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A SUSPENSION THEOREM FOR CONTINUOUS TRACE C^*-ALGEBRAS

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ABSTRACT. Let $\mathscr{B}$ be a stable continuous trace C^*-algebra with spectrum Y. We prove that the natural suspension map $S_*: [C_0(X), \mathscr{B}] \to [C_0(X) \otimes C_0(\mathbf{R}), \mathscr{B} \otimes C_0(\mathbf{R})]$ is a bijection, provided that both X and Y are locally compact connected spaces whose one-point compactifications have the homotopy type of a finite CW-complex and X is noncompact. This is used to compute the second homotopy group of $\mathscr{B}$ in terms of K-theory. That is, $[C_0(\mathbf{R}^2), \mathscr{B}] = K_0(\mathscr{B}_0)$, where $\mathscr{B}_0$ is a maximal ideal of $\mathscr{B}$ if Y is compact, and $\mathscr{B}_0 = \mathscr{B}$ if Y is noncompact.

1. INTRODUCTION

For C^*-algebras A, B let $\mathrm{Hom}(A, B)$ denote the space of (not necessarily unital) $*$-homomorphisms from A to B endowed with the topology of pointwise norm convergence. By definition $\varphi, \psi \in \mathrm{Hom}(A, B)$ are called homotopic if they lie in the same pathwise connected component of $\mathrm{Hom}(A, B)$. The set of homotopy classes of the $*$-homomorphisms in $\mathrm{Hom}(A, B)$ is denoted by $[A, B]$. For a locally compact space X let $C_0(X)$ denote the C^*-algebra of complex-valued continuous functions on X vanishing at infinity. The suspension functor S for C^*-algebras is defined as follows: it takes a C^*-algebra A to $A \otimes C_0(\mathbf{R})$ and $\varphi \in \mathrm{Hom}(A, B)$ to

$$\varphi \otimes \mathrm{id}_{C_0(\mathbf{R})} \in \mathrm{Hom}(A \otimes C_0(\mathbf{R}), B \otimes C_0(\mathbf{R})).$$

For commutative A, B this corresponds via Gelfand duality to the usual topological suspension functor. Therefore, it is natural to consider the induced map

$$S_*: [A, B] \to [A \otimes C_0(\mathbf{R}), B \otimes C_0(\mathbf{R})]$$

and to think about an analogue of the Freudenthal suspension theorem (see [E, R, Sc]). Let $\mathscr{K}$ denote the compact operators on an infinite separable Hilbert space. In this note we prove the following:

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Theorem 1. *Let X, Y be locally compact connected spaces whose one-point compactifications X^+, Y^+ have the homotopy type of a finite CW-complex, and let $\mathscr{B}$ be a continuous trace C^*-algebra with spectrum Y. Assume that either $\mathscr{B}$ is stable, i.e., $\mathscr{B} \simeq \mathscr{B} \otimes \mathscr{K}$, or $\mathscr{B}$ is n-homogeneous, $n \geq 4$, and Y^+ is homotopy equivalent to a CW-complex of dimension at most $[2n/3] - 3$. If X is noncompact then the suspension map*

$$S_*: [C_0(X), \mathscr{B}] \to [C_0(X) \otimes C_0(\mathbf{R}), \mathscr{B} \otimes C_0(\mathbf{R})]$$

is a bijection.

Theorem 2. *Let Y, $\mathscr{B}$ be as in Theorem 1. If $\mathscr{B}$ is stable then $[C_0(\mathbf{R}^2), \mathscr{B}] = K_0(\mathscr{B}_0)$ where $\mathscr{B}_0$ is the subalgebra of $\mathscr{B}$ consisting of all the sections vanishing at a base point if Y is compact and $\mathscr{B}_0$ is equal to $\mathscr{B}$ if Y is noncompact.*

