FIBERWISE KK-EQUIVALENCE OF CONTINUOUS FIELDS OF C*-ALGEBRAS

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Abstract. Let A and B be separable nuclear continuous $C(X)$-algebras over a finite dimensional compact metrizable space X. It is shown that an element σ of the parametrized Kasparov group $\mathrm{KK}_X(A,B)$ is invertible if and only all its fiberwise components $\sigma_x \in KK(A(x),B(x))$ are invertible. This criterion does not extend to infinite dimensional spaces since there exist nontrivial unital separable continuous fields over the Hilbert cube with all fibers isomorphic to the Cuntz algebra $\mathcal{O}_2$. Several applications to continuous fields of Kirchberg algebras are given. It is also shown that if each fiber of a separable nuclear continuous $C(X)$-algebra A over a finite dimensional locally compact space X satisfies the UCT, then A satisfies the UCT.

1. Introduction

Continuous C*-bundles arise naturally: any separable C*-algebra A with Hausdorff primitive space X is isomorphic to the C*-algebra of continuous sections of a C*-bundle over X with fibers the primitive quotients of A [14], [5]. A continuous C*-bundle (also called continuous field [14] or continuous $C(X)$-algebra [18]) needs not be locally trivial at any point. In his work on the Novikov conjecture [18], Kasparov has introduced parametrized KK-theory groups $\mathcal{R}KK(X;A,B)$ for $C(X)$-algebras A and B. These groups, for which we prefer the more compact notation $KK_X(A,B)$, admit a natural product structure $KK_X(A,B) \times KK_X(B,C) \to KK_X(A,C)$. The invertible elements in $KK_X(A,B)$ are denoted by $KK_X(A,B)^{-1}$. If $KK_X(A,B)^{-1} \neq \emptyset$ we say that A is KK_X-equivalent to B. The work of Kirchberg on the classification of purely infinite C*-algebras raises the question of determining when two $C(X)$-algebras are KK_X-equivalent. Indeed, by [19], if A and B are two separable nuclear C*-algebras with Hausdorff spectrum X, then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ if and only if A is KK_X-equivalent to B. A closely related problem is to characterize the invertible elements of $\mathrm{KK}_X(A,B)$. We give a criterion for KK_X-invertibility for spaces X of finite covering dimension.

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Theorem 1.1. *Let A and B be separable nuclear continuous $C(X)$-algebras over a finite dimensional compact metrizable space X. If $\sigma \in \mathrm{KK}_X(A,B)$, then $\sigma \in \mathrm{KK}_X(A,B)^{-1}$ if and only if $\sigma_x \in KK(A(x),B(x))^{-1}$ for all $x \in X$.*

Consequently, if the fibers of A and B satisfy the Universal Coefficient Theorem for the Kasparov groups (abbreviated UCT, [26]) and if there is a $C(X)$-linear morphism from A to B, or just an element of $\mathrm{KK}_X(A,B)$, such that all the induced maps $K_*(A(x)) \to K_*(B(x))$ are bijective, then $K_*(A) \cong K_*(B)$. This opens the way for the use of homological methods for the computation of the K-theory groups of continuous fields. Let us note that Mayer-Vietoris type arguments are not directly applicable due to lack of local triviality. The assumption on the finite dimensionality of X is essential:

Examples 1.2. There is a family $(E_P)_P$ which has the power of the continuum and which consists of mutually nonisomorphic unital separable continuous $C(Z)$-algebras over the Hilbert cube Z with all fibers isomorphic to the Cuntz algebra $\mathcal{O}_2$.

The K_0-groups of E_P are nonzero even though Z is contractible and $KK(\mathcal{O}_2,\mathcal{O}_2)=0$, see Section 3. The family $(E_P)_P$ is easily constructed starting from an example of a continuous $C(Y)$-algebra over $Y=\prod_{n=1}^{\infty} S^2$ with fibers isomorphic to the CAR algebra but which does not absorb the CAR algebra, exhibited in [16].

By specializing Theorem 1.1 to the case when A is a trivial $C(X)$-algebra, we obtain, based on results of Kirchberg [19], an explicit necessary and sufficient (Fell type) K-theory condition for triviality of continuous fields of arbitrary Kirchberg algebras: Corollary 2.8. In particular we no longer require KK-semiprojectivity of the fibers as in our earlier approach [8] which was not relying on [19].

Let us say that a unital C*-algebra D has the *automatic triviality property* if any separable unital continuous $C(X)$-algebra over a finite dimensional compact metrizable space X all of whose fibers are isomorphic to D is isomorphic to $C(X)\otimes D$. We proved in [8] that the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are the only Kirchberg algebras with the automatic triviality property among those Kirchberg algebras satisfying the UCT and having finitely generated K-theory. By combining Corollary 2.8 with our homotopy calculations from [9] and the recent absorption result of Hirshberg, Rørdam and Winter [16], we are now able to drop the finite generation condition.

Theorem 1.3. *A unital Kirchberg algebra D satisfying the UCT has the automatic triviality property if and only D is isomorphic to either $\mathcal{O}_2$, $\mathcal{O}_\infty$, or $\mathcal{O}_\infty \otimes U$ where U is a unital uniformly hyperfinite algebra of infinite type.*

The condition that X is finite dimensional cannot be dropped as shown by Example 1.2. Let us recall that a Kirchberg algebra is a purely infinite simple nuclear separable C*-algebra [25] and that a separable C*-algebra A satisfies the UCT if and only if A is

KK-equivalent to a commutative C*-algebra [26]. The class of C*-algebras satisfying the UCT is surprisingly large; it contains the C*-algebras of locally compact second countable amenable groupoids [30].

