ON HOMOMORPHISMS OF MATRIX ALGEBRAS OF CONTINUOUS FUNCTIONS

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If $X$ is a topological space we denote by $C(X) \otimes M_n$ the algebra of continuous functions from $X$ to the algebra $M_n$ of $n \times n$ complex matrices. A complete characterization of those topological spaces $Y$ is given (in terms of vector bundles on $Y$) such that each unital algebra-homomorphism $\Phi: C(X) \otimes M_n \to C(Y) \otimes M_{kn}$ is of the form $\alpha \circ (\Phi' \otimes \text{id}_{n})$ for some homomorphism $\Phi': C(X) \to C(Y) \otimes M_k$ and some suitable inner (or $C(Y)$-linear) automorphism $\alpha$ of the algebra $C(Y) \otimes M_{kn}$. In particular this decomposition is assured provided that $Y$ is a finite CW-complex of dimension $\leq 2k$ and $K^0(Y)$ does not have $n$-torsion.

Our interest in such homomorphisms arose in connection with a question of E. G. Effros [1] concerning the structure of inductive limits of $C^*$-algebras of the form $C(X) \otimes M_n$. In this context certain classes of homomorphisms related to a covering $X \to Y$ have been considered by C. Pasnicu [5]. When restricted to the case of automorphisms our results give nothing new (see [4], [6] and [7]).

1. Preliminaries. Let $\text{GL}_n(C)$ be the general linear group (nonsingular $n \times n$ matrices over the complex field) and denote by $1_n$ its unit. Let $\text{Vect}_m(Y)$ denote the set of isomorphism classes of complex vector bundles of rank $m$ on the topological space $Y$. In $\text{Vect}_m(Y)$ we have one naturally distinguished element—the class of the trivial bundle of rank $m$. Let $T_n \text{Vect}_m(Y)$ be the subset of $\text{Vect}_m(Y)$ given by all vector bundles $E$ such that the direct sum $E \oplus E \oplus \cdots \oplus E$ (n-times) is isomorphic to the trivial bundle of rank $nm$.

If $A$, $B$ are unital complex algebras we denote by $\text{Hom}(A, B)$ the set of all unital algebra-homomorphisms from $A$ to $B$. Two homomorphisms $\Phi_1, \Phi_2 \in \text{Hom}(A, B)$ are said to be inner equivalent if $\Phi_2 = u\Phi_1u^{-1}$ for some invertible element $u \in B$. Let $\text{Hom}(A, B)/\sim$ be the set of classes of inner equivalent homomorphisms from $A$ to $B$.

We need some elementary sheaf cohomology. Let $G$ be a Lie group and let $H$ be a closed subgroup of $G$. For each topological space $Y$ the fibration $H \to G \to G/H$ induces the following exact sequence of
pointed cohomology sets:
\[ H^0(Y,H) \to H^0(Y,G) \to H^0(Y,G/H) \xrightarrow{\delta} H^1(Y,H) \to H^1(Y,G). \]

We have \( H^0(Y,H) = C(Y,H) \) (continuous maps from \( Y \) to \( H \)) and \( H^0(Y,G) = C(Y,G) \). These sets are pointed by the constant map \( f = 1_G \) given by the unity of \( G \). Similarly \( H^0(Y,G/H) = C(Y,G/H) \) is pointed by the constant map \( f = \{H\} \). The cohomology sets \( H^1(Y,H) \) and \( H^1(Y,G) \) are pointed by the trivial cocycles \( \{Y, H\} \) and \( \{Y, G\} \) respectively [2]. Given \( f \in C(Y,G/H) \) the cocycle \( \delta(f) \) represents the obstruction for lifting \( f \) to a function in \( C(Y,G) \). By the exactness of the above sequence \( f \) has a continuous lifting if and only if \( \delta(f) = (Y, 1_H) \). The action of \( G \) on \( G/H \) induces an action of \( C(Y,G) \) on \( C(Y,G/H) \). If \( f_1, f_2 \in C(Y,G/H) \) then \( \delta(f_1) = \delta(f_2) \) if and only if \( f_2 = gf_1 \) for some \( g \in C(Y,G) \).