The idea of the proof of Theorem 1 is to use a bundle version of the classical Whitehead theorem (see Theorem 3) in order to reduce the problem to the cases $\mathscr{B} = C_0(\mathbf{R}^q) \otimes \mathscr{K}$ and respectively $\mathscr{B} = C_0(\mathbf{R}^q) \otimes M_n$ which are known. The stable case is implicitly contained in [Seg] (see also [R]) as it corresponds to the suspension isomorphism for complex connective K-theory. The unstable case follows from the stable one via a connectivity result in [DN]. Theorem 2 provides a computation of the second homotopy group of $\mathscr{B}$. The homotopy groups of C^*-algebras have been introduced in [R]. The proof of Theorem 2 is based on the fact that $\mathrm{Hom}(C_0(\mathbf{R}^2), \mathscr{K})$ is homotopy equivalent to BU, the classifying space for the infinite unitary group U. The computation of $[C_0(\mathbf{R}^n), \mathscr{B}]$ for $n \geq 3$ involves twisted connective K-theory and it is not discussed here.

2. Sections of locally trivial bundles

We review here some elementary bundle theory needed in order to state and prove Theorems 3 and 4. These theorems should be clear to anyone who is familiar with the classical Whitehead theorem, but sketched proofs are nonetheless included for the convenience of the reader.

We use the setting of [H]. Let G be a topological group, and let $\xi = (E, p, Y)$ be a principal G-bundle. Given a continuous left action $G \times F \to F$ the relation $(e, y)g = (eg, g^{-1}y)$ defines a right G-space structure on $E \times F$. Let E_F denote the quotient space $E \times F/G$, and let $p_F: E_F \to Y$ be induced by the map $E \times F \to E \to Y$. The triplet $\xi[F] = (E_F, p_F, Y)$ is a locally trivial bundle with fiber F and structural group G. It is particularly important for our purposes that the assignment $F \to \xi[F]$ is functorial and behaves naturally with respect to the induced bundles; any G-equivariant map

$$\sigma: F \to F', \qquad \sigma(g \cdot x) = g \cdot \sigma(x)$$

induces a G-equivariant map

$$\sigma_\xi: \xi[F] \to \xi[F'], \qquad \sigma_\xi(e \cdot g) = \sigma_\xi(e) \cdot g.$$

(To simplify notation sometimes we will make no difference between a bundle and its total space.) Furthermore given a continuous map $f: X \to Y$ there are natural bundleisomorphisms φ, φ' such that the following diagram is commutative:

$$\begin{array}{ccc}
\xi[F] & \xrightarrow{\sigma_\xi} & \xi[F'] \\
\uparrow & & \uparrow \\
f^*(\xi[F]) & \longrightarrow & f^*(\xi[F']) \\
\uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi'} \\
f^*(\xi)[F] & \xrightarrow{\sigma_{f^*(\xi)}} & f^*(\xi)[F']
\end{array}$$

Similarly a local trivialization $U \times G \to \xi_U$ above some subset U of Y induces a commutative diagram

$$\begin{array}{ccc}
U \times F & \longrightarrow & \xi[F]_U \\
\downarrow{\scriptstyle(\mathrm{id},\sigma)} & & \downarrow{\scriptstyle\sigma_\xi} \\
U \times F' & \longrightarrow & \xi[F']_U
\end{array}$$

relating the corresponding local trivializations of $\xi[F]$ and $\xi[F']$.