The list given by Theorem 1.3 coincides with the list of all strongly selfabsorbing Kirchberg algebras satisfying the UCT exhibited in [29]. In a forthcoming joint paper with Winter [13], we show that any K_1-injective separable strongly selfabsorbing C*-algebra has the automatic triviality property.

In the last part of the paper we prove a new permanence property for the class of nuclear C*-algebras which satisfy the UCT:

Theorem 1.4. *A separable nuclear continuous $C(X)$-algebra over a finite dimensional locally compact space satisfies the UCT if all its fibers satisfy the UCT.*

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2. $C(X)$-algebras and KK_X-equivalence

Kirchberg has shown that any nuclear separable C*-algebra is equivalent in KK-theory to a Kirchberg algebra [25, Prop. 8.4.5]. We extend his result in the context of continuous $C(X)$-algebras and KK_X-theory (see Theorem 2.5). *The space X is assumed to be compact and metrizable throughout this section.*

Lemma 2.1 ([3, Prop. 3.2])**.** *If A is a continuous $C(X)$-algebra, then there is a split short exact sequence of continuous $C(X)$-algebras*

$$0 \longrightarrow A \longrightarrow A^+ \underset{\alpha}{\rightleftarrows} C(X) \longrightarrow 0 \qquad (1)$$

where A^+ is unital, α is $C(X)$-linear and $\alpha(1) = 1$.

Consider the category of separable $C(X)$-algebras where the morphisms from A to B are the elements of $KK_X(A, B)$ with composition given by the Kasparov product. The isomorphisms in this category are the KK_X-invertible elements denoted by $KK_X(A, B)^{-1}$. Two $C(X)$-algebras are KK_X-equivalent if they are isomorphic objects in this category. In the sequel we shall use twice the following elementary observation (valid in any category). If composition with $\gamma \in KK_X(A, B)$ induces a bijection $\gamma^* : KK_X(B, C) \to KK_X(A, C)$ (or $\gamma_* : KK_X(C, A) \to KK_X(C, B)$) for $C = A$ and $C = B$, then $\gamma \in KK_X(A, B)^{-1}$.

Lemma 2.2. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear unital continuous $C(X)$-algebra $A^\flat$ and $C(X)$-linear monomorphisms $\alpha : C(X) \otimes \mathcal{O}_2 \to A^\flat$ and $\jmath : A \to A^\flat$ such that α is unital and* $\mathrm{KK}_X(\jmath) \in \mathrm{KK}_X(A, A^\flat)^{-1}$.

Proof. Let $p \in \mathcal{O}_\infty$ be a non-zero projection with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$. Then there is a unital $*$-homomorphism $\mathcal{O}_2 \to p\mathcal{O}_\infty p$ which induces a $C(X)$-linear unital $*$-monomorphism $\mu : C(X) \otimes \mathcal{O}_2 \to C(X) \otimes p\mathcal{O}_\infty p$. We tensor the exact sequence (1) by $p\mathcal{O}_\infty p$ and then take the pullback by μ. This gives a split exact sequence of $C(X)$-algebras:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r] & A^+ \otimes p\mathcal{O}_\infty p \arrow[r, shift left] & C(X) \otimes p\mathcal{O}_\infty p \arrow[l, shift left, "\alpha"] \arrow[r] & 0 \\
0 \arrow[r] & A \otimes p\mathcal{O}_\infty p \arrow[r, "j"] \arrow[u, equal] & A^\flat \arrow[r, shift left] \arrow[u] & C(X) \otimes \mathcal{O}_2 \arrow[l, shift left, "\alpha"] \arrow[r] \arrow[u, "\mu"'] & 0
\end{tikzcd}$$

The map $A^\flat \to A^+ \otimes p\mathcal{O}_\infty p$ is a unital $C(X)$-linear $*$-monomorphism, so that $A^\flat$ is a continuous $C(X)$-algebra. It is nuclear being an extension of nuclear C*-algebras. By [1, Thm. 5.4] for any separable nuclear continuous $C(X)$-algebra B there is an exact sequence of groups

$$0 \to \mathrm{KK}_X(B, A \otimes p\mathcal{O}_\infty p) \xrightarrow{j_*} \mathrm{KK}_X(B, A^\flat) \longrightarrow \mathrm{KK}_X(B, C(X) \otimes \mathcal{O}_2) \to 0.$$

$\mathrm{KK}_X(B, C(X) \otimes \mathcal{O}_2) = 0$ since the class of the identity map of $C(X) \otimes \mathcal{O}_2$ vanishes in KK_X. Therefore j_* is bijective and so $\mathrm{KK}_X(\jmath) \in \mathrm{KK}_X(A \otimes p\mathcal{O}_\infty p, A^\flat)^{-1}$. We conclude the proof by observing that map $A \to A \otimes p\mathcal{O}_\infty p$, $a \mapsto a \otimes e$, induces a KK_X-equivalence, if e is a subprojection of p equivalent to $1_{\mathcal{O}_\infty}$. $\square$

Let (A_i, φ_i) be an inductive system of separable nuclear unital continuous $C(X)$-algebras with unital (fiberwise) injective connecting maps. Let $A = \varinjlim (A_i, \varphi_i)$ be the inductive limit C*-algebra and let $\varphi_{i,\infty} : A_i \to A$ be the induced inclusion map.

