

# Strongly Self-Absorbing $C^*$ -algebras which contain a nontrivial projection

Marius Dadarlat and Mikael Rørdam

**Abstract.** It is shown that a strongly self-absorbing  $C^*$ -algebra is of real rank zero and absorbs the Jiang-Su algebra if it contains a non-trivial projection. We also consider cases where the UCT is automatic for strongly self-absorbing  $C^*$ -algebras, and  $K$ -theoretical ways of characterizing when Kirchberg algebras are strongly self-absorbing.

## 1. INTRODUCTION

Strongly self-absorbing  $C^*$ -algebras were first systematically studied by Toms and Winter in [11]. The classification program of Elliott had prior to that been seen to work out particularly well for those (separable, nuclear)  $C^*$ -algebras that tensorially absorb one of the Cuntz algebras  $\mathcal{O}_2$ ,  $\mathcal{O}_\infty$ , or the Jiang-Su algebra  $\mathcal{Z}$ . More precisely, thanks to deep theorems of Kirchberg, the classification of separable, nuclear, stable  $C^*$ -algebras that absorb the Cuntz algebra  $\mathcal{O}_2$  is complete (the invariant is the primitive ideal space); and separable, nuclear, stable  $C^*$ -algebras that absorb the Cuntz algebra  $\mathcal{O}_\infty$  are classified by an ideal related  $KK$ -theory. The situation for separable, nuclear  $C^*$ -algebras that absorb the Jiang-Su algebra is at present very promising (see for example [13]) but not as complete as in the purely infinite case.

The  $C^*$ -algebras  $\mathcal{O}_2$ ,  $\mathcal{O}_\infty$  and  $\mathcal{Z}$  are all examples of strongly self-absorbing  $C^*$ -algebras. They are in [11] defined to be those unital separable  $C^*$ -algebras  $D \neq \mathbb{C}$  for which there is an isomorphism  $D \rightarrow D \otimes D$  that is approximately unitarily equivalent to the  $*$ -homomorphism  $d \mapsto d \otimes 1$ . Strongly self-absorbing  $C^*$ -algebras are automatically simple and nuclear, and they have at most one tracial state. It is shown in [11] that if  $D$  is a strongly self-absorbing  $C^*$ -algebra in the UCT class, then it has the same  $K$ -theory as one of the  $C^*$ -algebras in the following list:  $\mathcal{Z}$ , UHF-algebras of infinite type,  $\mathcal{O}_\infty$ ,  $\mathcal{O}_\infty$  tensor a UHF-algebra of infinite type, or  $\mathcal{O}_2$ . It is an open problem if nuclear  $C^*$ -algebras always satisfy the UCT (and also if strongly self-absorbing  $C^*$ -algebras enjoy this property); and it is an intriguing problem, very much related to the Elliott classification program, if the list above exhausts all strongly self-absorbing

---

M.D. was partially supported by NSF grant #DMS-0801173.

M.R. was supported by a grant from the Danish Natural Science Research Council (FNU)

$C^*$ -algebras. Should the latter be the case, then it would in particular follow that every strongly self-absorbing  $C^*$ -algebra absorbs the Jiang-Su algebra  $\mathcal{Z}$ . By the Kirchberg-Phillips classification theorem, a strongly self-absorbing Kirchberg algebra belongs to the list above if and only if it belongs to the UCT class. Let us also remind the reader that a strongly self-absorbing  $C^*$ -algebra is a Kirchberg algebra if and only if it is not stably finite (or, equivalently, if and only if it is traceless).

In Section 2 of this paper we show that every strongly self-absorbing  $C^*$ -algebra which contains a non-trivial projection is of real rank zero and absorbs the Jiang-Su algebra. In Section 3 we consider  $K$ -theoretical conditions on strongly self-absorbing Kirchberg algebras. One such condition (phrased at the level of  $K$ -homology) characterizes the Kirchberg algebra  $\mathcal{O}_\infty$ , and other results in Section 3 give  $K$ -theoretical characterizations on when a Kirchberg algebra is strongly self-absorbing.