For topological spaces T, W let $\mathrm{Map}(T, W)$ denote the space of all continuous maps from T to W endowed with the compact-open topology. One defines $\Gamma(Y, \xi[F])$ to be the space of all continuous sections of $\xi[F]$, i.e., $\Gamma(Y, \xi[F])$ consists of all $\psi \in \mathrm{Map}(Y, \xi[F])$ satisfying $p_F\psi = \mathrm{id}_Y$. We let $\Gamma(Y, \xi[F])$ have the induced compact-open topology. The path components of $\Gamma(Y, \xi[F])$ correspond to the homotopy classes of sections of $\xi[F]$ which are denoted by $[Y, \xi[F]]_\Gamma$. There is an alternative description of $\Gamma(Y, \xi[F])$ in terms of G-equivariant maps. Let $\mathrm{Map}^G(E, F)$ consist of all $u \in \mathrm{Map}(E, F)$ satisfying $u(e \cdot g) = g^{-1} \cdot u(e)$, $e \in E$, $g \in G$. By [H, Theorem 8.1 p. 46] there is a bijection η from $\mathrm{Map}^G(E, F)$ to $\Gamma(Y, \xi[F])$. The section corresponding to u is $\eta(u)(eG) = (e, u(e))G$ in E_F for each $eG \in E/G \simeq Y$. Letting $\mathrm{Map}^G(E, F)$ have the compact-open topology one has:

Proposition 1. *The map* $\eta\colon \mathrm{Map}^G(E, F) \to \Gamma(Y, \xi[F])$ *is a homeomorphism.*

Proof. The continuity of η, η^{-1} is essentially a local problem so that one can reduce the proof to the case of trivial bundles. We leave the details to the reader.

Recall that a continuous map $f\colon T \to W$ is called an n-equivalence for $n \geq 1$ if f induces a one-to-one correspondence between the path components of T and W and if, for every $t \in T$, $f_*\colon \pi_q(T, t) \to \pi_q(W, f(t))$ is an isomorphisms for $0 < q < n$ and an epimorphism for $q = n$; f is called a weak homotopy equivalence if f is an n-equivalence for all $n \geq 1$. We need the following result, which is the core of Whitehead's theorem.

Proposition 2. *Let $f\colon T \to W$ be an n-equivalence, and let (P, Q) be a relative CW-complex with* $\dim(P - Q) \leq n$. *Given maps $g\colon Q \to T$ and $h\colon P \to W$ such that $h|Q = f \circ g$, there exists a map $g'\colon P \to T$ such that $g'|Q = g$ and $f \circ g'$ is homotopic to h relative to Q.*

Proof. This is [Sp, Theorem 22, p. 404].

Theorem 3. *Let $\xi = (E, p, Y)$ be a principal G-bundle whose base Y is a finite CW-complex. Let F, F' be left G-spaces, and let $\sigma\colon F \to F'$ be a*

G-equivariant map. Assume that σ *is an n-equivalence, and consider the induced map*

$$\sigma_{\xi *}\colon [Y, \xi[F]]_{\Gamma} \to [Y, \xi[F']]_{\Gamma}.$$

If $\dim Y \leq n$, *this map is surjective*; *if* $\dim Y \leq n-1$, *it is injective.*

Proof (sketch). One proves separately the surjectivity and the injectivity of $\sigma_{\xi *}$. The proof is done by induction on cells. At each step one uses Proposition 2. The characteristic map $f\colon B^q \to Y$ of a generic cell e^q gives rise to an induced bundle $f^*(\xi)$. Since the q-dimensional ball is contractible, there is a G-equivariant trivialization $B^q \times G \to f^*(\xi)$. This can be used, as explained in our preliminary discussion on bundles, to obtain the following commutative diagram:

$$\begin{array}{ccc}
\xi[F] & \xrightarrow{\sigma_\xi} & \xi[F'] \\
\uparrow & & \uparrow \\
f^*(\xi)[F] & & f^*(\xi)[F'] \\
\uparrow \wr & & \uparrow \wr \\
B^q \times F & \xrightarrow{(\mathrm{id}, \sigma)} & B^q \times F'
\end{array}$$

Therefore, by choosing good local trivialization of ξ, one has to deal with ordinary maps rather than with sections of bundles.