Lemma 2.3. *A is a unital nuclear continuous $C(X)$-algebra and there is a sequence $(\eta_i : A \to A_{n(i)})_i$ of unital completely positive $C(X)$-linear maps such that $(\varphi_{n(i),\infty} \circ \eta_i)_i$ converges to id_A in the point norm topology.*

Proof. The map $C(X) \to Z(A)$ is induced by the maps $C(X) \to Z(A_i)$, so that A clearly becomes a $C(X)$-algebra. The continuity of the map $x \mapsto \|a(x)\|$ for $a \in A$ is verified by approximating $a \in A$ by some $a_i \in A_i$ and using the fiberwise injectivity of $\varphi_{i,\infty}$. Since A is a nuclear C*-algebra, it follows by [1, Thm. 7.2] that A is $C(X)$-nuclear. This means that there are sequences of unital $C(X)$-linear completely positive maps $\alpha_i : A \to C(X) \otimes M_{k(i)}$ and $\beta_i : C(X) \otimes M_{k(i)} \to A$ such that $\beta_i \circ \alpha_i$ converges to id_A in the point-norm topology. After perturbing the restriction of β_i to $1_{C(X)} \otimes M_{k(i)}$ to a unital and completely positive map $\beta_i' : 1_{C(X)} \otimes M_{k(i)} \to A_{n(i)}$ and extending β_i' to a $C(X)$-linear map $\beta_i'' : C(X) \otimes M_{k(i)} \to A_{n(i)}$, we may assume that β_i factorizes as $\beta_i = \varphi_{n(i),\infty} \circ \beta_i''$. Then $\eta_i = \beta_i'' \circ \alpha_i$ satisfies the conclusion of the lemma. $\square$

Proposition 2.4. *Let (A_i, φ_i) be an inductive system of separable nuclear unital continuous $C(X)$-algebras with unital injective connecting maps. If $\varphi_i \in \mathrm{KK}_X(A_i, A_{i+1})^{-1}$ for all i, and $\Phi : A_1 \to \varinjlim(A_i, \varphi_i) = A$ is the induced map, then $\Phi \in \mathrm{KK}_X(A_1, A)^{-1}$.*

Proof. We use Milnor's $\varprojlim^1$-exact sequence in KK_X-theory applied to the inductive system (A_i, φ_i). The proof of σ-additivity of $KK(A, B)$ in the first variable given in [18, Thm. 2.9] applies with essentially no changes to show the corresponding property for $\mathrm{KK}_X(A, B)$. Lemma 2.3 verifies the assumptions of [24, Lemma 2.7]. Thus the system (A_i, φ_i) is admissible in the sense of [24, Def. 2.5] and hence by [24, Lemma 2.4 and Prop. 2.6] we have an exact sequence:

$$0 \to \varprojlim{}^1 \mathrm{KK}^1_X(A_i, B) \longrightarrow \mathrm{KK}_X(A, B) \longrightarrow \varprojlim \mathrm{KK}_X(A_i, B) \to 0$$

(One can also give a direct proof of this exact sequence which is essentially identical to the proof of the corresponding sequence in KK-theory. One argues as in [26] using the exact sequences from [1]. The maps from Lemma 2.3 are needed to verify that the mapping telescope extension of A is semisplit in the category of $C(X)$-algebras.)

Since $\varprojlim^1 \big(G_i, \lambda_i\big) = 0$ and $G_1 \cong \varprojlim \big(G_i, \lambda_i\big)$ for any sequence of abelian groups $(G_i)_{i=1}^{\infty}$ and group isomorphisms $\lambda_i : G_i \to G_{i+1}$, the $\varprojlim^1$-exact sequence shows that for any separable continuous $C(X)$-algebra B, the map $\mathrm{KK}_X(A, B) \to \mathrm{KK}_X(A_1, B)$ induced by Φ is bijective. Therefore $\mathrm{KK}_X(\Phi) \in \mathrm{KK}_X(A_1, A)^{-1}$. □

We need the following $C(X)$-equivariant construction which parallels a construction of Kirchberg as presented in [25]. A similar deformation technique has appeared in [10].

Theorem 2.5. *Let A be a separable nuclear continuous $C(X)$-algebra. Then there exist a separable nuclear continuous unital $C(X)$-algebra $A^\sharp$ whose fibers are Kirchberg C^*-algebras and a $C(X)$-linear $*$-monomorphism $\Phi : A \to A^\sharp$ such that Φ is a KK_X-equivalence. For any $x \in X$ the map $\Phi_x : A(x) \to A^\sharp(x)$ is a KK-equivalence.*

Proof. By Proposition 2.2 we may assume that A is unital and that there is a unital $C(X)$-linear $*$-monomorphism $\alpha : C(X) \otimes \mathcal{O}_2 \to A$. By [4, Thm. 2.5] there is a unital $C(X)$-linear $*$-monomorphism $\beta : A \to C(X) \otimes \mathcal{O}_2$. Let s_1, s_2 be the images in A of the canonical generators v_1, v_2 of $\mathcal{O}_2 \subset C(X) \otimes \mathcal{O}_2$ under the map α. Set $\theta = \alpha \circ \beta : A \to A$ and define $\varphi : A \to A$ by $\varphi(a) = s_1\, a\, s_1^* + s_2\, \theta(a)\, s_2^*$. The unital $*$-homomorphism $\varphi_x : A(x) \to A(x)$ induced by φ satisfies $\varphi_x \pi_x = \pi_x \varphi$ and $\varphi_x(b) = s_1(x)\, b\, s_1(x)^* + s_2(x)\, \theta_x(b)\, s_2(x)^*$. Moreover θ_x factors through $\mathcal{O}_2$ since $\theta_x = \alpha_x \circ \beta_x$. Let $A^\sharp$ be the continuous $C(X)$-algebra obtained as the limit of the inductive system