## 2. STRONGLY SELF-ABSORBING $C^*$ -ALGEBRAS WITH A NON-TRIVIAL PROJECTION

In this section we show that any strongly self-absorbing  $C^*$ -algebra that contains a non-trivial projection is automatically approximately divisible, of real rank zero, and absorbs the Jiang-Su algebra  $\mathcal{Z}$ .

**Lemma 2.1.** *There is a unital  $*$ -homomorphism from  $M_3 \oplus M_2$  into a unital  $C^*$ -algebra  $A$  if and only if  $A$  contains projections  $e, e'$  such that  $e \perp e'$ ,  $e \sim e'$ , and  $1 - e - e' \precsim e$ .*

*Proof.* It is easy to see that such projections  $e$  and  $e'$  exist in  $M_3 \oplus M_2$  and hence in any unital  $C^*$ -algebra  $A$  that is the target of a unital  $*$ -homomorphism from  $M_3 \oplus M_2$ .

Assume now that such projections  $e$  and  $e'$  exist. Let  $v \in A$  be a partial isometry such that  $v^*v = e$  and  $vv^* = e'$ . Put  $f_0 = 1 - e - e'$ . Find a subprojection  $f_1$  of  $e$  which is equivalent to  $f_0$ , and put  $f_2 = vf_1v^*$ . Put  $g_1 = e - f_1$  and put  $g_2 = e' - f_2 = vg_1v^*$ . The projections  $f_0, f_1, f_2, g_1, g_2$  then satisfy

$$1 = f_0 + f_1 + f_2 + g_1 + g_2, \quad f_0 \sim f_1 \sim f_2, \quad g_1 \sim g_2.$$

Extending the sets  $\{f_0, f_1, f_2\}$  and  $\{g_1, g_2\}$  to sets of matrix units for  $M_3$  and  $M_2$ , respectively, yields a unital  $*$ -homomorphism from  $M_3 \oplus M_2$  into  $A$ . (If the  $f_j$ 's are zero or if the  $g_j$ 's are zero, then this  $*$ -homomorphism will fail to be injective, and will instead give a unital embedding of  $M_2$  or  $M_3$  into  $A$ .)  $\square$

If  $D$  is any unital (nuclear<sup>1</sup>)  $C^*$ -algebra then we let  $D^{\otimes n}$  denote the  $n$ -fold tensor product  $D \otimes D \otimes \cdots \otimes D$  (with  $n$  tensor factors), and we let  $D^{\otimes \infty}$

---

<sup>1</sup>We shall here exclusively be concerned with nuclear  $C^*$ -algebras, where the tensor product is unique; otherwise we must specify a tensor product, for example the minimal one.

denote the infinite tensor product  $\bigotimes_{n=1}^{\infty} D$ . The latter is the inductive limit of the sequence

$$D \rightarrow D^{\otimes 2} \rightarrow D^{\otimes 3} \rightarrow D^{\otimes 4} \rightarrow \dots,$$

(with connecting mappings  $d \mapsto d \otimes 1_D$ ). We shall view  $D$  as a (unital) sub- $C^*$ -algebra of  $D^{\otimes n}$ ,  $D^{\otimes n}$  as a sub- $C^*$ -algebra of  $D^{\otimes m}$  (if  $n \leq m$ ), and finally  $D$  and  $D^{\otimes n}$  are viewed as subalgebras of  $D^{\otimes \infty}$ .

If  $x \in D^{\otimes n}$ , then  $x^{\otimes k}$  will denote the  $k$ -fold tensor product

$$x^{\otimes k} = x \otimes x \otimes x \otimes \dots \otimes x \in D^{\otimes kn}.$$

The proof of the lemma below resembles the proof of [9, Lemma 6.4].