Next we consider a bundle version for maps of pointed spaces. This is useful for the study of homomorphisms into continuous-trace C^*-algebras with non-compact spectrum. Assume that $s_0 \in F$ is fixed under the action of G. The space $\Gamma(Y, \xi[F])$ is pointed by the section γ_0 that corresponds to the constant equivariant map that maps E onto s_0. Assume further that F is a metric space with metric d and that G acts on F by isometries. Define $\mathrm{Map}_0^G(E, F)$ to consist of maps γ in $\mathrm{Map}^G(E, F)$ such that $d(\gamma(e), s_0)$ converges to 0 as the projection $p(e)$ of e on Y goes to infinity. Via the identification with $\mathrm{Map}^G(E, F)$, we define $\Gamma_0(Y, \xi[F]) = \eta(\mathrm{Map}_0^G(E, F))$. One has a pointed version of Proposition 1: $\mathrm{Map}_0^G(E, F)$ is homeomorphic to $\Gamma_0(Y, \xi[F])$. Since G acts on F by isometries, one can use d to measure distances between points on the same fiber of $\xi[F]$, and this is a well-defined operation. Thus $\Gamma_0(Y, \xi[F])$ consists of the all sections γ such that $d(\gamma(y), \gamma_0(y))$ vanishes at infinity. It is also clear that $d(\gamma_1, \gamma_2) = \sup_{y\in Y} d(\gamma(y), \gamma_0(y))$ is a metric on $\Gamma_0(Y, \xi[F])$.

Assume that there is a G-invariant open neighbourhood V of s_0 and a G-homotopy $h\colon F \times [0, 1] \to F$ contracting V at s_0 such that $h(s, 0) = s$, $h(v, 1) = s_0$, and $h(gs, t) = gh(s, t)$ for all $s \in F$, $v \in V$, $t \in [0, 1]$, and $g \in G$. The image of γ_0 denoted by Y_0 is a copy of Y embedded in $\xi[F]$. The homotopy h produces a homotopy H on $E \times_G F$ that contracts the tubular neighbourhood $E \times_G V$ of Y_0 onto $Y_0 = E \times_G \{s_0\}$. Denote by $\Gamma_{00}(Y, \xi[F])$ the subspace of $\Gamma_0(Y, \xi[F])$ consisting of all the sections that are equal to γ_0 on a neighbourhood of infinity. Using the homotopy H it is easily seen that the embedding $\Gamma_{00}(Y, \xi[F]) \hookrightarrow \Gamma_0(Y, \xi[F])$ is a homotopy equivalence. In particular, it induces a bijection between the path components of these spaces of sections $[Y, \xi[F]]_{\Gamma_{00}} \to [Y, \xi[F]]_{\Gamma_0}$.

One has a version of Theorem 3 for based homotopy. Let (F, s_0), (F', s'_0) be left G-spaces with base points. Assume that F, F' satisfy all the conditions considered above. Therefore, it is meaningful to consider the spaces $\Gamma_0(Y, \xi[F])$, $\Gamma_{00}(Y, \xi[F])$, $\Gamma_0(Y, \xi[F'])$, and $\Gamma_{00}(Y, \xi[F'])$.

Theorem 4. *Let $\xi = (E, p, Y)$ be a principal G-bundle whose base Y is a locally compact connected space whose one-point compactification is a finite CW-complex. Let F, F' be as above, and let $\sigma: F \to F'$ be a G-equivariant map that preserves the base points. Assume that σ is an n-equivalence, and consider the induced map*

$$\sigma_{\xi^*}: [Y, \xi[F]]_{\Gamma_0} \to [Y, \xi[F']]_{\Gamma_0}.$$

If $\dim Y \leq n$, this map is surjective; if $\dim Y \leq n-1$, it is injective.

Proof (sketch). By the discussion before Theorem 4 it is enough to prove the statement for the map $\sigma_{\xi^*}: [Y, \xi[F]]_{\Gamma_{00}} \to [Y, \xi[F']]_{\Gamma_{00}}$. This is very similar to the proof of Theorem 3.