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

and let $\Phi : A \to A^\sharp$ be the induced map. The commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & A \arrow[r, "\varphi"] \arrow[d, "\pi_x"] & \cdots \arrow[r] & A^\sharp \arrow[d, "\pi_x"] \\ A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & A(x) \arrow[r, "\varphi_x"] & \cdots \arrow[r] & A^\sharp(x) \end{tikzcd}$$

shows that the fiber $A^\sharp(x)$ of A is isomorphic to $\varinjlim(A(x), \varphi_x)$. By the proof of [25, Prop. 8.4.5] $A^\sharp(x)$ is a unital Kirchberg algebra. It remains to prove that the map $\Phi : A \to A^\sharp$ induces a KK_X-equivalence. By Proposition 2.4 it suffices to verify that $\mathrm{KK}_X(\varphi) = \mathrm{KK}_X(\mathrm{id}_A)$. This follows from the equation $\varphi(a) = s_1\, a\, s_1^* + s_2\, \theta(a)\, s_2^*$, since θ factors through $C(X) \otimes \mathcal{O}_2$ and hence $\mathrm{KK}_X(\theta) = 0$. $\square$

Theorem 2.6. *Let X be a compact metrizable finite dimensional space. Let A be a separable nuclear continuous $C(X)$-algebra. If all the fibers of A are* KK*-contractible, then A is* KK_X*-contractible.*

Proof. By Theorem 2.5, A is KK_X-equivalent to a separable nuclear unital continuous $C(X)$-algebra $A^\sharp$ whose fibers are KK-contractible Kirchberg algebras. Therefore $A^\sharp(x) \cong \mathcal{O}_2$ for all x [25]. By [8, Thm. 1.1] $A^\sharp$ is isomorphic to $C(X) \otimes \mathcal{O}_2$ and hence is KK_X-contractible. Alternately one can argue that $A^\sharp \cong A^\sharp \otimes \mathcal{O}_2$ by [16] and hence that $\mathrm{KK}_X(A^\sharp, A^\sharp) = 0$. $\square$

Proof of Theorem 1.1

Proof. By Theorem 2.5 we may assume that both A and B are stable continuous $C(X)$-algebras which absorb $\mathcal{O}_\infty$ tensorially and whose fibers are Kirchberg algebras. By [19, Hauptsatz 4.2], for any given $\sigma \in \mathrm{KK}_X(A, B)$, there is a $C(X)$-linear $*$-homomorphism $\varphi : A \to B$ such that $\mathrm{KK}_X(\varphi) = \sigma$. The mapping cone of φ,

$$C_\varphi = \{(a, f) \in A \oplus C_0[0, 1) \otimes B \,:\, f(0) = \varphi(a)\}$$

is a separable nuclear continuous $C(X)$-algebra with fibers $C_\varphi(x) \cong C_{\varphi_x}$, $x \in X$. Since each φ_x is a KK-equivalence, it follows from the Puppe sequence in KK-theory [2] that C_{φ_x} is KK-contractible for each $x \in X$. Then C_φ is KK_X-contractible by Theorem 2.6. Using the Puppe exact sequence for separable nuclear continuous $C(X)$-algebras (see [1])

$$\mathrm{KK}_X(C, C_\varphi) \longrightarrow \mathrm{KK}_X(C, A) \xrightarrow{\varphi_*} \mathrm{KK}_X(C, B) \longrightarrow \mathrm{KK}^1_X(C, C_\varphi)$$

we see that $\varphi_* : \mathrm{KK}_X(C, A) \to \mathrm{KK}_X(C, B)$ is bijective for all separable nuclear continuous $C(X)$-algebras C and hence σ is a KK_X-equivalence. Note that if we assume in the statement that $\sigma = KK_X(\varphi)$ for some morphism φ of $C(X)$-algebras, then the result from [19] is not required for the proof. $\square$

A remarkable isomorphism result for separable nuclear strongly purely infinite stable C*-algebras was announced (with an outline of the proof) by Kirchberg in [19]: two such C*-algebras A and B with the same primitive spectrum X are isomorphic if and only if they are KK_X-equivalent. In conjunction with Theorem 1.1 we derive the following.

Theorem 2.7. *Let X be a compact metrizable finite dimensional space. Let A and B be separable continuous $C(X)$-algebras all of whose fibers are Kirchberg algebras. Suppose that there is $\sigma \in \mathrm{KK}_X(A,B)$ such that $\sigma_x \in KK(A(x),B(x))^{-1}$ for all $x \in X$. Then there is an isomorphism of $C(X)$-algebras $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ such that $\mathrm{KK}_X(\varphi) = \sigma$. Moreover if A and B are unital and if $K_0(\sigma)[1_A] = [1_B]$, then $A \cong B$.*

Proof. Since X is finite dimensional, $A \otimes \mathcal{K} \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{K}$ and $B \otimes \mathcal{K} \otimes \mathcal{O}_\infty \cong B \otimes \mathcal{K}$ by [6, Cor. 5.11]. In the unital case, one also has $A \otimes \mathcal{O}_\infty \cong A$ and $B \otimes \mathcal{O}_\infty \cong B$ by [6, Cor. 5.11], [20, Thm. 4.23] and [21, Thm. 8.6] as explained for example in [12, Lemma 3.4] or by [16]. By Theorem 1.1 σ is a KK_X-equivalence. This enables us to apply Kirchberg's result [19, Folgerung 4.3] to obtain an isomorphism of $C(X)$-algebras $\varphi : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ such that $\mathrm{KK}_X(\varphi) = \sigma$. In the unital case, since both $\varphi(1_A \otimes e_{11})$ and $1_B \otimes e_{11}$ are full and properly infinite projections in $B \otimes \mathcal{K}$, the condition $\varphi_*[1_A] = [1_B]$ will allows us to arrange that $\varphi(1_A \otimes e_{11}) = 1_B \otimes e_{11}$ after conjugating φ by a suitable unitary in $M(B \otimes \mathcal{K})$ (see [7]) and hence conclude that $A \cong B$. □