**Lemma 2.2.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra, and let  $p$  be a projection in  $D$ . Consider the following projections in  $D \otimes D$ ,*

$$e_1 = p \otimes (1 - p), \quad e'_1 = (1 - p) \otimes p, \quad f = p \otimes p + (1 - p) \otimes (1 - p).$$

*For each natural number  $n$  consider also the following projections in  $D^{\otimes 2(n+1)}$ ,*

$$e_{n+1} = f^{\otimes n} \otimes p \otimes (1 - p), \quad e'_{n+1} = f^{\otimes n} \otimes (1 - p) \otimes p.$$

*It follows that the projections  $e_1, e_2, \dots, e'_1, e'_2, \dots$  are pairwise orthogonal in  $D^{\otimes \infty}$ , and that  $e_j \sim e'_j$ . Moreover, for each natural number  $n$ , set*

$$E_n = e_1 + e_2 + \dots + e_n, \quad E'_n = e'_1 + e'_2 + \dots + e'_n.$$

*Then  $E_n \perp E'_n$ ,  $E_n \sim E'_n$ , and*

$$(1) \quad 1 - E_n - E'_n = f^{\otimes n}.$$

*Proof.* The equivalence  $e_j \sim e'_j$  comes from the fact that the flip automorphism  $a \otimes b \mapsto b \otimes a$  on  $D \otimes D$  is approximately inner when  $D$  is strongly self-absorbing. The projections  $e_1, e_2, \dots, e'_1, e'_2, \dots$  are pairwise orthogonal by construction. The only thing left to prove is (1). We prove this by induction after  $n$ , and note first that (1) for  $n = 1$  follows from the fact that  $e_1 + e'_1 + f = 1$ . Suppose that (1) holds for some  $n \geq 1$ . Then

$$\begin{aligned} & 1 - E_{n+1} - E'_{n+1} \\ &= 1 - E_n - E'_n - e_{n+1} - e'_{n+1} \\ &= f^{\otimes n} \otimes 1_D \otimes 1_D - f^{\otimes n} \otimes p \otimes (1 - p) - f^{\otimes n} \otimes (1 - p) \otimes p \\ &= f^{\otimes(n+1)}. \end{aligned}$$

□

**Lemma 2.3.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra and let  $p$  be a projection in  $D$  such that  $p \neq 1$ . Then there exists a natural number  $n$  such that  $p^{\otimes n} \lesssim 1 - p^{\otimes n}$  in  $D^{\otimes n}$ .*

*Proof.* To simplify the notation we express our calculations in terms of the monoid  $V(D)$  of Murray-von Neumann equivalence classes of projections in  $D$  and in matrix algebras over  $D$ . Let  $[e] \in V(D)$  denote the equivalence class containing the projection  $e$  in (a matrix algebra over)  $D$ .

Since  $D$  is simple and  $p \neq 1$  there is a natural number  $n$  such that  $n[1-p] \geq [p]$ . It follows that

$$\begin{aligned} & [1 - p^{\otimes n}] \\ & \geq [(1-p) \otimes p \otimes \cdots \otimes p] + [p \otimes (1-p) \otimes \cdots \otimes p] + [p \otimes p \otimes \cdots \otimes (1-p)] \\ & = n[(1-p) \otimes p \otimes \cdots \otimes p] \\ & \geq [p \otimes p \otimes p \otimes \cdots \otimes p] = [p^{\otimes n}], \end{aligned}$$

where the equality between the second and third expression holds because the flip on a strongly self-absorbing  $C^*$ -algebra is approximately inner.  $\square$

**Lemma 2.4.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra, let  $p$  be a projection in  $D^{\otimes k}$ , and let  $e$  be a projection in  $D^{\otimes l}$  for some natural numbers  $k$  and  $l$ . Assume that  $p \neq 1$  and that  $e \neq 0$ . It follows that there exists a natural number  $n$  such that  $p^{\otimes n} \lesssim e$  in  $D^{\otimes \infty}$ .*