3. The proofs of Theorems 1 and 2

Recall from [D] that any stable continuous trace C^*-algebra with locally compact, separable, spectrum Y is of the form $\Gamma_0(Y, \xi[\mathscr{K}])$ for some principal $\mathrm{Aut}(\mathscr{K})$-bundle ξ over Y. This bundle is uniquely determined up to isomorphism by its Dixmier-Douady class $\delta(\xi) \in H^3(Y, \mathbf{Z})$. The action $\mathrm{Aut}(\mathscr{K}) \times \mathscr{K} \to \mathscr{K}$ is the canonical one.

In order to make our discussion clearer we choose to work in a slightly more general context. In addition, this will be useful in order to handle simultaneously both the stable and the unstable cases. To this purpose we start our discussion with some separable C^*-algebras A, B, D and a principal $\mathrm{Aut}(B)$-bundle $\xi = (E, p, Y)$. The canonical action $\mathrm{Aut}(B) \times B \to B$, $(\alpha, b) \to \alpha(b)$ induces left actions:

$\mathrm{Aut}(B) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B)$, $(\alpha, \varphi) \to \alpha \circ \varphi$,
$\mathrm{Aut}(B) \times B \otimes D \to B \otimes D$, $(\alpha, b \otimes d) \to \alpha(b) \otimes d$,
$\mathrm{Aut}(B) \times \mathrm{Hom}(A \otimes D, B \otimes D) \to \mathrm{Hom}(A \otimes D, B \otimes D)$, $(\alpha, \varphi) \to (\alpha \otimes \mathrm{id}(D)) \circ \varphi$,

which give rise to various bundles associated with ξ, namely, $\xi[B]$, $\xi[B \otimes D]$, $\xi[\mathrm{Hom}(A, B)]$, $\xi[\mathrm{Hom}(A \otimes D, B \otimes D)]$. (We work with minimal tensor products.) The C^*-algebras B, $B \otimes D$ are pointed by the zero element, and $\mathrm{Hom}(A, B)$, $\mathrm{Hom}(A \otimes D, B \otimes D)$ are pointed by the null homomorphism.

It is clear that these base points are fixed points for the action of $\mathrm{Aut}(B)$. Since A is separable, the topology of $\mathrm{Hom}(A, B)$ is metrizable. If (a_n) is a dense sequence in the unit ball of A, then $d(\varphi, \psi) = \sum_{n=1}^{\infty} \|\varphi(a_n) - \psi(a_n)\|/2^n$ is an $\mathrm{Aut}(B)$-invariant metric on $F = \mathrm{Hom}(A, B)$. Thus it makes sense to consider in this context spaces of type $\Gamma_0(Y, \xi[F])$ and $\Gamma_{00}(Y, \xi[F])$. If Y is compact, we identify $\Gamma(Y, \xi[F])$ with $\Gamma_0(Y, \xi[F])$.

Note that $\Gamma_0(Y, \xi[B])$ has a natural structure of C^*-algebra. Moreover, it is easily seen that the norm topology of $\Gamma_0(Y, \xi[B])$ coincides with the compact-open topology. The relevance of the above bundles for our problem is made clear by Proposition 3, which describes a map between Hom—*spaces* as a map between spaces of sections of bundles. This is the common idea in bundle (sheaf)-theory. Let

$$\tau_D\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A \otimes D, B \otimes D),$$
$$\overline{\tau}_D\colon \mathrm{Hom}(A, \Gamma_0(Y, \xi[B])) \to \mathrm{Hom}(A \otimes D, \Gamma_0(Y, \xi[B]) \otimes D)$$

be the maps induced by the tensor product with $\mathrm{id}(D)$. Note that τ_D is $\mathrm{Aut}(B)$-equivariant. Let

$$\beta'\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(D, B),$$
$$\overline{\beta}\colon \mathrm{Hom}(A, \Gamma_0(Y, \xi[B])) \to \mathrm{Hom}(D, \Gamma_0(Y, \xi[B]))$$

be induced by the composition with some homomorphism $\beta \in \mathrm{Hom}(D, A)$.