Corollary 2.8. *Let X be a compact metrizable finite dimensional space. Let B be a separable continuous unital $C(X)$-algebra all of whose fibers are Kirchberg algebras and let D be a unital Kirchberg algebra. Suppose that there is $\sigma \in KK(D,B)$ such that $\sigma_x \in KK(D,B(x))^{-1}$ for all $x \in X$ and $K_0(\sigma)[1_D] = [1_B]$. Then $B \cong C(X) \otimes D$.*

Proof. This follows from the previous theorem since $\mathrm{KK}_X(C(X) \otimes D, B) \cong KK(D,B)$. To explain this isomorphism of groups, let us recall from [18, 2.19] that if A and B are separable $C(X)$-algebras, then $\mathrm{KK}_X(A,B)$ is defined in the same way as $\mathrm{KK}(A,B)$ except that one replaces the set $\mathcal{E}(A,B)$ of Kasparov A-B-bimodules $(\Phi : A \to \mathcal{L}(E), T)$ by its subset $\mathcal{E}_X(A,B)$ consisting of those elements satisfying an additional assumption: for any $f \in C(X)$, $a \in A$, $b \in B$ and $\xi \in E$ one has the equality

$$\Phi(fa)\xi b = \Phi(a)\xi(fb). \tag{2}$$

Setting $A = C(X) \otimes D$, it is then easily verified that there is a bijection

$$\eta : \mathcal{E}_X(C(X) \otimes D, B) \to \mathcal{E}(D,B),$$

which maps $(\Phi : C(X) \otimes D \to \mathcal{L}(E), T)$ to $(\varphi : D \to \mathcal{L}(E), T)$, where $\varphi(d) = \Phi(1 \otimes d)$ for all $d \in D$. The map η is injective since by condition (2) $\Phi(f \otimes d)\xi = \varphi(d)\xi(f1_B)$. Let us check now that η is surjective. Let $(\varphi : D \to \mathcal{L}(E), T) \in \mathcal{E}(D,B)$ be given. If $\theta : C(X) \to$

$\mathcal{L}(E)$ is the central $*$-homomorphism $\theta(f)\xi = \xi(f1_B)$, define $\Phi : C(X)\otimes D \to \mathcal{L}(E)$ by $\Phi(f\otimes d) = \theta(f)\varphi(d)$. Then Φ satisfies (2) and the operators $[\Phi(f\otimes d), T] = \theta(f)[\varphi(d), T]$, $\Phi(f\otimes d)(T - T^*) = \theta(f)\varphi(d)(T-T^*)$, $\Phi(f\otimes d)(T^2-1) = \theta(f)\varphi(d)(T^2-1)$ are in $\mathcal{K}(E)$ since $[\varphi(d), T]$, $\varphi(d)(T-T^*)$ and $\varphi(d)(T^2-1)$ are in $\mathcal{K}(E)$. Therefore $(\Phi : C(X)\otimes D \to \mathcal{L}(E), T) \in \mathcal{E}_X(C(X)\otimes D, B)$ and its image under η is equal to $(\varphi : D \to \mathcal{L}(E), T)$. $\square$

3. Nontrivial $\mathcal{O}_2$-bundles

Let Z denote the Hilbert cube. Let K be the Cantor set. If G is a discrete group let $C(K,G)$ denote the continuous functions from K to G. For any two countable, abelian, torsion groups G_0 and G_1, we exhibit a unital separable continuous $C(Z)$-algebra E with all fibers isomorphic to $\mathcal{O}_2$ such that $K_i(E) = C(K, G_i)$, $i = 0, 1$.

A crucial ingredient of the construction is an example from [16] which we now recall (with a minor variation). Let e be the unit of $C(S^2)$ and let $f \in M_2(C(S^2))$ be the Bott projection. For each $n \geq 1$ we let $e_n = e$ and $f_n = (e \oplus ... \oplus e) \oplus f$ ($n-1$ copies of e) be realized as orthogonal projections in $M_{n+2}(C(S^2))$. Set

$$B_n = (e_n + f_n)M_{n+2}(C(S^2))(e_n+f_n), \quad A = \bigotimes_{n=1}^{\infty} B_n.$$

Let U be the universal UHF algebra with $K_0(U) = \mathbb{Q}$. Let $Y = \prod_{n=1}^{\infty} S^2$. Arguing as in [16] one shows that A is a continuous $C(Y)$-algebra with all fibers isomorphic to U. Since each B_n is Morita equivalent to $C(S^2)$, $K_0(B_n)$ is freely generated as a $\mathbb{Z}$-module by the classes of $[e_n]$ and $[f_n]$. It is clear that $K_1(A) = 0$. Using the Künneth formula one shows that $K_0(A)$ is isomorphic to the limit of the inductive system of groups

$$\mathbb{Z}^2 \to \mathbb{Z}^4 \to \cdots \to \mathbb{Z}^{2^n} \to \mathbb{Z}^{2^{n+1}} \to \cdots$$

where the n-th connecting morphism maps $x \in \mathbb{Z}^{2^n}$ to $(x,x) \in \mathbb{Z}^{2^n} \oplus \mathbb{Z}^{2^n} \cong \mathbb{Z}^{2^{n+1}}$. Consequently $K_0(A)$ is isomorphic to $C(K,\mathbb{Z})$. Let D be a unital Kirchberg C*-algebra such that $K_i(D) = G_i$, $i = 0,1$ and D satisfies the UCT. Then $F = A \otimes D$ is a unital separable continuous $C(Y)$-algebra whose fibers are isomorphic to $A(x)\otimes D \cong U \otimes D$. By the Künneth formula $K_i(U\otimes D) \cong \mathbb{Q}\otimes G_i = 0$ since both G_0 and G_1 are torsion groups. It follows that all the fibers $F(x)$ of F are isomorphic to $\mathcal{O}_2$ by the Kirchberg-Phillips classification theorem [25]. On the other hand $K_i(F) = K_0(A)\otimes K_i(D) = C(K,\mathbb{Z})\otimes G_i \cong C(K, G_i)$, $i = 0, 1$.