*Proof.* Let  $d$  be a natural number such that  $dk \geq l$ . Upon replacing  $p$  with  $p^{\otimes d}$ ,  $e$  with  $e \otimes 1_D^{\otimes (dk-l)}$ , and  $D$  with  $D^{\otimes dk}$  we can assume that  $p$  and  $e$  both belong to  $D$ . Use Lemma 2.3 to find  $m$  such that  $p^{\otimes m} \lesssim 1-p^{\otimes m}$ . By replacing  $p$  with  $p^{\otimes m}$ ,  $e$  with  $e \otimes 1_D^{\otimes (m-1)}$ , and  $D$  with  $D^{\otimes m}$  we can assume that  $p$  and  $e$  both belong to  $D$  and that  $p \lesssim 1-p$ .

Now,  $p \sim q \leq 1-p$  for some projection  $q$  in  $D$ . In the language of the monoid  $V(D)$  we have

$$[1_D^{\otimes k}] \geq [(p+q)^{\otimes k}] = 2^k [p^{\otimes k}]$$

for any natural number  $k$ . Using simplicity of  $D$  we can choose  $n$  such that  $2^{n-1}[e] \geq [p]$ . Then

$$[e] = [e \otimes 1_D^{\otimes (n-1)}] \geq 2^{n-1}[e \otimes p^{\otimes (n-1)}] \geq [p^{\otimes n}],$$

in  $V(D^{\otimes n})$  as desired, where we in the first identity have used that the embedding of  $D$  into  $D^{\otimes n}$  maps  $e$  onto  $e \otimes 1_D^{\otimes (n-1)}$ .  $\square$

**Theorem 2.5.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra. Then the following three conditions are equivalent:*

- (1)  $D$  contains a non-trivial projection (i.e., a projection other than 0 and 1).
- (2)  $D$  is approximately divisible.
- (3)  $D$  is of real rank zero.

*If any of the three equivalent conditions are satisfied, then  $D$  absorbs the Jiang-Su algebra, i.e.,  $D \cong D \otimes \mathcal{Z}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). If  $D$  is strongly self-absorbing, then there is an asymptotically central sequence of embeddings of  $D$  into itself, i.e., a sequence  $\rho_k: D \rightarrow D$  of unital  $*$ -homomorphisms such that  $\|\rho_k(x)y - y\rho_k(x)\| \rightarrow 0$  as  $k \rightarrow \infty$  for all  $x, y \in D$ .

Identify  $D$  with  $D_0^{\otimes \infty}$  where  $D_0 \cong D$ . Take a non-trivial projection  $p$  in  $D_0$ . For each natural number  $n$  let  $E_n, E'_n \in D_0^{\otimes 2n}$  be as in Lemma 2.2

(corresponding to our non-trivial projection  $p$ ). Then  $e_n \neq 0$ ,  $E_n \neq 0$ , and so  $0 \neq f^{\otimes n} \neq 1$ . Use (1) and Lemma 2.4 to find  $n$  such that  $1 - E_n - E'_n \lesssim p \otimes (1 - p) \leq E_n$ . It then follows from Lemma 2.1 that there is an injective unital  $*$ -homomorphism from  $M_3 \otimes M_2$  into  $D_0^{\otimes 2n} \subseteq D$ . Composing this unital  $*$ -homomorphism with the unital  $*$ -homomorphism  $\rho_k$  yields an asymptotically central sequence of unital  $*$ -homomorphisms from  $M_3 \otimes M_2$  into  $D$ . This shows that  $D$  is approximately divisible.

(ii)  $\Rightarrow$  (iii). It is shown in [2] that a simple approximately divisible  $C^*$ -algebra is of real rank zero if and only if projections in the  $C^*$ -algebra separate the quasitraces. As quasitraces on an exact  $C^*$ -algebra are traces, [7], a result that applies to our case since strongly self-absorbing  $C^*$ -algebras are nuclear and hence exact, and since a strongly self-absorbing  $C^*$ -algebra has at most one tracial state, quasitraces are automatically separated by just one projection, say the unit.