Proposition 3. *There are natural commutative diagrams*

$$\begin{array}{ccc} \mathrm{Hom}(A, \Gamma_0(Y, \xi[B])) & \xrightarrow{\overline{\tau}_D} & \mathrm{Hom}(A \otimes D, \Gamma_0(Y, \xi[B]) \otimes D) \\ \downarrow & & \downarrow \\ \Gamma_0(Y, \xi[\mathrm{Hom}(A, B)]) & \xrightarrow{(\tau_D)_{\xi *}} & \Gamma_0(Y, \xi[\mathrm{Hom}(A \otimes D, B \otimes D)]), \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(A, \Gamma_0(Y, \xi[B])) & \xrightarrow{\overline{\beta}} & \mathrm{Hom}(D, \Gamma_0(Y, \xi[B])) \\ \downarrow & & \downarrow \\ \Gamma_0(Y, \xi[\mathrm{Hom}(A, B)]) & \xrightarrow{\beta'_{\xi *}} & \Gamma_0(Y, \xi[\mathrm{Hom}(D, B)]) \end{array}$$

where the vertical arrows are homeomorphisms.

While the content of this proposition is intuitively clear, the technical aspects of the proof require some care. We need two preliminary lemmas. Note that by putting $G = \mathrm{Aut}(B)$ and $F = B$ in Proposition 1 one gets an isomorphism of C^*-algebras $\eta\colon \Gamma_0(Y, \xi[B]) \to \mathrm{Map}_0^G(E, B)$.

Lemma 1. *There is an isomorphism of C^*-algebras*

$$\mu\colon \mathrm{Map}_0^G(E, B) \otimes D \to \mathrm{Map}_0^G(E, B \otimes D),$$

which takes $u \otimes d$ to the map $e \to u(e) \otimes d$.

Proof. This is a straightforward generalization of the isomorphism $\mathrm{Map}_0(Y, B) \otimes D \sim \mathrm{Map}_0(Y, B \otimes D)$.

Lemma 2. *There is a natural homeomorphism*

$$\theta\colon \mathrm{Hom}(A, \mathrm{Map}_0^G(E, B)) \to \mathrm{Map}_0^G(E, \mathrm{Hom}(A, B)).$$

Proof. For $\psi\colon A \to \mathrm{Map}_0^G(E, B)$ define $\theta(\psi)\colon E \to \mathrm{Hom}(A, B)$ by $\theta(\psi)(e)(a) = \psi(a)(e)$. θ and θ^{-1} are easily seen to be continuous. This is the analogue of the homeomorphism $\mathrm{Hom}(A, \mathrm{Map}_0(Y, B)) \simeq \mathrm{Map}_0(Y, \mathrm{Hom}(A, B))$.

Using the above lemmas, we prove the first part of Proposition 3 by putting together the following three commutative diagrams:

$$
\begin{array}{ccc}
\operatorname{Hom}(A, \Gamma_0(Y, \xi[B])) & \xrightarrow{\;\overline{\tau}_D\;} & \operatorname{Hom}(A \otimes D, \Gamma_0(Y, \xi[B]) \otimes D) \\
\uparrow{\scriptstyle \eta_*} & & \uparrow{\scriptstyle (\eta \otimes \operatorname{id}(D))_*} \\
\operatorname{Hom}(A, \operatorname{Map}_0^G(E, B)) & \xrightarrow{\;\otimes \operatorname{id}(D)\;} & \operatorname{Hom}(A \otimes D, \operatorname{Map}_0^G(E, B) \otimes D) \\
 & & \downarrow{\scriptstyle \mu_*} \\
\downarrow{\scriptstyle \theta_*} & & \operatorname{Hom}(A \otimes D, \operatorname{Map}_0^G(E, B \otimes D)) \\
 & & \downarrow{\scriptstyle \theta_*} \\
\operatorname{Map}_0^G(E, \operatorname{Hom}(A, B)) & \xrightarrow{\;(\tau_D)_*\;} & \operatorname{Map}_0^G(E, \operatorname{Hom}(A \otimes D, B \otimes D)) \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\Gamma_0(Y, \xi[\operatorname{Hom}(A, B)]) & \xrightarrow{\;(\tau_D)_{\xi^*}\;} & \Gamma_0(Y, \xi[\operatorname{Hom}(A \otimes D, B \otimes D)])
\end{array}
$$

The maps η, μ, θ were defined in Proposition 1 and Lemmas 1, 2. The proof for the second diagram in Proposition 3 is similar.