By Blanchard's embedding theorem [4], there is a unital $C(Y)$-linear monomorphism $\eta : F \to C(Y)\otimes \mathcal{O}_2$. Let us regard Y as a compact subset of the Hilbert cube Z. Define

$$E = \{f \in C(Z, \mathcal{O}_2) \, : \, f|_Y \in \eta(F)\}.$$

Then E is a separable unital continuous $C(Z)$-algebra with all fibers isomorphic to $\mathcal{O}_2$. Using the exact sequence:

$$0 \longrightarrow C_0(Z \setminus Y, \mathcal{O}_2) \longrightarrow E \longrightarrow F \longrightarrow 0$$

we see that $K_i(E) \cong K_i(F) \cong C(K, G_i)$, $i = 0, 1$. In particular E is not isomorphic to $C(Z) \otimes \mathcal{O}_2$ if $G_0 \neq 0$ or $G_1 \neq 0$, for example if $D = \mathcal{O}_n$, $2 < n < \infty$. For a nonempty set P of prime numbers, let $\mathbb{Z}(P)$ be the subgroup of $\mathbb{T}$ consisting of all elements whose orders have all prime factors in P. One verifies that $C(K, \mathbb{Z}(P))$ is not isomorphic to $C(K, \mathbb{Z}(P'))$ if $P \neq P'$. Indeed if $p \in P \setminus P'$, then $C(K, \mathbb{Z}(P'))$ does not have elements of order p, unlike $C(K, \mathbb{Z}(P))$. Consequently, the family $(E_P)_P$ obtained by choosing D with $K_0(D) = \mathbb{Z}(P)$ consists of mutually non-isomorphic $C(Z)$-algebras and has the power of the continuum.

4. Automatic triviality

For a C*-algebra D, let $\mathrm{Aut}(D)^0$ denote the path-component of the identity in the automorphism group of D endowed with the point-norm topology.

Proposition 4.1. *Let D be a unital Kirchberg algebra satisfying the UCT. Suppose that $[X, \mathrm{Aut}(D)^0]$ reduces to singleton for any path connected compact metrizable space X. Then D is isomorphic to either $\mathcal{O}_2$, $\mathcal{O}_\infty$, or $\mathcal{O}_\infty \otimes U$ where U is a unital UHF algebra of infinite type.*

Proof. We show that D has the same pointed K-theory groups as one of the listed C*-algebras. Let C_ν denote the mapping cone C*-algebra of the unital inclusion $\nu : \mathbb{C} \to D$. By [9, Thm. 5.9] there is a bijection $[X, \mathrm{Aut}(D)^0] \to KK(C_\nu, SC_0(X \setminus x_0) \otimes D)$ for some (any) point $x_0 \in X$. Since the K-theory groups of $C(X, x_0)$ can be arbitrary countable abelian groups it follows that $KK(C_\nu, A \otimes D) = 0$ for all separable C*-algebras A satisfying the UCT. Using the Puppe sequence in KK-theory [2] we see that the restriction map $\nu^* : KK(D, A \otimes D) \to KK(\mathbb{C}, A \otimes D)$ is bijective for all separable C*-algebras A satisfying the UCT. By the UCT (applied for A and its suspension), this implies that (i) $K_1(D) = 0$, (ii) $\mathrm{Ext}(K_0(D), K_0(A \otimes D)) = 0$ and (iii) the map $\nu^* : \mathrm{Hom}(K_0(D), K_0(A \otimes D)) \to \mathrm{Hom}(K_0(\mathbb{C}), K_0(A \otimes D))$ is bijective for all separable C*-algebras A satisfying the UCT.

First we are going to show that $G = K_0(D)$ is torsion free. We shall use the observation that if M, N are abelian groups and M' is a subgroup of M and N' is a quotient of N, then $\mathrm{Ext}(M', N')$ is a quotient of $\mathrm{Ext}(M, N)$ [15]. Fix a prime p and set $G[p] = \{x \in G : px = 0\}$. Using (ii) for $A = \mathcal{O}_{p+1}$, we obtain that $\mathrm{Ext}(G, G[p]) = 0$ since $G[p] = \mathrm{Tor}(G, \mathbb{Z}/p)$ is a quotient of $K_0(\mathcal{O}_{p+1} \otimes D)$ and hence $\mathrm{Ext}(G, G[p])$ is a quotient of $\mathrm{Ext}(K_0(D), K_0(\mathcal{O}_{p+1} \otimes D)) = 0$. Assuming that $G[p] \neq 0$ we find a subgroup of G isomorphic to $\mathbb{Z}/p$ and hence $\mathrm{Ext}(\mathbb{Z}/p, G[p])$ vanishes since it is a quotient of $\mathrm{Ext}(G, G[p]) = 0$. On the other hand

$\mathrm{Ext}(\mathbb{Z}/p, G[p])$ is isomorphic to $G[p]/pG[p] = G[p]$ and hence $G[p] = 0$. This contradiction shows that G is torsion free.