(iii)  $\Rightarrow$  (i). This is trivial. The only  $C^*$ -algebra of real rank zero that does not have a non-trivial projection is  $\mathbb{C}$ , the algebra of complex numbers. This  $C^*$ -algebra is not strongly self-absorbing by convention.

Finally, any simple approximately divisible  $C^*$ -algebra is  $\mathcal{Z}$ -absorbing, cf. [12].  $\square$

**Lemma 2.6.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra. Then  $K_0(D)$  has a natural structure of a commutative unital ring with unit  $[1_D]$ . If  $\tau$  is a unital trace on  $D$ , then  $\tau$  induces a morphism of unital rings  $\tau_*: K_0(D) \rightarrow \mathbb{R}$ .*

*Proof.* Fix an isomorphism  $\gamma: D \otimes D \rightarrow D$ . The multiplication on  $K_0(D)$  is defined by composing  $\gamma_*: K_0(D \otimes D) \rightarrow K_0(D)$  with the canonical map  $K_0(D) \otimes K_0(D) \rightarrow K_0(D \otimes D)$ . Since any two unital  $*$ -homomorphisms from  $D \otimes D$  to  $D$  are approximately unitarily equivalent, the above multiplication is well-defined and commutative. We leave the rest of proof for the reader, but note that if  $D$  has a unital trace, then  $\tau \otimes \tau$  is the unique unital trace of  $D \otimes D$ .  $\square$

**Proposition 2.7.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra. Suppose that  $D$  is quasidiagonal and that  $K_0(D)$  is torsion free. Then either  $K_0(D) \cong \mathbb{Z}$  or there is a UHF algebra  $B$  of infinite type such that  $K_0(D) \cong K_0(B)$ . If, in addition,  $D$  is assumed to contain a nontrivial projection, then  $D \otimes B \cong D$ , where  $B$  is as above.*

*Proof.* Since  $D$  is quasidiagonal it embeds unitaly in the universal UHF algebra  $B_{\mathbb{Q}}$  and  $D \otimes B_{\mathbb{Q}} \cong B_{\mathbb{Q}}$ , as explained in [5, Rem. 3.10]. The restriction of the unital trace of  $B_{\mathbb{Q}}$  to  $D$  is denoted by  $\tau$ . Thus we have an exact sequence

$$0 \longrightarrow H \longrightarrow K_0(D) \xrightarrow{\tau_*} \tau_* K_0(D) \longrightarrow 0$$

where  $H$  is the kernel of  $\tau_*$ . Since  $\mathbb{Z} \subseteq \tau_* K_0(D) \subseteq \mathbb{Q}$ , and  $K_0(D) \otimes \mathbb{Q} \cong \mathbb{Q}$ , the map  $\tau_* \otimes \text{id}_{\mathbb{Q}}: K_0(D) \otimes \mathbb{Q} \rightarrow \tau_* K_0(D) \otimes \mathbb{Q}$  is an isomorphism. Therefore  $H \otimes \mathbb{Q} = 0$  and so  $H$  is a torsion subgroup of  $K_0(D)$ . But we assumed that

$K_0(D)$  is torsion free and hence  $H = \{0\}$  and  $\tau_*: K_0(D) \rightarrow \tau_*K_0(D) \subseteq \mathbb{Q}$  is an isomorphism of unital rings. The unital subrings of  $\mathbb{Q}$  are easily determined and well-known. They are parametrized by arbitrary sets  $P$  of prime numbers. For each  $P$  the corresponding ring  $R_P$  consists of rational numbers  $r/s$  with  $r$  and  $s$  relatively prime and such that all prime factors of  $s$  are in  $P$ . If  $P = \emptyset$  then  $R_P = \mathbb{Z}$ , otherwise  $R_P$  is isomorphic to the  $K_0$ -ring associated to a UHF algebra  $B$  of infinite type.