Proposition 4. *Let A, B, D be C^*-algebras, let ξ be a principal* $\operatorname{Aut}(B)$*-bundle over a finite connected CW-complex Y, and let $\mathscr{B} = \Gamma(Y, \xi[B])$.*

(a) *Assume that $\tau_D = \bigotimes \operatorname{id}(D)\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(A \otimes D, B \otimes D)$ is an n-equivalence.*

Then $\overline{\tau}_D = \bigotimes \operatorname{id}(D)$ induces a map $[A, \mathscr{B}] \to [A \otimes D, \mathscr{B} \otimes D]$ which is surjective if $\dim Y \leq n$ and injective if $\dim Y \leq n-1$.

(b) *Let $\beta \in \operatorname{Hom}(D, A)$, and assume that $\beta'\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(D, B)$, $\beta'(\varphi) = \varphi \circ \beta$, is an n-equivalence. Then the map $\overline{\beta}_*\colon [A, \mathscr{B}] \to [D, \mathscr{B}]$, $\overline{\beta}(\psi) = \psi \circ \beta$, is surjective if $\dim Y \leq n$ and injective if $\dim Y \leq n-1$.*

(c) *Assume that A, B, and respectively D are continuous trace C^*-algebras with spectra X, Z, and respectively W and trivial Dixmier-Douady class. Then the statements* (a) *and* (b) *are still valid for $\mathscr{B} = \Gamma_0(Y, \xi[B])$ with noncompact Y, provided that the one-point compactifications of Y, X, Z, and W are finite CW-complexes.*

Proof. (a) and (b) follow from Theorem 3 and Proposition 3.

(c) For noncompact Y the difficulty comes from the fact that the principal bundle ξ may be not trivial when restricted to the complement of any compact subspace of Y. One way to overcome this difficulty is to impose restrictions on A, B, and D. For instance assume that A and B are continuous trace C^*-algebras with trivial Dixmier-Douady class and nice spectrum. Then it is not hard to see that the pointed space $(F, s_0) = (\operatorname{Hom}(A, B)$, *the null homomorphism*)) satisfies all the conditions that are needed in Theorem 4. This is explained in what follows. Let A have spectrum X. Assume that X^+ is a finite CW-complex. There is a homotopy $h'\colon X^+ \times [0, 1] \to X^+$ such that $h'(x, 0) = x$ and $h'(u, 1) = \infty$ for all x in X^+ and u in some open neighbourhood of ∞ in X^+. The homotopy h' defines a homotopy $h_t\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B)$, $t \in [0, 1]$, such that h_0 is the identity map and h_1 contracts some neighbourhood of the null homomorphism onto the null homomorphism. Moreover, h_t is $\mathrm{Aut}(B)$-equivariant for any $t \in [0, 1]$. Having all above, one uses Theorem 4 and Proposition 3 to derive the statement.

Proof of Theorem 1. For n finite or infinite let $\mathscr{K}_n$ denote the compact operators on an n-dimensional complex Hilbert space. Represent $\mathscr{B}$ in the form $\Gamma_0(Y, \xi[\mathscr{K}_n])$ for some principal $\mathrm{Aut}(\mathscr{K}_n)$-bundle ξ over Y. In virtue of Proposition 4 it is enough to prove that the suspension map

$$S\colon \mathrm{Hom}(C_0(X), \mathscr{K}_n) \to \mathrm{Hom}(C_0(X) \otimes C_0(\mathbf{R}), \mathscr{K}_n \otimes C_0(\mathbf{R}))$$

is an $(m-2)$-equivalence with $m = [2n/3]$.