2. Results.

**Proposition 1.** Let \( Y \) be a topological space. Then there is a bijection \( \text{Hom}(M_n, C(Y) \otimes M_{kn}) \sim T_n \text{Vect}_k(Y) \).

**Proof.** We describe the exact sequence induced by the following fibration:
\[ \text{GL}_k(C) \to \text{GL}_{kn}(C) \to \text{GL}_{kn}(C)/\text{GL}_k(C) \]
where the imbedding \( \gamma \) is given by
\[ \gamma(u) = u \otimes 1_n, \quad M_{kn} = M_k \otimes M_n. \]

There is a commutative diagram of pointed sets:
\[
\begin{array}{ccc}
C(Y, \text{GL}_{kn}(C)) & \to & C(Y, \text{GL}_{kn}(C)/\text{GL}_k(C)) \\
\| & \downarrow \alpha & \downarrow \beta \\
C(Y, \text{GL}_{kn}(C)) & \to & \text{Hom}(M_n, C(T) \otimes M_{kn}) \xrightarrow{\delta'} \text{Vect}_k(Y) \\
\end{array}
\]

The vertical arrows are bijections. To describe \( \alpha \) recall that
\[ \text{Hom}(M_n, M_{kn}) \simeq \text{GL}_{kn}(C)/\text{GL}_k(C) \]
as topological spaces, the homeomorphism being induced by the map \( \eta : \text{GL}_{kn}(C) \to \text{Hom}(M_n, M_{kn}) \) given by \( \eta(v)(a) = v(1_k \otimes a)v^{-1}, \quad a \in M_n \). Let \( \eta_1 \) be the map
\[ C(Y, \text{GL}_{kn}(C)/\text{GL}_k(C)) \to C(Y, \text{Hom}(M_n, M_{kn})) \]
induced by \( \eta \). By definition we set \( \alpha = \alpha_1 \eta_1 \) where
\[ \alpha_1 : C(Y, \text{Hom}(M_n, M_{kn})) \to \text{Hom}(M_n, C(Y) \otimes M_{kn}) \]
is given by $\alpha_1(\Psi)(a)(y) = \Psi(y)(a)$, $a \in M_n, y \in Y$. If in

$$\text{Hom}(M_n, C(Y) \otimes M_{kn})$$

we distinguish the homomorphism $a \mapsto a \otimes 1_k$, $\alpha$ will be an isomorphism of pointed sets. The maps $\beta$ and $\beta_1$ are the natural ones. Namely if $(U_i, g_{ij})$ is a $\text{GL}_k$-cocycle, then $\beta(U_i, g_{ij})$ is the isomorphism class of the vector bundle obtained by clutching the trivial bundles $U_i \times C^k$ with the transition functions $(g_{ij})$. The map $\beta_1$ is defined in a similar way. The other maps are defined to make the diagram commutative.

If $v \in C(\text{Y}, \text{GL}_{kn}(C))$ then $j'(v): M_n \rightarrow C(Y) \otimes M_{kn}$ is defined by

$$j'(v)(a)(y) = v(y)(1_k \otimes a)v(y)^{-1}, \quad a \in M_n, \; y \in Y.$$ 

The map $\gamma'$ takes the vector bundle $E$ to the direct sum $E \oplus E \oplus \cdots \oplus E$ ($n$-times). After the above identifications, it follows that two homomorphisms $\Phi_1, \Phi_2 \in \text{Hom}(M_n, C(Y) \otimes M_{kn})$ are inner equivalent if and only if $\delta'(\Phi_1) = \delta'(\Phi_2)$. The isomorphism class of the vector bundle $\delta'(\Phi_1)$ represents the obstruction for lifting $\Phi_1$ to an invertible element in $C(Y) \otimes M_{kn}$. Also, by the exactness of the second row in the above diagram, the image of $\delta'$ is equal to $T_n \text{Vect}_k(Y)$.