Suppose now that  $D$  contains a nontrivial projection. By Theorem 2.5,  $D$  has real rank zero and absorbs the Jiang-Su algebra  $\mathcal{Z}$ . In particular,  $K_0(D)$  is not  $\mathbb{Z}$  and is hence isomorphic (as a scaled abelian group) to  $K_0(B)$  for some UHF-algebra  $B$  of infinite type. It follows from [8] that  $D$  has stable rank one and that  $K_0(D)$  is weakly unperforated. Moreover, by [1, Sect. 6.9],  $K_0(D)$  has the strict order induced by  $\tau_*$ . The isomorphism  $K_0(B) \cong K_0(D)$  of scaled abelian groups is therefore an order isomorphism, and by the properties of  $D$  established above we can conclude that  $B$  embeds unitaly into  $D$ , whence  $D \otimes B \cong D$ .  $\square$

**Corollary 2.8.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra with torsion free  $K_0$ -group. Suppose that  $D$  contains a non-trivial projection and that  $D$  embeds unitaly into the UHF algebra  $M_{p^\infty}$  for some prime number  $p$ . Then  $D \cong M_{p^\infty}$ .*

*Proof.* By Proposition 2.7 there is a prime  $q$  such that  $M_{q^\infty}$  is contained unitaly in  $D$  and hence in  $M_{p^\infty}$ . From this we deduce that  $q = p$ . Finally since  $M_{p^\infty} \subseteq D \subseteq M_{p^\infty}$  we conclude that  $D \cong M_{p^\infty}$ .  $\square$

### 3. STRONGLY SELF-ABSORBING ALGEBRAS AND K-THEORY

The class of strongly self-absorbing Kirchberg algebras satisfying the UCT was completely described in [11]. In this section we give properties and characterizations of strongly self-absorbing Kirchberg algebras which can be derived without assuming the UCT. For a unital  $C^*$ -algebra  $D$  we denote by  $\nu_D$  the unital  $*$ -homomorphism  $\mathbb{C} \rightarrow D$ . When the  $C^*$ -algebra  $D$  is clear from context we will write  $\nu$  instead of  $\nu_D$ .

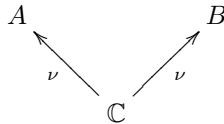
**Proposition 3.1.** *Let  $D$  be a strongly self-absorbing  $C^*$ -algebra. If  $D$  is not finite and the unital  $*$ -homomorphism  $\mathbb{C} \rightarrow D$  induces a surjection  $K^0(D) \rightarrow K^0(\mathbb{C})$ , then  $D \cong \mathcal{O}_\infty$ .*

*Proof.* By [11, Prop. 5.12], two strongly self-absorbing  $C^*$ -algebras are isomorphic if and only if they embed unitaly into each other. Thus it suffices to show the existence of unital  $*$ -homomorphisms  $\mathcal{O}_\infty \rightarrow D$  and  $D \rightarrow \mathcal{O}_\infty$ . Since  $D$  is not finite, it must be a Kirchberg algebra, see [11, Sec. 1], and hence  $\mathcal{O}_\infty$  embeds unitaly in  $D$  by [10, Prop. 4.2.3]. It remains to show that  $D$  embeds unitaly in  $\mathcal{O}_\infty$ .

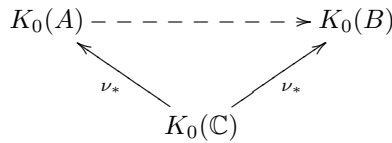
By assumption, the map  $\nu^*: KK(D, \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C})$  is surjective. By multiplying with the  $KK$ -equivalence class given by the unital morphism

