$.

By hypothesis, there is a unital subalgebra S of $A^{\otimes\infty}$ which is separable, subhomogeneous, and has no characters. By the main result of [7], S is the limit of an inductive system (R_i, ϕ_i), where each R_i is a (unital) separable RSH algebra of finite topological dimension and each ϕ_i is injective and unital. Suppose, contrary to our desire, that $X_i := \mathrm{Prim}_1(R_i) \neq \emptyset$ for each $i \in \mathbb{N}$. Each $\phi_i : R_i \to R_{i+1}$ induces a continuous map $\phi_i^\sharp : X_{i+1} \to X_i$. Since each X_i is compact, the limit of the inverse system $(X_i, \phi_{i-1}^\sharp)$ is not empty. In other words, there is an element of X_1 which has a pre-image in X_{i+1} under each composed map $\phi_1^\sharp \circ \cdots \circ \phi_i^\sharp$. It follows that S has a character, contrary to our assumption. We therefore conclude that R_i has no characters for some $i \in \mathbb{N}$. Since the ϕ_i are injective, we have that $A^{\otimes\infty}$ contains, unitally, a recursive subhomogeneous algebra $R := \phi_{i\infty}(R_i)$ of finite topological dimension d which has no characters.

Let $n > 1$ be the minimum matrix size of R. Find a natural number m such that $n^m/(md+1) > L(pq)^2$. It follows from [9, Proposition 3.4] that the topological dimension of $R^{\otimes m}$ is at most md, while the minimum matrix size of $R^{\otimes m}$ is at least n^m. Applying Theorem 6.2 we obtain a unital $*$-homomorphism

$$\gamma : \mathrm{I}_{p,q} \to R^{\otimes m} \hookrightarrow \left(A^{\otimes\infty}\right)^m \cong A^{\otimes\infty},$$

as required. $\square$

6.4. Examples

We close by explaining why the examples (a)–(f) in the introduction satisfy the hypotheses of Theorem 1.1.

(a) Let A be a unital simple separable exact C*-algebra containing an infinite projection. It follows from a result of Kirchberg [5] that $A^{\otimes\infty}$—even $A^{\otimes 2}$—is purely infinite and simple, and so has real rank zero. By Proposition 5.7 of [8], there is a unital $*$-homomorphism $\phi : F \to A^{\otimes\infty}$, where F is a finite-dimensional (hence subhomogeneous) C*-algebra without characters.

(b) Let A be a unital separable ASH algebra without characters. Following the arguments in the proof of Theorem 1.1, we see that there must be a unital subalgebra of A which is subhomogeneous without characters.

(c) If A is properly infinite, then there is a unital embedding of $\mathcal{O}_\infty$ into A; $\mathcal{O}_\infty$ is $\mathcal{Z}$-stable by the Kirchberg–Phillips classification, and so any $\mathrm{I}_{p,q}$ embeds unitally into A.

(d) Let A be a unital separable C*-algebra of real rank zero without characters. By Proposition 5.7 of [8], there is a unital map $\phi : F \to A$, where F is a finite-dimensional C*-algebra without characters.

(e) Let X be a compact connected infinite metric space, and $\alpha : X \to X$ a minimal homeomorphism. It follows from Theorem 2.7 of [6] that the crossed product $\mathrm{C}^*(X, \mathbb{Z}, \alpha)$ contains a recursive subhomogeneous algebra without characters.

(f) There are several examples which show that Banach algebra K-theory and traces do not form a complete invariant for simple unital separable amenable C*-algebras. The first of these is due to Rørdam, and consists of a simple unital separable amenable C*-algebra A containing both a finite and an infinite projection. By the theorem of Kirchberg cited in (a) we have that $A \otimes A$ is purely infinite, and so following the arguments of (a) we see that A satisfies the hypotheses of Theorem 1.1. Other examples were produced by the second named author in [14] and [12]. These algebras are ASH and non-type-I, and so satisfy the hypotheses of Theorem 1.1 by the arguments of (b) above.

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