If n is infinite, this is proved in Corollary 3.1.8 in [DN]. If n is finite, we use Theorem 6.4.2 in [DN], which tells us that the natural map $\mathrm{Hom}(C_0(X), \mathscr{K}_n) \to \mathrm{Hom}(C_0(X), \mathscr{K})$ is an m-equivalence for $n \geq 3$. Therefore, in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(C_0(X), \mathscr{K}_n) & \longrightarrow & \mathrm{Hom}(C_0(X) \otimes C_0(\mathbf{R}), \mathscr{K}_n \otimes C_0(\mathbf{R})) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(C_0(X), \mathscr{K}) & \longrightarrow & \mathrm{Hom}(C_0(X) \otimes C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R})) \end{array}$$

the left vertical arrow is an m-equivalence, the right vertical arrow is an $(m-1)$-equivalence, and the bottom horizontal arrow is an ∞-equivalence. Consequently, the suspension map is an $(m-2)$-equivalence.

Proof of Theorem 2. By Theorem 1 one can identify $[C_0(\mathbf{R}^2), \mathscr{B}_0]$ with $[C_0(\mathbf{R}^3), \mathscr{B}_0 \otimes C_0(\mathbf{R})]$. The space $\mathrm{Hom}(C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R}))$ is homotopy equivalent to $\mathbf{Z} \times BU$. The connected component of the null homomorphism, denoted by $\mathrm{Hom}(C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R}))_0$, consists of all the homomorphisms that induce the identity map on K-theory. This component is homotopy equivalent to BU. The composition with the Bott map $\beta\colon C_0(\mathbf{R}) \to \mathscr{K} \otimes C_0(\mathbf{R}^3)$ induces maps

$$\beta'\colon \mathrm{Hom}(\mathscr{K} \otimes C_0(\mathbf{R}^3), \mathscr{K} \otimes C_0(\mathbf{R})) \to \mathrm{Hom}(C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R}))_0,$$
$$\overline{\beta}\colon \mathrm{Hom}(\mathscr{K} \otimes C_0(\mathbf{R}^3), \mathscr{B}_0 \otimes C_0(\mathbf{R})) \to \mathrm{Hom}(C_0(\mathbf{R}), \mathscr{B}_0 \otimes C_0(\mathbf{R})).$$

By Proposition 3 one can identify $\overline{\beta}$ with

$$\beta'_{\xi *}\colon \Gamma_0(Y, \xi[\mathrm{Hom}(\mathscr{K} \otimes C_0(\mathbf{R}^3), \mathscr{K} \otimes C_0(\mathbf{R}))]) \to \Gamma_0(Y, \xi[\mathrm{Hom}(C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R}))_0]).$$

One uses here the connectedness of Y. The fact that β' is a weak homotopy equivalence [Seg, DN] in conjunction with Theorem 4 show that

$$\beta'_{\xi *}\colon [Y, \xi[\mathrm{Hom}(\mathscr{K} \otimes C_0(\mathbf{R}^3), \mathscr{K} \otimes C_0(\mathbf{R}))]]_{\Gamma_0} \to [Y, \xi[\mathrm{Hom}(C_0(\mathbf{R}), \mathscr{K} \otimes C_0(\mathbf{R}))_0]]_{\Gamma_0}$$

is a bijection.

It follows that $\overline{\beta}$ induces a bijection at the level of the homotopy classes of homomorphisms. Therefore, the suspension map followed by the composition with the Bott homomorphism and by the Bott isomorphism induce bijections

$$[C_0(\mathbf{R}^2), \mathscr{B}_0)] \to [C_0(\mathbf{R}^3), \mathscr{B}_0 \otimes C_0(\mathbf{R})] \to [C_0(\mathbf{R}), \mathscr{B}_0 \otimes C_0(\mathbf{R})] \simeq K_1(\mathscr{B}_0 \otimes C_0(\mathbf{R})) \simeq K_0(\mathscr{B}_0).$$

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