**Theorem 2.** Let $X, Y$ be topological spaces. Then the following assertions are equivalent:

(i) The set $T_n \text{Vect}_k(Y)$ reduces to the trivial bundle of rank $k$.

(ii) Each homomorphism $\Phi \in \text{Hom}(C(X) \otimes M_n, C(Y) \otimes M_{kn})$ is inner equivalent to a homomorphism of the form $\Phi' \otimes \text{id}_n$ for some $\Phi' \in \text{Hom}(C(X), C(Y) \otimes M_k)$.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows easily from Proposition 1. Indeed, if we choose a point $x$ in $X$ and a homomorphism $\Phi_1$ in $\text{Hom}(M_n, C(Y) \otimes M_{kn})$ which is not inner equivalent to the homomorphism $a \mapsto 1_k \otimes a$ then the homomorphism $C(X) \otimes M_n \ni F \mapsto \Phi_1(F(x)) \in C(Y) \otimes M_{kn}$ failed to satisfy (ii).

To prove the other implication we assume, as a preliminary step, that $\Phi$ acts on matrices as an amplification:

$$\Phi(1_{C(X)} \otimes a) = 1_{C(Y)} \otimes 1_k \otimes a, \quad a \in M_n.$$ 

Under this assumption we get

$$\Phi(f \otimes a) = \Phi(f \otimes 1_n)\Phi(1 \otimes a) = \Phi(1 \otimes a)\Phi(f \otimes 1_n)$$

$$= 1 \otimes 1_k \otimes a \cdot \Phi(f \otimes 1_n), \quad a \in M_n, \; f \in C(X).$$
The previous computation shows us that the algebra $\Phi(C(X) \otimes 1_n)$ lies in the relative commutant of $1_{C(Y)} \otimes 1_k \otimes M_n$ in $C(Y) \otimes M_k \otimes M_n$ which is equal to $C(Y) \otimes M_k \otimes 1_n$. It follows that there is a unique homomorphism $\Phi' \in \text{Hom}(C(X), C(Y) \otimes M_k)$ such that $\Phi(f \otimes 1_n) = \Phi'(f) \otimes 1_n$. Using again our assumption on $\Phi$ we get $\Phi = \Phi' \otimes \text{id}_n$.

Consider now an arbitrary homomorphism $\Phi$ and let $\Phi_1$ denote its restriction to $M_n$. Using (i) it follows by Proposition 1 that there is some invertible element $u \in C(Y) \otimes M_{kn}$ such that

$$\Phi_1(a) = \Phi(1 \otimes a) = u(1 \otimes 1_k \otimes a)u^{-1}, \quad a \in M_n.$$ 

Hence the homomorphism $u^{-1}\Phi u$ acts on matrices as an amplification.

**Remark 3.** The assertion (i) in the above theorem holds provided that $Y$ is homotopy equivalent to a finite CW-complex of dimension $\leq 2k$ and the $K$-theory group $K^0(Y)$ does not have $n$-torsion. This follows from the stability properties of vector bundles (see [3, Ch. 8, Th. 1.5]).

Note that $T_n \text{Vect}_1(Y)$ is a subgroup of the group $(\text{Vect}_1(Y), \otimes)$. We have a natural action of $T_n \text{Vect}_1(Y)$ on $T_n \text{Vect}_k(Y)$ given by $(L, E) \mapsto L \otimes E$. By similar methods one can prove the following

**Theorem 4.** Let $X, Y$ be topological spaces. Then the following assertions are equivalent:

(i) $T_n \text{Vect}_1(Y)$ acts transitively on $T_n \text{Vect}_k(Y)$.

(ii) For any homomorphism $\Phi \in \text{Hom}(C(X) \otimes M_n, C(Y) \otimes M_{kn})$ there is an automorphism $\alpha$ of $C(Y) \otimes M_{kn}$ which is $C(Y)$-linear such that $\alpha \circ \Phi = \Phi' \otimes \text{id}_n$ for some homomorphism $\Phi' \in \text{Hom}(C(X), C(Y) \otimes M_k)$.

**References**


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