$\mathbb{C} \rightarrow \mathcal{O}_\infty$ , we obtain that the map  $\nu^*: KK(D, \mathcal{O}_\infty) \rightarrow KK(\mathbb{C}, \mathcal{O}_\infty)$  is surjective. If  $\varphi: D \rightarrow \mathcal{O}_\infty \otimes \mathcal{K}$  is a  $*$ -homomorphism, then, after identifying  $KK(\mathbb{C}, \mathcal{O}_\infty) \cong K_0(\mathcal{O}_\infty)$ , the map  $\nu^*$  sends  $[\varphi]$  to the class  $[\varphi(1_D)] \in K_0(\mathcal{O}_\infty)$ . By [10, Thm. 8.3.3] each element of  $KK(D, \mathcal{O}_\infty)$  is represented by a  $*$ -homomorphism. Therefore, by the surjectivity of  $\nu^*$ , there is a  $*$ -homomorphism  $\varphi: D \rightarrow \mathcal{O}_\infty \otimes \mathcal{K}$  such that  $[\varphi(1_D)] = [1_{\mathcal{O}_\infty}]$ . Since these are both full projections, by [10, Prop. 4.1.4] there is a partial isometry  $v \in \mathcal{O}_\infty \otimes \mathcal{K}$  such that  $v^*v = \varphi(1_D)$  and  $vv^* = 1_{\mathcal{O}_\infty}$ . Then  $\nu\varphi\nu^*$  is a unital embedding  $D \rightarrow \mathcal{O}_\infty$ .  $\square$

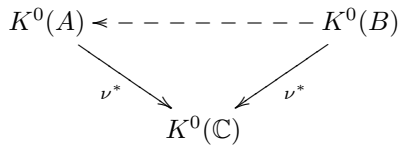
**Remark 3.2.** Note that the isomorphism  $D \cong \mathcal{O}_\infty$  was obtained without assuming that  $D$  satisfies the UCT. Let us argue that assumptions of Proposition 3.1 are natural. Let  $A$  and  $B$  be unital  $C^*$ -algebras and let  $\nu: \mathbb{C} \rightarrow A$  and  $\nu: \mathbb{C} \rightarrow B$  be the corresponding unital  $*$ -homomorphisms. The condition that there is a morphism of pointed groups  $(K_0(A), [1_A]) \rightarrow (K_0(B), [1_B])$  can be viewed as the condition that the diagram



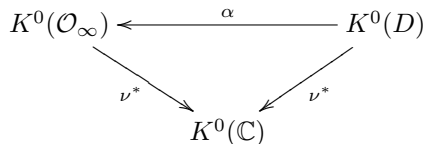
can be completed to a commutative diagram after passing to  $K$ -theory:



It would then be completely natural to use  $K$ -homology instead of  $K$ -theory and ask that the first diagram can be completed to a commutative diagram after passing to  $K$ -homology.



Now let us observe that the condition, imposed in Proposition 3.1, that  $\nu^*: K^0(D) \rightarrow K^0(\mathbb{C})$  is surjective clearly is equivalent to the existence of a commutative diagram





**Proposition 3.4.** *Let  $D$  be a unital Kirchberg algebra such that  $D \cong D \otimes D$ . The following assertions are equivalent:*

- (1)  $D$  is strongly self-absorbing.
- (2)  $KK(C_\nu, SD) = 0$ .
- (3)  $KK(C_\nu, D \otimes A) = 0$  for all separable  $C^*$ -algebras  $A$ .
- (4) The map  $KK(D, D \otimes A) \rightarrow KK(\mathbb{C}, D \otimes A)$  is bijective for all separable  $C^*$ -algebras  $A$ .

*Proof.* (iii)  $\Leftrightarrow$  (iv). This equivalence is verified by using the Puppe sequence associated to  $\nu: \mathbb{C} \rightarrow D$ , arguing as in the proof of Proposition 3.3.

(i)  $\Rightarrow$  (iv). This implication is proved in [5, Thm. 3.4].

(iii)  $\Rightarrow$  (ii). This follows by taking  $A = S\mathbb{C}$  in (iii).