Let e denote the class of $[1_D] \in K_0(D) = G$. If $e = 0$, then $G = 0$ by applying (iii) for $A = \mathbb{C}$. For the rest of the proof we may assume that $e \neq 0$ and hence that $\mathbb{Z} \cong \mathbb{Z}e \subset G$.

Let A be a commutative C*-algebra such that $K_0(A) = G/\mathbb{Z}$ [27]. Since G is torsion is free, by the Künneth formula $K_0(A \otimes D) \cong K_0(D) \otimes K_0(A) \cong G \otimes G/\mathbb{Z}$. The exact sequence $0 \to \mathbb{Z} \to G \to G/\mathbb{Z} \to 0$ induces an exact sequence

$0 \to \mathrm{Hom}(G/\mathbb{Z}, K_0(A \otimes D)) \to \mathrm{Hom}(G, K_0(A \otimes D)) \to \mathrm{Hom}(\mathbb{Z}, K_0(A \otimes D))$. ¿From (iii) we obtain that $\mathrm{Hom}(G/\mathbb{Z}, K_0(A \otimes D)) = 0$ and hence $\mathrm{Hom}(G/\mathbb{Z}, G \otimes G/\mathbb{Z}) = 0$. Therefore for each $y \in G$, the morphism $G/\mathbb{Z} \to G \otimes G/\mathbb{Z}$, $x \mapsto y \otimes x$ is the zero map and hence $G \otimes G/\mathbb{Z} = 0$. In particular $G/\mathbb{Z}$ must be a torsion group since G contains a copy of $\mathbb{Z}$. We also have that $\mathrm{Tor}(G/\mathbb{Z}, G) = 0$ since G is torsion free. The exact sequence $0 \to \mathbb{Z} \to G \to G/\mathbb{Z} \to 0$ induces an exact sequence $\mathrm{Tor}(G/\mathbb{Z}, G) \to \mathbb{Z} \otimes G \to G \otimes G \to G/\mathbb{Z} \otimes G$. Therefore the map $\theta : G \to G \otimes G$, $\theta(x) = x \otimes e$ is an isomorphism of groups. We also know that G is torsion free, $\mathbb{Z}e \subset G$ and that $G/\mathbb{Z}e$ is a torsion group. Under these conditions it was proved in [29] that G is isomorphic to either $\mathbb{Z}$ or to a subgroup H of $\mathbb{Q}$ with the property that $1/n^2 \in H$ whenever $1/n \in H$ for some nonzero integer n; in both cases e corresponds to 1. Indeed, for every $x \in G$, $x \neq 0$, there is a unique pair of relatively prime integers m and n with $n > 0$ such that $nx = me$. One verifies immediately that $\gamma : G \to \mathbb{Q}$, $\gamma(x) = m/n$, if $x \neq 0$ and $\gamma(0) = 0$ is an injective morphism of groups. Moreover, if $x \in G$ satisfies $\gamma(x) = 1/n$, then $y = \theta^{-1}(x \otimes x)$ satisfies $\gamma(y) = 1/n^2$. □

Proof of Theorem 1.3

Proof. Let D be isomorphic to either $\mathcal{O}_2$, $\mathcal{O}_\infty$, or $\mathcal{O}_\infty \otimes U$ where U is a unital UHF algebra of infinite type. Let X be a finite dimensional compact metrizable space and let A be a unital separable continuous $C(X)$-algebra with all fibers isomorphic to D. Let $\varphi : D \to A \otimes D$ be defined by $\varphi(d) = 1_A \otimes d$ for all $d \in D$ and let $\widetilde{\varphi} : C(X) \otimes D \to A \otimes D$ be the $C(X)$-linear extension of φ. By the UCT, every unital $*$-endomorphism of D is a KK-equivalence. Therefore each $\varphi_x : D \to A(x) \otimes D \cong D$ is a KK-equivalence. It follows that $\mathrm{KK}_X(\widetilde{\varphi})$ satisfies the assumptions of Theorem 2.7 and hence $A \otimes D$ is isomorphic to $C(X) \otimes D$. We conclude the first half of the proof by invoking a recent result of Hirshberg, Rørdam and Winter [16, Thm. 4.3] which shows that if D is strongly selfabsorbing, then $A \otimes D \cong A$.

Conversely, suppose that D is a unital Kirchberg algebra which has the automatic triviality property and satisfies the UCT. Let Y be a finite dimensional compact metric space and let SY be its unreduced suspension. The locally trivial $C(SY)$-algebras with fibers D and structure group $\mathrm{Aut}(D)^0$ are classified by the homotopy classes $[Y, \mathrm{Aut}(D)^0]$

(see [17]) and hence $[Y, \mathrm{Aut}(D)^0]$ must reduce to a singleton since D has the automatic triviality property. We conclude the proof by applying Proposition 4.1. □

5. $C(X)$-algebras and the Universal Coefficient Theorem

Let us recall the notion of *category* of a $C(X)$-algebra with respect to a class $\mathcal{C}$ of C*-algebras [8]. A $C(Z)$-algebra E satisfies $\mathrm{cat}_{\mathcal{C}}(E) = 0$ if there is a finite partition of Z into closed subsets $Z_1, \ldots, Z_r$ $(r \geq 1)$ and there exist C*-algebras $D_1, \ldots, D_r$ in $\mathcal{C}$ such that $E \cong \bigoplus_{i=1}^r C(Z_i) \otimes D_i$. We write $\mathrm{cat}_{\mathcal{C}}(A) \leq n$ if there are closed subsets Y and Z of X with $X = Y \cup Z$ and there exist a $C(Y)$-algebra B with $\mathrm{cat}_{\mathcal{C}}(B) \leq n-1$ and a $C(Z)$-algebra E with $\mathrm{cat}_{\mathcal{C}}(E) = 0$ and a $*$-monomorphism of $C(Y \cap Z)$-algebras $\gamma : E(Y \cap Z) \to B(Y \cap Z)$ such that A is isomorphic to

$$B \oplus_{\pi, \gamma\pi} E = \{(b, e) \in B \oplus E : \pi^Y_{Y \cap Z}(b) = \gamma \pi^Z_{Y \cap Z}(e)\}.$$