(ii)  $\Rightarrow$  (i). Fix an isomorphism  $\gamma: D \rightarrow D \otimes D$ . Since  $KK_1(C_\nu, D \otimes D) = 0$  by hypothesis, it follows from the Puppe sequence that the map  $\nu^*: KK(D, D \otimes D) \rightarrow KK(\mathbb{C}, D \otimes D)$  is injective. Therefore  $\gamma$  and  $\nu \otimes \text{id}_D$  induce the same class in  $KK(D, D \otimes D)$  since they are both unital. It follows that  $\nu \otimes \text{id}_D$  is asymptotically unitarily equivalent to  $\gamma$  and so  $D$  is strongly self-absorbing.  $\square$

**Corollary 3.5.** *Let  $D$  be a unital Kirchberg algebra such that  $D \cong D \otimes D$ . Then  $D$  is strongly self-absorbing if and only if  $\pi_2 \text{Aut}(D) = 0$ .*

*Proof.* Since  $\pi_2 \text{Aut}(D) \cong KK(C_\nu, SD)$  by [5, Cor. 3.1], the conclusion follows from Proposition 3.4.  $\square$

It was shown in [4, Prop. 4.1] that if a unital Kirchberg algebra satisfies the UCT, then  $D$  is strongly self-absorbing if and only if the homotopy classes  $[X, \text{Aut}(D)]$  reduces to a singleton for any path connected compact metrizable space  $X$ .

## REFERENCES

- [1] Blackadar, B. *K-theory for operator algebras*, Second edition, Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.
- [2] Blackadar, B.; Kumjian, A. and Rørdam, M. *Approximately central matrix units and the structure of noncommutative tori*, *K-Theory* 6 (1992), no. 3, 267–284.
- [3] Dadarlat, M. *The homotopy groups of the automorphism group of Kirchberg algebras*, *J. Noncommut. Geom.* 1 (2007), no. 1, 113–139.
- [4] Dadarlat, M. *Fiberwise KK-equivalence of continuous fields of  $C^*$ -algebras*, to appear in *J. K-theory*.
- [5] Dadarlat, M. and Winter, W. *On the K-theory of strongly self-absorbing  $C^*$ -algebras*, *Math. Scand.* 104 (2009), no. 1, 95–107.
- [6] Gong, G.; Jiang, X. and Su, H. *Obstructions to  $\mathcal{Z}$ -stability for unital simple  $C^*$ -algebras*, *Canad. Math. Bull.* 43 (2000), no. 4, 418–426.
- [7] Haagerup, U. *Every quasi-trace on an exact  $C^*$ -algebra is a trace*, preprint 1991.
- [8] Rørdam, M. *The stable and the real rank of  $\mathcal{Z}$ -absorbing  $C^*$ -algebras*, *International J. Math.*, 15, No. 10 (2004), 1065–1084.
- [9] Rørdam, M. and Winter, W. *The Jiang-Su algebra revisited*, to appear in *J. Reine Angew. Math.*

- [10] Rørdam, M. *Classification of nuclear, simple  $C^*$ -algebras*, Classification of nuclear  $C^*$ -algebras. Entropy in operator algebras, 1–145, Encyclopaedia Math. Sci., 126, Springer, Berlin, 2002.
- [11] Toms, A. S. and Winter, W. *Strongly self-absorbing  $C^*$ -algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.
- [12] Toms, A. S. and Winter, W.  *$\mathcal{Z}$ -stable *ASH* algebras*, Canad. J. Math. 60 (2008), no. 3, 703–720.
- [13] Winter, W. *Localizing the Elliott conjecture at strongly self-absorbing  $C^*$ -algebras*, preprint.

Department of Mathematics  
Purdue University  
West Lafayette  
IN 47906, USA

*E-mail address:* [mdd@math.purdue.edu](mailto:mdd@math.purdue.edu)  
*Internet home page:* [www.math.purdue.edu/~mdd](http://www.math.purdue.edu/~mdd)

Department of Mathematics  
University of Copenhagen  
Universitetsparken 5  
2100 Copenhagen Ø  
Denmark

*E-mail address:* [rordam@math.ku.dk](mailto:rordam@math.ku.dk)  
*Internet home page:* [www.math.ku.dk/~rordam](http://www.math.ku.dk/~rordam)