One has an exact sequence

$$0 \longrightarrow \{b \in B : \pi_{Y \cap Z}(b) = 0\} \longrightarrow B \oplus_{\pi, \gamma\pi} E \xrightarrow{\pi_Z} E \longrightarrow 0. \tag{3}$$

By definition $\mathrm{cat}_{\mathcal{C}}(A) = n$ if n is the smallest number with the property that $\mathrm{cat}_{\mathcal{C}}(A) \leq n$. If no such n exists, then $\mathrm{cat}_{\mathcal{C}}(A) = \infty$.

Lemma 5.1. *Let A be a $C(X)$-algebra such that $\mathrm{cat}_{\mathcal{C}}(A) = n < \infty$ where $\mathcal{C}$ is the class of all Kirchberg algebras satisfying the UCT. Then A satisfies the UCT.*

Proof. We shall prove by induction on n that if $\mathrm{cat}_{\mathcal{C}}(A) \leq n$, then A and all its closed two-sided ideals satisfy the UCT. If $n = 0$, then $A \cong \oplus_i C(Z_i) \otimes D_i$ and all its closed two-sided ideals satisfy the UCT since each D_i is simple and satisfies the UCT. By a result of [26], if two out of three separable nuclear C*-algebras in a short exact sequence satisfy the UCT, then all three of them satisfy the UCT. For the inductive step we use the exact sequence (3), with E elementary and $\mathrm{cat}_{\mathcal{C}}(B) \leq n-1$. □

Theorem 5.2 ([11])**.** *Let A be a nuclear separable C*-algebra. Assume that for any finite set $\mathcal{F} \subset A$ and any $\varepsilon > 0$ there is a C*-subalgebra B of A satisfying the UCT and such that $\mathcal{F} \subset_\varepsilon B$. Then A satisfies the UCT.*

Proof. For the convenience of the reader we sketch an alternative proof in the case when B is nuclear. It is just this case that is needed in the sequel. By assumption, A admits an exhaustive sequence (A_n) consisting of nuclear separable C*-subalgebras which satisfy the UCT. This means that for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$ there is n such that $\mathcal{F}$ is ε-contained in A_n.

We may assume that A is unital and its unit is contained in each A_n. Let us replace the pair $A_n \subseteq A$ by $A_n \otimes p\mathcal{O}_\infty p \subseteq A \otimes p\mathcal{O}_\infty p$ with p as in Lemma 2.2. If we use the map $\theta : A \otimes p\mathcal{O}_\infty p \hookrightarrow \mathcal{O}_2 \subset 1_A \otimes p\mathcal{O}_\infty p$ and $s_1, s_2 \in 1_A \otimes p\mathcal{O}_\infty p$ to construct $\varphi : A \otimes p\mathcal{O}_\infty p \to A \otimes p\mathcal{O}_\infty p$ as in the proof of Theorem 2.5, then $\varphi(B \otimes p\mathcal{O}_\infty p) \subset B \otimes p\mathcal{O}_\infty p$ for any subalgebra B of A. Therefore we can make the construction $A \mapsto A^\sharp$ functorial with respect to subalgebras. This shows that $A^\sharp$ admits an exhaustive sequence $(A_n^\sharp)$ consisting of nuclear separable C*-subalgebras which satisfy the UCT since each $A_n^\sharp$ is KK-equivalent to A_n. We can write each $A_n^\sharp$ as an inductive limit of a sequence of Kirchberg algebras satisfying the UCT and having finitely generated K-theory groups [25]. Those algebras are weakly semiprojective ([22],[28]; see also [8, Thm. 3.11] for a short proof). Thus $A^\sharp$ admits an exhaustive sequence (B_n) consisting of weakly semiprojective C*-algebras which satisfy the UCT. By a standard perturbation argument ([23]) we see that $A^\sharp$ is isomorphic to the inductive limit of a subsequence (B_{i_n}) of (B_n) and hence $A^\sharp$ satisfies the UCT [26]. Therefore A satisfies the UCT since it is KK-equivalent to $A^\sharp$. □

Proof of Theorem 1.4

Proof. Let A be as in the statement and consider the open set $Y = \{x \in X : A(x) \neq 0\}$. By replacing X by Y and viewing $A \cong C_0(Y)A$ as a $C(Y)$-algebra we may assume that all the fibers of A are nonzero. Let X^+ be the one-point compactification of X. Then $C(X^+)$ is separable by [8, Lemma 2.2]. By [3, Prop. 3.2], there is a unital $C(X^+)$-algebra A^+ which contains A as an ideal and such that $A^+/A \cong C(X^+)$. Thus we have reduced the proof to the case when X is compact and metrizable and A is unital. By Theorem 2.5 we may assume that the fibers of A are Kirchberg C*-algebras satisfying the UCT. By [8, Thm. 4.6], A admits an exhaustive sequence (A_k) such that each A_k verifies the assumptions of Lemma 5.1 and hence A_k satisfies the UCT. We conclude the proof by applying Theorem 5.2. Let us note that the above proof only requires a weaker version of Theorem 2.5 which states that $\Phi : A \to A^\sharp$ and each Φ_x are KK-equivalences. Its proof requires only the usual $\varprojlim^1$-sequence for KK-theory. □

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